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Article

Stancu Type Baskakov–Durrmeyer Operators and Approximation Properties

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Abstract: In this article, we introduce Stancu type generalization of Baskakov–Durrmeyer operators by using inverse Pólya–Eggenberger distribution. We discuss some basic results and approximation properties. Moreover, we study the statistical convergence for these operators.

Keywords: Pólya–Eggenberger distribution; Euler functions; weighted approximation; statistical convergence

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1. Preliminaries and Introduction

The approximation of functions by positive linear operators is an important research area in the classical approximation theory. It provides us key tools for exploring the computer-aided geometric design, numerical analysis and the solutions of ordinary and partial differential equations that arise in the mathematical modeling of real world phenomena. In the last four decades, several operators have been modified and their approximation properties has been discussed in real and complex domain.

A fundamental result in development of functions approximation theory is known as First Weierstrass approximation theorem, given by K. Weierstrass [1] in 1885, namely: for any function $h \in C[a,b]$ and $\varepsilon > 0$, there exists an algebraic polynomial $P(x)$ with real coefficients, such that $|h(u) - P(u)| < \varepsilon$, for any $u \in [a,b]$, where $C[0,1]$ is the space of all continuous functions defined on $[0,1]$. The first proof of the Weierstrass approximation theorem was long and complicated, and provoked many famous mathematicians to find simpler and more instructive proofs. In 1905, E. Borel [2] proposed determination of an interpolation process, that allows finding polynomials $P(u)$, which converge uniformly to the function $h \in C[a,b]$. In 1912, S.N. Bernstein was able to give an outstanding solution for the problem proposed by E. Borel.

In [3], Bernstein polynomial was proposed to provide an easy and elegant proof of the famous Weierstrass theorem which is defined by

$$\mathcal{B}_r(h)(u) = \mathcal{B}_r(h;u) = \sum_{j=0}^{r} \binom{r}{j} u^j (1-u)^{r-j} h\left(\frac{j}{r}\right) \tag{1}$$

$h \in C[0,1]$.

To deal with Lebesgue integrable functions, for which Bernstein operators are unsuitable, Kantorovich [4] studied the following operators:

$$\mathcal{K}_r(h;u) = (j+1) \sum_{k=0}^{j} \binom{j}{k} u^k (1-u)^{j-k} \int_{\frac{k}{j+1}}^{\frac{k+1}{j+1}} h(\theta)d\theta, \tag{2}$$

for $h \in L^p[0,1]$.

For functions $h \in C[0,\infty)$, the Szász-Mirakjan operators

$$\mathcal{S}_r(h;u) = e^{-rx} \sum_{k=0}^{\infty} \frac{(ru)^k}{k!} h\left(\frac{k}{r}\right) \tag{3}$$

were introduced by Mirakjan [5] and Szász [6].

Durrmeyer [7] proposed

$$\mathcal{D}_r(h;u) = (r+1) \sum_{k=0}^{r} p_{rk}(u) \int_0^1 p_{rk}(\theta)h(\theta)d\theta,\ p_{rk}(u) = \binom{r}{k} u^k (1-u)^{r-k}. \tag{4}$$

Since then a large number of such operators have been introduced to approximate functions of different classes in different settings and spaces. One of such operators are the Baskakov operators [8] defined by

$$\mathcal{V}_r(h;u) == \sum_{j=0}^{\infty} \left(\begin{array}{c} r+j-1 \\ j \end{array} \right) \frac{u^j}{(1+u)^{r+j}} h\left(\frac{j}{r}\right). \tag{5}$$

The Pólya–Eggenberger distribution (P-E) and the inverse Pólya–Eggenberger (I-P-E) (see [9]) are defined by

$$Pr(X=r) = \left(\begin{array}{c} m \\ r \end{array} \right) \frac{\prod\limits_{j=0}^{r-1}(a+js) \prod\limits_{j=0}^{m-r-1}(b+js)}{\prod\limits_{j=0}^{m-1}(a+b+js)}, \tag{6}$$

$$Pr(X=r) = \left(\begin{array}{c} m+r-1 \\ r \end{array} \right) \frac{\prod\limits_{j=0}^{r-1}(a+js) \prod\limits_{j=0}^{m-1}(b+js)}{\prod\limits_{j=0}^{m+r-1}(a+b+js)}, \tag{7}$$

$r = 0,1,2,...,m$.

Stancu [10] constructed new operators by using the Pólya–Eggenberger distribution (P-E) (6).

$$P_r^{[\alpha]}(h;u) = \sum_{j=0}^{r} p_{r,j}^{[\alpha]}(u) h\left(\frac{j}{r}\right) \tag{8}$$

where $p_{r,j}^{[\alpha]}(u) = \left(\begin{array}{c} r \\ j \end{array} \right) \frac{1}{1^{[r,-\alpha]}} u^{[j,-\alpha]}(1-u)^{[r-j,-\alpha]}$ and $u^{[r,h]} = u(u-h)...(u-(r-1)h)$, $u^{[0,h]} = 1$.

Further, in 1970, Stancu [11] introduced the following operators using (I-P-E) distribution (7)

$$V_r^{[\alpha]}(h;u) = \sum_{j=0}^{\infty} v_{r,j}^{[\alpha]}(u) h\left(\frac{j}{r}\right) \tag{9}$$

where $v_{r,j}^{[\alpha]}(u) = \left(\begin{array}{c} r+j-1 \\ j \end{array} \right) \frac{1^{[r,-\alpha]} u^{[j,-\alpha]}}{(1+u)^{[r+j,-\alpha]}}$. For $\alpha = 0$, operators (9) are reduced to (5).

