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Article

Results in wt-Distance over b-Metric Spaces

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Abstract: In this manuscript, we introduce Meir-Keeler type contractions and Geraghty type contractions in the setting of the wt-distances over b-metric spaces. We examine the existence of a fixed point for such mappings. Under some additional assumption, we proved the uniqueness of the found fixed point.

Keywords: wt-distance; Meir-Keeler type contraction; Geraghty type contraction; b-metric

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1. Introduction and Preliminaries

The concept of distance is one of the first concepts discovered by mankind. The concept of distance was first formulated by Euclid. The general form and more axiomatic version was considered by Maurice René Frechét under the name of "L-space" and later was redefined by Felix Hausdorff as "metric space." Since then, the distance notion has been discussed, refined and generalized in various ways. Among all such generalizations, in this manuscript, we focus on b-metric and wt-distance.

Before starting to examine the subject in detail, we shall fix some notations as well as notions. Throughout the manuscript, we presume that all considered sets and subsets are non-empty. A mapping δ, defined from the cross-product of a set X to non-negative reals, is called *distance function*, if it is symmetric ($\delta(v,\nu) = \delta(\nu,v)$, for all $v,\nu \in X$) and has a zero-self distance ($\delta(v,\nu) = 0$ if and only if $v = \nu$).

A distance function δ forms a (standard) metric if

$$\delta(v,\omega) \leq \delta(v,\nu) + \delta(\nu,\omega), \text{ for all } v,\nu,\omega \in X.$$

As it is well known, the metric notion has been extended in several ways. One of the interesting extensions is called b-metric that was invented by several authors, in different time periods, involving Bakhtin [1] and Czerwik [2]. Indeed, after Czerwik [2], it has attracted the attention of researchers.

Definition 1. *Let $s \in [1,\infty)$ and d be a distance function on X. If the following inequality holds for all $v,\nu,\omega \in X$,*

$$d(v,\omega) \leq s[d(v,\nu) + d(\nu,\omega)],$$

then d is called b-metric over constant s.

In short, (X,d,s) (respectively, (X^*,d,s)) denotes a b-metric space over s (respectively, a complete b-metric space over s).

An immediate and simple observation is that b-metric turns to be a metric for $s = 1$. Moreover, despite the standard metric, b-metric may not be continuous functional, see, e.g., [3–6].

Below is the first conceivable example:

Example 1. *Let d be a distance function on $X = [0, \infty)$ that is defined as $d(\upsilon, \nu) = |\upsilon - \nu|^p$, $p > 1$. Then, d forms a b-metric over $s = 2^p$, but not a metric.*

Throughout the paper, a function μ, defined from $[0, \infty)$ to itself, is called *auxiliary distance function* that is, $\mu : [0, \infty) \to [0, \infty)$.

Lemma 1 ([3,7])**.** *Let μ be a non-decreasing auxiliary distance function so that*

$$\text{for any } t \in [0, \infty) \text{ we have } \lim_{n \to \infty} \mu^n(t) = 0.$$

Then,

(a) at $t = 0$, auxiliary distance function μ is continuous.
(b) for any $t > 0$, we have $\mu(t) < t$.

The mapping μ, introduced in Lemma 1, is called comparison. Note also that, for each $k \geq 1$, iteration μ^k forms a comparison function, see [3].

Definition 2 ([3,7])**.** *Let $s \in [1, \infty)$. A monotone increasing auxiliary distance function is said to be a b-comparison if there are positive integers k_0, $a \in (0, 1)$ and a convergent series $\sum_{k=1}^{\infty} \upsilon_k$ with $\upsilon_k \geq 0$ such that $s^{k+1}\mu^{k+1}(t) \leq a s^k \mu^k(t) + \upsilon_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.*

The letter $\mathcal{B}$ denotes the set of all $b-$comparison functions. If we take $s = 1$ in Definition 2, then μ is named as c-comparison function.

The given lemma below has an important place in the proof of the results discussed here.

Lemma 2 ([3])**.** *For a b-comparison function μ, the following holds:*

(1) for any $t \in [0, \infty)$, the series $\sum_{k=0}^{\infty} s^k \mu^k(t)$ is convergent;
(2) An auxiliary distance function S_b, formulated by $S_b(t) = \sum_{k=0}^{\infty} s^k \mu^k(t)$, $t \in [0, \infty)$, is continuous at 0 and increasing.