For the operators based on (P-E) distribution, one can see the details in [12–14]. Gupta et al. [15] introduced Durrmeyer type modification of (9) as follows:

$$\mathcal{D}_r^{[m]}(h;u) = (r-1) \sum_{j=0}^{\infty} v_{r,j}^{[m]}(u) \int_0^{\infty} b_{r,j}(\theta)h(\theta)d\theta, \tag{10}$$

where $b_{r,j}(\theta)=\begin{pmatrix} r+j-1 \\ j \end{pmatrix}\frac{\theta^j}{(1+\theta)^{r+j}}$.

Approximation theory is very crucial subject which is used in various fields by researchers. In which of them, a part of approximation theory is linear positive operators having an important role for studying various properties. There has been an extensive study on the approximation by these operators. Many mathematicians has inspired so far from past.

The computation of the test functions by Stancu operators was done long time ago and can be found in [10]. Based on the fact that many properties of Bernstein operators can be transferred to the Stancu operators (see [16–18]), we define Stancu type generalization of Baskakov type Pólya-Durrmeyer operators (10) as follows:

$$D_p^{[m](\lambda,\mu)}(h;u)=(p-1)\sum_{k=0}^{\infty}v_{p,k}^{[m]}(u)\int_0^{\infty}b_{p,k}(t)h\left(\frac{pt+\lambda}{p+\mu}\right)dt, \tag{11}$$

where λ,μ are any non negative real numbers such that $\lambda\le\mu$. If $\lambda=\mu=0$, the operators (11) reduce to (10).

2. Basic Results

The following results will be needed.

Lemma 1. *For $m>0$ and $0\le x<\infty$, we get*

$$D_p^{[m](\lambda,\mu)}(h;u)=\frac{1}{\beta(\frac{u}{m},\frac{1}{m})}\int_0^{\infty}\frac{t^{\frac{u}{m}-1}}{(1+t)^{\frac{1+u}{m}}}K_p^{(\lambda,\mu)}(h;t)dt,$$

where

$$K_p^{(\lambda,\mu)}(h;t)=(p-1)\sum_{k=0}^{\infty}b_{p,k}(t)\int_0^{\infty}b_{p,k}(s)h\left(\frac{ps+\lambda}{p+\mu}\right)ds,$$

and $\beta(l,p)$, $l,p>0$ is the Beta function of second kind.

Proof. We use the relationship

$$\beta(l,p)=\frac{\Gamma(l)\Gamma(p)}{\Gamma(l+p)},$$

where $\beta(l,p)$ and $\Gamma(r)(r>0)$ are defined by

$$\beta(l,p)=\int_0^{\infty}\frac{u^{l-1}}{(1+u)^{l+p}}du,\qquad \Gamma(r)=\int_0^{\infty}u^{r-1}e^{-u}du.$$

By simple calculation, we get

$$\begin{aligned}
\beta\left(\frac{u}{m}+k,\frac{1}{m}+p\right) &= \frac{\Gamma\left(\frac{u}{m}+k\right)\Gamma\left(\frac{1}{m}+p\right)}{\Gamma\left(\frac{u+1}{m}+p+k\right)}\\
&= \frac{\frac{u}{m}\left(\frac{u}{m}+1\right)\ldots\left(\frac{u}{m}+k-1\right)\Gamma\left(\frac{u}{m}\right)\cdot\frac{1}{m}\left(\frac{1}{m}+1\right)\ldots\left(\frac{1}{m}+p-1\right)\Gamma\left(\frac{1}{m}\right)}{\left(\frac{u+1}{m}\right)\left(\frac{u+1}{m}+1\right)\ldots\left(\frac{u+1}{m}+p+k-1\right)\Gamma\left(\frac{u+1}{m}\right)}\\
&= \begin{pmatrix} p+k-1 \\ k \end{pmatrix}^{-1}v_{p,k}^{[m]}(u)\beta\left(\frac{u}{m},\frac{1}{m}\right).
\end{aligned}$$

Hence

$$v_{p,k}^{[m]}(u) = \binom{p+k-1}{k} \beta\left(\frac{u}{m}, \frac{1}{m}\right)^{-1} \beta\left(\frac{u}{m}+k, \frac{1}{m}+p\right)$$

and it follows

$$\begin{aligned} D_p^{[m](\lambda,\mu)}(h;u) &= \frac{(p-1)}{\beta\left(\frac{u}{m}, \frac{1}{m}\right)} \sum_{k=0}^{\infty} \binom{p+k-1}{k} \beta\left(\frac{u}{m}+k, \frac{1}{m}+p\right) \int_0^\infty b_{p,k}(s) h\left(\frac{ps+\lambda}{p+\mu}\right) ds \\ &= \frac{(p-1)}{\beta\left(\frac{u}{m}, \frac{1}{m}\right)} \left(\sum_{k=0}^{\infty} \binom{p+k-1}{k} \int_0^\infty \frac{t^{\frac{u}{m}+k-1}}{(1+t)^{\frac{1+u}{m}+p+k}} dt \int_0^\infty b_{p,k}(s) h\left(\frac{ps+\lambda}{p+\mu}\right) ds \right) \\ &= \frac{1}{\beta(\frac{u}{m}, \frac{1}{m})} \int_0^\infty \frac{t^{\frac{u}{m}-1}}{(1+t)^{\frac{1+u}{m}}} K_p^{(\lambda,\mu)}(h;t) dt. \end{aligned}$$