Remark 1. *Each b-comparison (and hence, c-comparison) function forms also a comparison function.*

For $\beta : X \times X \to [0, \infty)$, we say that $f : X \to X$ is β-orbital admissible ([8]) if

$$\beta(\upsilon, f(\upsilon)) \geq 1 \Rightarrow \beta(f(\upsilon), f^2(\upsilon)) \geq 1, foreach \upsilon \in X. \tag{1}$$

In addition to (1), if the implication below is also fulfilled, then f is named triangular β-orbital admissible ([8], see also [9]):

$$\beta(\upsilon, \nu) \geq 1 \text{ and } \beta(\nu, f(\nu)) \geq 1 \Rightarrow \beta(\upsilon, f(\nu)) \geq 1, \text{ for every } \upsilon, \nu \in X.$$

Lemma 3 ([8])**.** *Let f be a self-mapping on (X^*, d, s) and form a triangular β-orbital admissible mapping. If there exists $\upsilon_0 \in X$ such that $\beta(\upsilon_0, f(\upsilon_0)) \geq 1$, then,*

$$\beta(\upsilon_n, \upsilon_m) \geq 1, \quad \textit{for all } n, m \in \mathbb{N},$$

where $v_{n+1} = f(v_n)$ *for each* $n \in \mathbb{N}$.

Very recently, an interesting auxiliary function (to unify the different type contraction) was defined by Khojasteh [10] under the name of *simulation function*.

Definition 3 ([10])**.** *We say that a function* σ*, defined from the cross-product of non-negatives real numbers to real line, is* simulation function *if*

(S_1) *for all* $t, s > 0$*, we have* $\sigma(t,s) < s - t$*, and,*
(S_2) *for* $(t_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}} \subset (0,\infty)$*, if* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$*, then*

$$\limsup_{n\to\infty} \sigma(t_n, s_n) < 0. \tag{2}$$

In the original definition, given in [10], there was an additional but a superfluous condition $\sigma(0,0) = 0$. We underline the observation that a function $\sigma(t,s) := ks - t$, where $k \in [0,1)$ for each $t, s \in [0,\infty)$, is an instantaneous example of a simulation function. For further and more interesting examples, we refer to, e.g., [10–17] and relates references therein.

We say that f, defined on a metric space (X,d) to itself, is a Σ-*contraction* over $\sigma \in \Sigma$ (see [10]), if

$$\sigma(d(fv, f\nu), d(v,\nu)) \geq 0 \text{ for all } v, \nu \in X. \tag{3}$$

The theorem below is the main result of [10]:

Theorem 1. *Each* Σ*-contraction admits a unique fixed point in the setting of a complete metric space.*

Definition 4 ([18])**.** *We say that* $q : X \times X \to [0,\infty)$ *is wt-distance over* (X,d,s)*, if,*

(i) *for all* $v, \nu, \omega \in X$ *the s-weighted triangle inequality holds, that is,*

$$q(v,\omega) \leq s[q(v,\nu) + q(\nu,\omega)];$$

(ii) $q(v,\cdot) : X \to [0,\infty)$ *is s-lower semicontinuous, for any* $v \in X$*, that is,*

$$q(v,\nu) \leq \lim\inf_{n\to\infty} sq(v,\nu_n), \text{ for } v \in X; \text{ and } \nu_n \to \nu \in X,$$

(iii) *for each* $\varepsilon > 0$*, there is* $\delta > 0$*,*

$$q(v,\nu) \leq \delta \text{ and } q(v,\omega) \leq \delta, \text{ yields } d(v,\nu) \leq \varepsilon.$$

Lemma 4 ([18])**.** *Let* $q : X \times X \to [0,\infty)$ *be a wt-distance over* (X,d,s)*. Suppose the sequences* $(v_k), (\nu_k) \subset X$ *and the sequence* $(a_k), (c_k) \subset [0,\infty)$ *such that* $a_k, c_k \to 0$*. Then, the following holds:*