By the definition of Γ, we obtain

$$\begin{aligned} \int_0^\infty b_{p,k}(t) t^r dt &= \int_0^\infty \binom{p+k-1}{k} \frac{t^{k+r}}{(1+t)^{p+k}} dt \\ &= \binom{p+k-1}{k} \beta(k+r+1, p-r-1) \\ &= \frac{(k+r)!(p-r-2)!}{k!(p-1)!}. \end{aligned}$$

□

Lemma 2. *Let $e_r(t) = t^r$, $r \in \mathbb{N}_0$. For $m > 0$, we have*

$$\begin{aligned} (a)\ D_p^{[m](\lambda,\mu)}(e_0;u) &= 1 \\ (b)\ D_p^{[m](\lambda,\mu)}(e_1;u) &= \frac{p^2}{(p+\mu)(p-2)(1-m)} u + \frac{p+\lambda(p-2)}{(p+\mu)(p-2)} \\ (c)\ D_p^{[m](\lambda,\mu)}(e_2;u) &= \frac{(p^4+p^3)x^2 + [m+4p^3(1-2m)+2p^2\lambda(p-3)(1-2m)]u}{(p+\mu)^2(p-2)(p-3)(1-m)(1-2m)} \\ &\quad + \frac{p^2}{(p+\mu)^2(p-2)(p-3)} + \frac{2p\lambda}{(p+\mu)^2(p-2)} + \frac{\lambda}{(p+\mu)^2}. \end{aligned}$$

Proof. Using Lemma 1 for $r = 0,1,2$, we have

$$\begin{aligned} (i)\ K_p^{(\lambda,\mu)}(e_0;t) &= 1 \\ (ii)\ K_p^{(\lambda,\mu)}(e_1;t) &= \frac{p^2 t}{(p+\mu)(p-2)} + \frac{p+\lambda(p-2)}{(p+\mu)(p-2)} \\ (iii)\ K_p^{(\lambda,\mu)}(e_2;t)dt &= \frac{(p^4+p^3)t^2 + 4p^3 t + p^2}{(p+\mu)^2(p-2)(p-3)} + \frac{2p^2\lambda t + 2p\lambda + \lambda^2(p-2)}{(p+\mu)^2(p-2)}. \end{aligned}$$

Using Equality (i) first moment can be found trivially. Also, by using Equality (ii), we get

$$\begin{aligned} D_p^{[m](\lambda,\mu)}(e_1;u) &= \frac{1}{\beta(\frac{u}{m}, \frac{1}{m})} \int_0^\infty \frac{t^{\frac{u}{m}-1}}{(1+t)^{\frac{1+u}{m}}} K_p^{(\lambda,\mu)}(e_1;t) dt \\ &= \frac{1}{\beta(\frac{u}{m}, \frac{1}{m})} \int_0^\infty \frac{t^{\frac{u}{m}-1}}{(1+t)^{\frac{1+u}{m}}} \left(\frac{p^2 t}{(p+\mu)(p-2)} + \frac{p+\lambda(p-2)}{(p+\mu)(p-2)} \right) dt \\ &= \frac{p^2}{(p+\mu)(p-2)(1-m)} x + \frac{p+\lambda(p-2)}{(p+\mu)(p-2)}. \end{aligned}$$

Finally, using Equality (iii), we have

$$\begin{aligned} D_p^{[m](\lambda,\mu)}(e_2;x) &= \frac{1}{\beta(\frac{u}{m},\frac{1}{m})}\int_0^\infty \frac{t^{\frac{u}{m}-1}}{(1+t)^{\frac{1+u}{m}}} K_p^{(\lambda,\mu)}(e_2;t)dt \\ &= \frac{1}{\beta(\frac{u}{m},\frac{1}{m})}\int_0^\infty \frac{t^{\frac{u}{m}-1}}{(1+t)^{\frac{1+u}{m}}}\bigg(\frac{(p^4+p^3)t^2+4p^3t+p^2}{(p+\mu)^2(p-2)(p-3)} \\ &\quad +\frac{2p^2\lambda t+2p\lambda+\lambda^2(p-2)}{(p+\mu)^2(p-2)}\bigg)dt \\ &= \frac{(p^4+p^3)x^2+[m+4p^3(1-2m)+2p^2\lambda(p-3)(1-2m)]u}{(p+\mu)^2(p-2)(p-3)(1-m)(1-2m)} \\ &\quad +\frac{p^2}{(p+\mu)^2(p-2)(p-3)}+\frac{2p\lambda}{(p+\mu)^2(p-2)}+\frac{\lambda}{(p+\mu)^2}. \end{aligned}$$

□

Lemma 3. *Using Lemma 2, we have*

$$\begin{aligned} (i)\ D_p^{[m](\lambda,\mu)}(e_1-u;u) &= \left(\frac{p^2}{(p+\mu)(p-2)(1-m)}-1\right)u+\frac{p+\lambda(p-2)}{(p+\mu)(p-2)} \\ (ii)\ D_n^{[m](\alpha,\beta)}((e_1-u)^2;u) &= \left(\frac{p^3(p+1)-2p^2(p+\mu)(p-3)(1-2m)}{(p+\mu)^2(p-2)(p-3)(1-m)(1-2m)}+1\right)u^2 \\ &\quad +\bigg(\frac{m+4p^3(1-2m)+2p^2\lambda(p-3)(1-2m)}{(p+\mu)^2(p-1)(p-3)(1-m)(1-2m)} \\ &\quad -\frac{2p+2\lambda(p-2)}{(p+\mu)(p-2)}\bigg)u \\ &\quad +\frac{p^2+2p\lambda(p-3)+\lambda(p-2)(p-3)}{(p+\mu)^2(p-2)(p-3)}. \end{aligned}$$

3. Approximation Properties

Having a sequence of operators which approximate a given function arises the question of evaluation of the committed error. This is given by the approximation order, which depends on the smoothness properties of functions. In estimates of the approximation degree a convenient tool for measuring the smoothness of functions is represented by the modulus of continuity. The next question is: how can we evaluate the committed error in the function approximation process? A convenient tool is the modulus of continuity. Another important tool for evaluating of the committed error is the modulus of smoothness of second order. Estimates using combinations of first and second order modulus of smoothness are more refined then estimates using only the modulus of continuity.

Let

$$C_B[0,\infty) := \left\{h:[0,\infty)\to R;\ \|h\|_{C_B}=\sup_{u\in[0,\infty)}|h(u)|<\infty\right\}.$$

Further, consider the $\mathcal{K}_2$-functional:

$$\mathcal{K}_2(h,\delta)=\inf_{\tau\in\mathcal{W}^2}\{\| h-\tau \| +\delta \| \tau'' \|\},\ \delta>0,$$

and $\mathcal{W}^2=\{\tau\in C_B[0,\infty):\tau',\tau''\in C_B[0,\infty)\}$. There is an absolute constant $C>0$ such that (see Devore and Lorentz [19], p. 177, Theorem 2.4)

$$\mathcal{K}_2(h,\delta)\le\mathcal{C}\omega_2(h,\sqrt{\delta}) \tag{12}$$

where

$$\omega_2(h,\sqrt{\delta}) = \sup_{0<l\leq\sqrt{\delta}}\ \sup_{u\in[0,\infty)} \mid h(u+2l)-2h(u+l)+h(u)\mid$$

$$\omega_1(h,\delta) = \sup_{0<l\leq\delta}\ \sup_{u\in[0,\infty)} \mid h(u+l)-h(u)\mid,$$

are respectively the usual and second order modulus of continuity of $h \in C_B[0,\infty)$.

Theorem 1. *We have*

$$\mid D_p^{[m](\lambda,\mu)}(h;u)-h(u)\mid \leq\ \mathcal{C}\omega_2(h,\delta)+\omega_1\left(h,\left|\left(\frac{p^2}{(p+\mu)(p-2)(1-m)}-1\right)u+\frac{p+\lambda(p-2)}{(p+\mu)(p-2)}\right|\right),$$

for $h \in C_B[0,\infty)$, *where*

$$\delta = \sqrt{D_p^{[m](\lambda,\mu)}((e_1-u)^2;u)+\left(\frac{p^2}{(p+\mu)(p-2)(1-m)}u+\frac{p+\lambda(p-2)}{(p+\mu)(p-2)}-u\right)^2}.$$

Proof. Write

$$\widehat{D}_p^{[m](\lambda,\mu)}(h;u) = D_p^{[m](\lambda,\mu)}(h;u)-h\left(\frac{p^2}{(p+\mu)(p-2)(1-m)}u+\frac{p+\lambda(p-2)}{(p+\mu)(p-2)}\right)+h(u).$$

Let $\tau \in W_\infty^2$ and $x,t \in [0,\infty)$. Then from the Taylor's expansion, we get

$$\tau(t) = \tau(u)+\tau'(u)(t-u)+\int_x^t (t-v)\tau''(v)\mathrm{d}v.$$

Therefore

$$\widehat{D}_p^{[m](\lambda,\mu)}(\tau;u)-\tau(u) = \tau'(u)\widehat{D}_p^{[m](\lambda,\mu)}(t-u;u)+\widehat{D}_p^{[m](\lambda,\mu)}\left(\int_x^t (t-v)\tau''(v)\mathrm{d}v;u\right),$$

and hence

$$\begin{aligned}
\left|\widehat{D}_p^{[m](\lambda,\mu)}(\tau;u)-\tau(u)\right| &\leq \widehat{D}_p^{[m](\lambda,\mu)}\left(\left|\int_x^t \mid t-v\mid\ \mid\tau''(v)\mid \mathrm{d}v\right|;u\right)\\
&\leq D_p^{[m](\lambda,\mu)}\left(\left|\int_x^t \mid t-u\mid\mid\tau''(v)\mid \mathrm{d}v\right|;u\right)\\
&\quad+\int_x^{\frac{p^2}{(p+\mu)(p-2)(1-m)}u+\frac{p+\lambda(p-2)}{(p+\mu)(p-2)}}\left|\left(\frac{p^2}{(p+\mu)(p-2)(1-m)}u\right.\right.\\
&\quad\left.\left.+\frac{p+\lambda(p-2)}{(p+\mu)(p-2)}-v\right)\tau''(v)\mathrm{d}v\right|\\
&\leq D_p^{[m](\lambda,\mu)}\left((t-u)^2;u\right)\parallel\tau''\parallel\\
&\quad+\left(\left(\frac{p^2}{(p+\mu)(p-2)(1-m)}-1\right)u+\frac{p+\lambda(p-2)}{(p+\mu)(p-2)}\right)^2\parallel\tau''\parallel\\
&\leq \left[D_n^{[m](\lambda,\mu)}\left((t-u)^2;u\right)\right.\\
&\quad\left.+\left(\left(\frac{p^2}{(p+\mu)(p-2)(1-m)}-1\right)u+\frac{p+\lambda(p-2)}{(p+\mu)(p-2)}\right)^2\right]\parallel\tau''\parallel\\
&= \delta^2\parallel\tau''\parallel.
\end{aligned}$$