(0) d *is also a wt-distance over* (X,d,s)*,*
(1) *if* $q(v_k,\nu) \leq c_k$ *and* $q(v_k,\omega) \leq a_k$*, for all* $k \in \mathbb{N}$*, then* $\nu = \omega$*;*
(2) *if* $q(v_k,\nu_k) \leq c_k$ *and* $q(v_k,\omega) \leq a_k$*, for all* $k \in \mathbb{N}$*, then* (ν_k) *converges to* ω*;*
(3) *if* $q(v_k,v_l) \leq c_k$ *for all* $k, l \in \mathbb{N}$ *with* $l > k$*, then* (v_k) *is Cauchy sequence;*
(4) *if* $q(\nu,v_k) \leq c_k$ *for all* $k \in \mathbb{N}$*, then* (v_k) *is Cauchy sequence.*

2. Existence and Uniqueness Results for Geragthy Type Operators

We say that $\gamma : [0,\infty) \to [0,\frac{1}{s}), s \geq 1$, is a Geraghty function if

$$\limsup_{n\to\infty} \gamma(t_n) = 1 \Rightarrow t_n \to 0, \text{ as } n \to \infty.$$

We reserve Γ to denote the set of all Geragthy functions.

Theorem 2. *Let q be a wt-distance on (X^*, d, s) and f be a self-mapping on X. We presume that*

(i) *there exist $\sigma \in \Sigma$, $\mu \in \mathcal{B}$ and $\gamma \in \Gamma$ such that*

$$\sigma(\beta(v,\nu)\, q(f(v), f(\nu)), \gamma(\mu(q(v,\nu)))\, \mu(q(v,\nu))) \geq 0, \text{ for all } v,\nu \in X; \tag{4}$$

(ii) *f is triangular $\beta-$orbital admissible;*

(iii) *there is $v_0 \in X$ with $\beta(v_0, f(v_0)) \geq 1$;*

(iv) *f is continuous, or,*

(iv') *for all $v \in X$, with $\beta(v, f(v)) \geq 1$,*

$$\inf\{q(v,\nu) + q(v, f(v))\} > 0,$$

for all $\nu \in X, \nu \neq f(\nu)$.

Then, f has a fixed point.

Proof. Existence of a point $v_0 \in X$ is guaranteed by (iii). A sequence $(v_n)_{n\in\mathbb{N}}$ is defined by $v_n = f^n(v_0)$, for all $n \in \mathbb{N}$. If there exists $k_0 \in \mathbb{N}$ with $f(v_{k_0}) = v_{k_0+1} = v_{k_0}$, then v_{k_0} is a fixed point of f that terminate the proof. Suppose that $v_n \neq v_{n-1}$ for any $n \in \mathbb{N}$. From (ii) and Lemma 3, we have $\beta(v_{n-1}, v_n) \geq 1$.

Under the assumption (i), we have

$$\begin{aligned} 0 \leq\ & \sigma(\beta(v,\nu)q(f(v), f(\nu)), \gamma(\mu(q(v,\nu)))\,\mu(q(v,\nu))) \\ <\ & \gamma(\mu(q(v,\nu)))\,\mu(q(v,\nu)) - \beta(v,\nu)q(f(v), f(\nu)) \end{aligned}$$

which is equivalent to

$$\beta(v,\nu)q(f(v), f(\nu)) < \gamma(\mu(q(v,\nu)))\,\mu(q(v,\nu)).$$

From here, using the properties of γ we obtain

$$\beta(v,\nu)q(f(v), f(\nu)) < \frac{1}{s}\mu(q(v,\nu)), \text{ for all } v,\nu \in X \tag{5}$$

If we consider in (5) $v = v_{n-1}$ and $\nu = v_n$, we get

$$\beta(v_{n-1}, v_n)q(f(v_{n-1}), f(v_n)) \leq \frac{1}{s}\mu(q(v_{n-1}, v_n)). \tag{6}$$

In this way, we have

$$\begin{aligned} q(v_n, v_{n+1}) \leq\ & \beta(v_{n-1}, v_n)q(f(v_{n-1}), f(v_n)) < \tfrac{1}{s}\mu(q(v_{n-1}, v_n)) \\ <\ & \mu(q(v_{n-1}, v_n)) < q(v_{n-1}, v_n). \end{aligned}$$

Hence, we obtain

$$q(v_n, v_{n+1}) < q(v_{n-1}, v_n).$$

Thus, we deduce that $(q(v_n, v_{n+1}))_{n\in\mathbb{N}}$ is a non-increasing sequence. Attendantly, there exists $r \geq 0$ with $\lim_{n\to\infty} q(v_n, v_{n+1}) = r$.