Since

$$| D_p^{[m](\lambda,\mu)}(h;u) | \leq (p-1)\sum_{k=0}^{\infty} v_{p,k}^{[m]}(u)\int_0^{\infty} b_{p,k}(t)h\left(\frac{pt+\lambda}{p+\mu}\right)dt \leq \| h \|,$$

it follows,

$$\begin{aligned} | D_p^{[m](\lambda,\mu)}(h;u) - h(u) | \;&\leq\; | \widehat{D}_p^{[m](\lambda,\mu)}(h-\tau;u) - (h-\tau)(u) | + | \widehat{D}_p^{[m](\lambda,\mu)}(\tau;u) - \tau(u) | \\ &\quad + \left| h\left(\frac{p^2}{(p+\mu)(p-2)(1-m)}u + \frac{p+\lambda(p-2)}{(p+\mu)(p-2)}\right) - h(u)\right| \\ &\leq\; \| h-\tau \| + \delta^2 \| \tau'' \| + \omega_1\left(h, \left|\left(\frac{p^2}{(p+\mu)(p-2)(1-m)} - 1\right)u \right.\right. \\ &\quad \left.\left. + \frac{p+\lambda(p-2)}{(p+\mu)(p-2)}\right|\right). \end{aligned}$$

Taking infimum over all $\tau \in \mathcal{W}^2$, we get

$$| D_p^{[m](\lambda,\mu)}(h;u) - h(u) | \leq \mathcal{C}\mathcal{K}_2(h,\delta^2) + \omega_1\left(h, \left|\left(\frac{p^2}{(p+\mu)(p-2)(1-m)} - 1\right)u + \frac{p+\lambda(p-2)}{(p+\mu)(p-2)}\right|\right).$$

In view of (12),

$$| D_p^{[m](\lambda,\mu)}(h;u) - h(u) | \leq \mathcal{C}\omega_2(h,\delta) + \omega_1\left(h, \left|\left(\frac{p^2}{(p+\mu)(p-2)(1-m)} - 1\right)u + \frac{p+\lambda(p-2)}{(p+\mu)(p-2)}\right|\right),$$

Hence the proof. □

Now we consider some approximation properties in the weighted spaces $B_\rho[0,\infty) := \{h : [0,\infty) \to R;\ | h(u) | \leq M_h\rho(u)\}$, where the constant M_h depends only on h and $\rho(u) = 1+u^2$. Let $C_\rho[0,\infty) = \{h \in B_\rho[0,\infty) :\ h \text{ is continuous on } [0,\infty)\}$ with $\|h\|_\rho = \sup_{u\in[0,\infty)} \frac{|h(u)|}{\rho(u)} < \infty$ and

$$C_\rho^0 = \left\{h \in C_\rho[0,\infty) : \lim_{u\to\infty} \frac{| h(u) |}{\rho(u)} \text{ exists}\right\}.$$

Theorem 2. *Let $m = m(p) \to 0$ as $p \to \infty$. Then for $h \in C_\rho^0$, we have*

$$\lim_{p\to\infty} \|D_p^{[m](\alpha,\beta)}(h;u) - h(u)\|_\rho = 0.$$

Proof. It suffices to verify that

$$\lim_{p\to\infty} \|D_p^{[m](\lambda,\mu)}(t^i;u) - u^i\|_\rho = 0,\ i = 0,1,2,$$

holds. Observe that these conditions hold by Lemma 2. Now using the Korovkin's theorem [20], we have $\lim_{p\to\infty} \|D_p^{[m](\alpha,\beta)}(h;u) - h(u)\|_\rho = 0.$ □

Theorem 3. *Let $m = m(p) \to 0$ as $p \to \infty$. Then for $h \in C_\rho^0$ and $a > 0$, we have*

$$\lim_{p\to\infty} \sup_{u\in[0,\infty)} \frac{| D_p^{[m](\lambda,\mu)}(h;u) - h(u) |}{(1+u^2)^{1+a}} = 0.$$

Proof. Let $u_0 > 0$ be fixed.

$$\begin{aligned}
\sup_{u\in[0,\infty)} \frac{| D_p^{[m](\lambda,\mu)}(h;u) - h(u) |}{(1+u^2)^{1+a}} &\leq \sup_{u\leq u_0} \frac{| D_p^{[m](\lambda,\mu)}(h;u) - h(u) |}{(1+u^2)^{1+a}} + \sup_{u\geq u_0} \frac{| D_p^{[m](\lambda,\mu)}(h;u) - h(u) |}{(1+u^2)^{1+a}} \\
&\leq \|D_p^{[m](\lambda,\mu)}(h;u) - h(u)\|_{C[0,u_0]} + \|h\|_\rho \sup_{u\geq u_0} \frac{| D_p^{[m](\lambda,\mu)}(1+t^2;u) |}{(1+u^2)^{1+a}} \\
&\quad + \sup_{u\geq u_0} \frac{| h(u) |}{(1+u_0^2)^{1+a}} \\
&= I_1 + I_2 + I_3. \qquad (13)
\end{aligned}$$