Let us suppose that $r > 0$, and let $r_n = q(v_n, v_{n+1})$. We have

$$\begin{array}{rl} 0 \leq & \sigma(\beta(v_{n-1}, v_n)r_n, \gamma(\mu(r_{n-1}))\mu(r_{n-1})) \\ < & \gamma(\mu(r_{n-1}))\mu(r_{n-1}) - \beta(v_{n-1}, v_n)r_n \\ < & \frac{1}{s}r_{n-1} - \beta(v_{n-1}, v_n)r_n \\ < & r_{n-1} - r_n \end{array}$$

Now, taking the limit when $n \to \infty$, we reach a contradiction, and hence $q(v_n, v_{n+1}) \to 0$, as $n \to \infty$.

We shall prove now that the sequence$(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

First, we have

$$\begin{array}{rl} q(v_n, v_{n+1}) \leq & \beta(v_{n-1}, v_n)q(f(v_{n-1}), f(v_n)) < \mu(q(v_{n-1}, v_n)) \\ < & \mu^2(q(v_{n-2}, v_{n-1})) < ... < \mu^n(q(v_0, v_1)). \end{array}$$

Now, let $m, p \in \mathbb{N}, p > m$

$$\begin{aligned} q(v_m, v_p) &\leq sq(v_m, v_{m+1}) + s^2 q(v_{m+1}, v_{m+2}) + \ldots + s^{p-m} q(v_{p-1}, v_p) \\ &\leq s\mu^m((q(v_0, v_1)) + s^2\mu^{m+1}(q(v_0, v_1)) + \ldots + s^{p-m+1}\mu^p(q(v_0, v_1)) \\ &= \frac{1}{s^{m-1}}\left(s^m\mu^m((q(v_0, v_1)) + s^{m+1}\mu^{m+1}(q(v_0, v_1)) + \ldots + s^p\mu^p(q(v_0, v_1))\right) \\ &= \frac{1}{s^{m-1}}\sum_{j=m}^{p} s^j\mu^j((q(v_0, v_1)). \end{aligned}$$

Since μ is a b-comparison function the series $\sum_{j=0}^{\infty} s^j\mu^j(q(v_0, v_1))$ is convergent. If we denote by $\mathcal{S}_n = \sum_{j=0}^{n} s^j\mu^j(q(v_0, v_1))$, then the above inequality becomes

$$q(v_m, v_p) \leq \frac{1}{s^{m-1}}\left(\mathcal{S}_{p-1} - \mathcal{S}_{m-1}\right), \tag{7}$$

Denoting $\frac{1}{s^{m-1}}\left(\mathcal{S}_{p-1} - \mathcal{S}_{m-1}\right) = \beta_m$, then (7) becomes

$$q(v_m, v_p) \leq \beta_m, \text{ for all } m, p \in \mathbb{N} \text{ with } p > m.$$

Since (β_m) converges to 0, as $m \to \infty$, using (3) from Lemma 4, we obtain that $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence on a complete b-metric space, so there exists $v^* \in X$ such that $v_n \to v^*$. We shall prove that v^* is a fixed point of f.

Suppose that (iv) takes place and f is continuous. In this case, we have

$$v^* = \lim_{n\to\infty} f^{n+1}(v_0) = f\left(\lim_{n\to\infty} f^n(v_0)\right) = f(v^*).$$

Now, suppose that (iv') take place.