Since $| h(u) | \leq \|h\|_\rho (1+u^2)$, we have

$$I_3 = \sup_{u\geq u_0} \frac{| h(u) |}{(1+u^2)^{1+a}} \leq \sup_{u\geq u_0} \frac{\|h\|_\rho}{(1+u^2)^a} \leq \frac{\|h\|_\rho}{(1+u_0^2)^a}.$$

Let $\epsilon > 0$. Then, there exists $p_1 \in \mathbb{N}$ such that

$$\begin{aligned}
\|h\|_\rho \frac{| D_p^{[m](\lambda,\mu)}(1+t^2;u) |}{(1+u^2)^{1+a}} &< \frac{1}{(1+u^2)^{1+a}} \|h\|_\rho \left((1+u^2) + \frac{\epsilon}{3\|h\|_\rho} \right), \qquad \forall p \geq p_1. \\
&< \frac{\|h\|_\rho}{(1+u^2)^a} + \frac{\epsilon}{3} \qquad \forall p \geq p_1. \qquad (14)
\end{aligned}$$

Hence

$$\|h\|_\rho \sup_{u\geq u_0} \frac{| D_p^{[m](\alpha,\beta)}(1+t^2;u) |}{(1+u^2)^{1+a}} < \frac{\|h\|_\rho}{(1+u_0^2)^a} + \frac{\epsilon}{3}, \qquad \forall p \geq p_1.$$

Thus

$$I_2 + I_3 < \frac{2\|h\|_\rho}{(1+u_0^2)^a} + \frac{\epsilon}{3}, \qquad \forall p \geq p_1.$$

Now, let u_0 to be such that $\frac{\|h\|_\rho}{(1+u_0^2)^a} < \frac{\epsilon}{6}$.

Then,

$$I_2 + I_3 < \frac{2\epsilon}{3}, \qquad \forall p \geq p_1. \qquad (15)$$

$$I_1 = \|D_p^{[m](\lambda,\mu)}(h) - h\|_{C[0,u_0]} < \frac{\epsilon}{3}, \qquad \forall n \geq p_2. \qquad (16)$$

Let $p_0 = \max(p_1, p_2)$. Then, combining (13)–(16)

$$\sup_{u\in[0,\infty)} \frac{| D_p^{[m](\lambda,\mu)}(h;u) - h(u) |}{(1+u^2)^{1+a}} < \epsilon, \qquad \forall p \geq p_0.$$

Hence the proof. □

The Lipschitz type space is defined by

$$lip_{\mathcal{M}}(\gamma) = \left\{ h \in C_B[0,\infty) : | h(t) - h(u) | \leq \mathcal{M} \frac{| t-u |^\gamma}{(t+u)^{\frac{\gamma}{2}}} \right\}, \qquad (17)$$

where $\mathcal{M} > 0$ and $0 < \gamma \leq 1$.

Theorem 4. *For $h \in lip_{\mathcal{M}}(\gamma)$, we have*

$$|D_p^{[m](\lambda,\mu)}(h;u) - h(u)| \leq \mathcal{M}\left(\frac{\varphi_p^{[m](\lambda,\mu)}(u)}{u}\right)^{\frac{\gamma}{2}},$$

where $\varphi_p^{[m](\lambda,\mu)}(u) = D_p^{[m](\lambda,\mu)}((e_1-u)^2;u)$.

Proof. We show the caser $\gamma = 1$. For $h \in lip_{\mathcal{M}}(\gamma)$, we get

$$\begin{aligned} |D_p^{[m](\lambda,\mu)}(h;u) - h(u)| &\leq (p-1)\sum_{k=0}^{\infty} v_{p,k}^{[m]}(u)\int_0^{\infty} b_{p,k}(t)\left|h\left(\frac{pt+\lambda}{p+\mu}\right) - h(u)\right|dt \\ &\leq \mathcal{M}(p-1)\sum_{k=0}^{\infty} v_{p,k}^{[m]}(u)\int_0^{\infty} b_{p,k}(t)\frac{\left|\frac{pt+\lambda}{p+\mu} - u\right|}{\sqrt{\frac{pt+\lambda}{p+\mu} + u}}dt. \end{aligned}$$

Using the Cauchy–Schwarz inequality and $\sqrt{u} < \sqrt{\frac{pt+\lambda}{p+\mu} + u}$, we get

$$\begin{aligned} |D_p^{[m](\lambda,\mu)}(h;u) - h(u)| &\leq \frac{\mathcal{M}}{\sqrt{u}}(p-1)\sum_{k=0}^{\infty} v_{p,k}^{[m]}(u)\int_0^{\infty} b_{p,k}(t)\left|\frac{pt+\lambda}{p+\mu} - u\right|dt \\ &= \frac{\mathcal{M}}{\sqrt{u}} D_p^{[m](\lambda,\mu)}((e_1-u)^2;u) \leq \mathcal{M}\sqrt{\frac{\varphi_p^{[m](\lambda,\mu)}(u)}{u}}. \end{aligned}$$

That is the result holds for $\gamma = 1$. Now we prove for $0 < \gamma < 1$. Applying Holder's inequality, we get

$$\begin{aligned} |D_p^{[m](\lambda,\mu)}(h;u) - h(u)| &\leq (p-1)\sum_{k=0}^{\infty} v_{p,k}^{[m]}(u)\int_0^{\infty} b_{p,k}(t)\left|h\left(\frac{pt+\lambda}{p+\mu}\right) - h(u)\right|dt \\ &\leq (p-1)\sum_{k=0}^{\infty}\left\{v_{p,k}^{[m]}(u)\left(\int_0^{\infty} b_{p,k}(t)\left|h\left(\frac{pt+\lambda}{p+\mu}\right) - h(u)\right|dt\right)^{\frac{1}{\gamma}}\right\}^{\gamma} \\ &\leq (p-1)\sum_{k=0}^{\infty}\left\{v_{p,k}^{[m]}(u)\int_0^{\infty} b_{p,k}(t)\left|h\left(\frac{pt+\lambda}{p+\mu}\right) - h(u)\right|^{\frac{1}{\gamma}}dt\right\}^{\gamma}. \end{aligned}$$

Since $h \in lip_M(\gamma)$, we have

$$\begin{aligned} |D_p^{[m](\lambda,\mu)}(h;u) - h(u) &\leq \frac{\mathcal{M}}{u^{\frac{\gamma}{2}}}\left\{(p-1)\sum_{k=0}^{\infty} v_{p,k}^{[m]}(u)\int_0^{\infty} b_{p,k}(t)\left|\frac{pt+\lambda}{p+\mu} - u\right|dt\right\}^{\gamma} \\ &= \frac{\mathcal{M}}{u^{\frac{\gamma}{2}}} D_p^{[m](\lambda,\mu)}((e_1-u)^2;u)^{\gamma} \leq \mathcal{M}\left(\sqrt{\frac{\varphi_p^{[m](\lambda,\mu)}(u)}{u}}\right)^{\gamma}. \end{aligned}$$

Therefore, we get the result. □

4. Statistical Approximation

We prove here a result on statistical approximation.

The asymptotic density of $A \subseteq \mathbb{N}$ is defined by

$$\delta(A) = \lim_{p} \frac{1}{p}\#\{r \leq p : r \in A\},$$

where # denotes the cardinality of the set.

A sequence $\xi = (\xi_r)$ is said to be statistically convergent (see [21,22]) to the number $\mathcal{L}$ if $\delta(A(\varepsilon)) = 0$ for each $\varepsilon > 0$, where

$$A(\varepsilon) = \{r \leq p : |\xi_r - \mathcal{L}| > \varepsilon\}$$

and we write $st\text{-}\lim \xi = \mathcal{L}$.

Theorem 5. *Let $m = m(p) \to 0$ as $p \to \infty$. Then, for all $h \in C^0_\rho$ we have*

$$st - \lim_p \|D_p^{[m](\lambda,\mu)}(h;u) - h\|_\rho = 0.$$

where $\rho(u) = 1 + u^2$, $u \in [0, \infty)$.

Proof. Let us consider $e_i(u) = u^i$, $i = 0, 1, 2$. It is sufficient to show that

$$st - \lim_p \|D_p^{[m](\lambda,\mu)}(e_i;u) - e_i\|_\rho = 0,$$

for $i = 0, 1, 2$. It is clear that

$$st - \lim_p \|D_p^{[m](\lambda,\mu)}(e_0;u) - e_0\|_\rho = 0.$$

From Lemma 2, we have

$$\begin{aligned} st - \lim_p \|D_p^{[m](\lambda,\mu)}(e_1;u) - e_1\|_\rho &= \sup_{u\in[0,\infty)} \Bigg| \left(\frac{p^2}{(p+\mu)(p-2)(1-m)} - 1 \right) u \\ &\quad + \frac{p}{(p+\mu)(p-2)} + \frac{\lambda}{p+\mu} \Bigg| \frac{1}{1+u^2} \\ &\leq \left(\frac{p^2}{(p+\mu)(p-2)(1-m)} - 1 \right) + \frac{p}{(p+\mu)(p-2)} + \frac{\lambda}{(p+\mu)}. \end{aligned}$$

For $\varepsilon > 0$, define the sets:

$$\begin{aligned} \mathcal{E}_1 &:= \left\{ p \in \mathbb{N} : \|D_p^{[m](\lambda,\mu)}(e_1;u) - e_1\| \geq \varepsilon \right\} \\ \mathcal{E}_2 &:= \left\{ p \in \mathbb{N} : \left(\frac{p^2}{(p+\mu)(p-2)(1-m)} - 1 \right) \geq \frac{\varepsilon}{3} \right\} \\ \mathcal{E}_3 &:= \left\{ p \in \mathbb{N} : \frac{p}{(p+\mu)(p-2)} \geq \frac{\varepsilon}{3} \right\} \\ \mathcal{E}_4 &:= \left\{ p \in \mathbb{N} : \frac{\lambda}{(p+\mu)} \geq \frac{\varepsilon}{3} \right\}. \end{aligned}$$

Then, we obtain $\mathcal{E}_1 \subseteq \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4$ which implies that $\delta(\mathcal{E}_1) \leq \delta(\mathcal{E}_2) + \delta(\mathcal{E}_3) + \delta(\mathcal{E}_4)$ and hence

$$st - \lim_p \|D_p^{[m](\lambda,\mu)}(e_1;u) - e_1\|_\rho = 0.$$

Again, using Lemma 2, we obtain

$$\begin{aligned}
st-\lim_{p}\|D_p^{[m](\lambda,\mu)}(e_2;u)-e_2\|_\rho &= \sup_{x\in[0,\infty)}\Bigg|\left(\frac{p^3(p+1)}{(p+\mu)^2(p-2)(p-3)(1-m)(1-2m)}-1\right)u^2 \\
&\quad+\left(\frac{m+4p^3(1-2m)+2p^2\lambda(p-3)(1-2m)}{(p+\mu)^2(p-2)(p-3)(1-m)(1-2m)}\right)u \\
&\quad+\frac{p^2}{(p+\mu)^2(p-2)(p-3)}+\frac{2p\lambda}{(p+\mu)^2(p-2)} \\
&\quad+\frac{\lambda}{(p+\mu)^2}\Bigg|\frac{1}{1+u^2} \\
&\le \left(\frac{p^3(p+1)}{(p+\mu)^2(p-2)(p-3)(1-m)(1-2m)}-1\right) \\
&\quad+\left(\frac{m+4p^3(1-2m)+2p^2\lambda(p-3)(1-2m)}{(p+\mu)^2(p-2)(p-3)(1-m)(1-2m)}\right) \\
&\quad+\frac{p^2}{(p+\mu)^2(p-2)(p-3)} \\
&\quad+\frac{2p\lambda}{(p+\mu)^2(p-2)}+\frac{\lambda}{(p+\mu)^2}.
\end{aligned}$$

For $\varepsilon > 0$, define the sets:

$$\begin{aligned}
\mathcal{D}_1 &:= \left\{p\in\mathbb{N} : \|D_p^{[m](\lambda,\mu)}(e_2;u)-e_2\| \ge \varepsilon\right\} \\
\mathcal{D}_2 &:= \left\{p\in\mathbb{N} : \left(\frac{p^3(p+1)}{(p+\mu)^2(p-2)(p-3)(1-m)(1-2m)}-1\right) \ge \frac{\varepsilon}{5}\right\} \\
\mathcal{D}_3 &:= \left\{p\in\mathbb{N} : \frac{m+4p^3(1-2m)+2p^2\lambda(p-3)(1-2m)}{(p+\mu)^2(p-2)(p-3)(1-m)(1-2m)} \ge \frac{\varepsilon}{5}\right\} \\
\mathcal{D}_4 &:= \left\{p\in\mathbb{N} : \frac{p^2}{(p+\mu)^2(p-2)(p-3)} \ge \frac{\varepsilon}{5}\right\} \\
\mathcal{D}_5 &:= \left\{p\in\mathbb{N} : \frac{2p\lambda}{(p+\mu)^2(p-2)} \ge \frac{\varepsilon}{5}\right\} \\
\mathcal{D}_6 &:= \left\{p\in\mathbb{N} : \frac{\lambda}{(p+\mu)^2} \ge \frac{\varepsilon}{5}\right\}.
\end{aligned}$$

Then, we obtain

$$\mathcal{D}_1 \subseteq \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}_5 \cup \mathcal{D}_6$$

which implies that

$$\circ(\mathcal{D}_1) \le \circ(\mathcal{D}_2)+\circ(\mathcal{D}_3)+\circ(\mathcal{D}_4)+\circ(\mathcal{D}_5)+\circ(\mathcal{D}_6)$$

and hence

$$st-\lim_{p}\|D_p^{[m](\lambda,\mu)}(e_2;u)-e_2\|_\rho = 0.$$

Hence, the proof is completed. □

Example 1. *Consider the sequence* $\eta = (\eta_p)_{p=0}^{\infty}$

$$\eta_p = \begin{cases} 1 & \textit{if } p \textit{ is square,} \\ 0 & \textit{otherwise.} \end{cases}$$

Then this sequence is statistically convergent but not convergent. Define $A_p^{[m](\lambda,\mu)} = (1+\eta_p)D_p^{[m](\lambda,\mu)}$. *Then obviously* $st - \lim_p \|A_p^{[m](\lambda,\mu)}(e_i;u) - e_i\|_\rho = 0$, *for* $i = 0,1,2$. *Applying Theorem 2, we have*

$$st - \lim_p \|A_p^{[m](\lambda,\mu)}(h;u) - h\|_\rho = 0 \text{ for all } h \in C_\rho^0.$$

On the other hand, since $D_p^{[m](\lambda,\mu)}(h;0) = h(0)$ *and* $A_p^{[m](\lambda,\mu)}(h;0) = (1+\eta_p)h(0)$, *and hence*

$$\|A_p^{[m](\lambda,\mu)}(h;u) - h\|_\rho \geq \mid A_p^{[m](\lambda,\mu)}(h;0) - h(0) \mid = \eta_p \mid h(0) \mid$$

so that the sequence $(A_p^{[m](\lambda,\mu)})$ *can not be convergent; while it is statistically convergent.*

5. Conclusions

Here, Stancu type generalization of Baskakov–Durrmeyer operators are constructed which are based on inverse Pólya–Eggenberger distribution. We calculated moments and central moments for these operators and have investigated convergence properties. We have also determined the rate of convergence by using the modulus of continuity and Lipschitz type class of functions. Moreover, the statistical approximation for these operators have also been studied.

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