Equation (7) implies that $q(v_m, v_p) \to 0$, as $m, p \to \infty$. Then, for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$, such that, if $n > n_\varepsilon$, then

$$q(v_{n_\varepsilon}, v_n) < \varepsilon.$$

Since $q(v, \cdot)$ is s-lower semicontinuous, we have

$$q(v_{n_\varepsilon}, v^*) \leq \liminf_{n\to\infty} sq(v_{n_\varepsilon}, v_n) \leq s\varepsilon. \tag{8}$$

Choosing $\varepsilon = \frac{1}{k}$ and $n_\varepsilon = n_k$, we have from (8)

$$\lim_{k\to\infty} q(v_{n_\varepsilon}, v^*) = 0. \tag{9}$$

We shall prove that v^* is a fixed point of f. Suppose that $f(v^*) \neq v^*$. We have

$$0 < \inf\{q(v_{n_k}, v^*) + q(v_{n_k}, f(v_{n_k}))\} \to 0, \text{ as } k \to \infty,$$

a contradiction. Therefore, $f(v^*) = v^*$. □

Example 2. $X = [0,1]$, $d(x,y) = |x-y|^2$, $q(x,y) = |y|^2$

$$f(x) = \begin{cases} x^2, & \text{if } x \in [0, \frac{1}{4}] \\ \frac{1-x}{3} & \text{if } x \in (\frac{1}{4}, 1] \end{cases}$$

$\gamma(t) = \frac{1}{1+t}$, $\mu(t) = \frac{t}{2}$, $\sigma(t,s) = \frac{15}{16}s - t$,

$$\beta(x,y) = \begin{cases} 1, & \text{if } x,y \in [0, \frac{1}{4}] \\ 3, & \text{if } x \in [0, \frac{1}{4}], y = 1 \\ 0, & \text{otherwise .} \end{cases}$$

With these choices, the inequality (4), Theorem 2 becomes:

$$\begin{aligned} \beta(x,y)q(fx,fy) &\leq \frac{15}{16}\frac{1}{1+\mu(q(x,y))}\mu(q(x,y)) \\ &= \frac{15}{16}\frac{2}{2+q(x,y)}\frac{q(x,y)}{2} \end{aligned}$$

or, since $q(x,y) = y^2$,

$$\beta(x,y)(fy)^2 \leq \frac{15}{16}\frac{2}{2+y^2}\frac{y^2}{2}.$$

- *If* $x, y \in [0, \frac{1}{4}]$,

$$\beta(x,y)(fy)^2 = y^4 \leq \frac{15}{16}\frac{2}{2+y^2}\frac{y^2}{2} \iff y^2\left(y^2 - \frac{15}{16}\cdot\frac{1}{2+y^2}\right) \leq 0 \iff$$

$$16y^4 + 32y^2 - 15 \leq 0.$$

- *If* $x \in [0, \frac{1}{4}]$, $y = 1$,

$$\beta(x,1)(f1)^2 = 0 \leq \frac{15}{16}\cdot\frac{2}{2+1}\cdot\frac{1}{2}$$

Other cases are obvious because of the choice of β. *It is clear that* 0 *is the fixed point of the given map.*

Theorem 3. *Suppose that the condition of Theorem 2 holds. If we suppose that, for* $v^*, \nu^* \in Fix(f)$ *we have* $\beta(v^*, \nu^*) \geq 1$, *then* $v^* = \nu^*$.

Proof. Let $v^*, \nu^* \in Fix(f)$ with $\beta(v^*, \nu^*) \geq 1$. Suppose $v^* \neq \nu^*$. Using (5), we have

$$\begin{aligned} q(v^*, \nu^*) \leq\ & \beta(v^*, \nu^*)q(f(v^*), f(\nu^*)) < \frac{1}{s}\mu(q(v^*, \nu^*)) \\ <\ & \mu(q(v^*, \nu^*)) < q(v^*, \nu^*), \end{aligned}$$

a contradiction. It concludes that f possesses a unique fixed point. □

3. Existence and Uniqueness Results for Meir-Keeler Type Operators

In this section, we shall give a similar result for Meir-Keeler type operators.

Theorem 4. *Let $q : X \times X \to [0, \infty)$ be a wt-distance on (X^*, d, s), and f be a self-mapping on X. We presume that*

(i) for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \le \mu\left(d\left(v, \nu\right)\right) < \varepsilon + \delta \text{ implies } \sigma(\beta\left(v, \nu\right) q(f\left(v\right), f\left(\nu\right)), \varepsilon) \ge 0, \text{ for all } v, \nu \in X, \tag{10}$$

where $\sigma \in \Sigma$, $\mu \in \mathcal{B}$ with $\mu\left(t\right) < \frac{t}{s}$, for all $t > 0$,

(ii) *f is triangular $\beta-$orbital admissible;*

(iii) *there exists $v_0 \in X$ such that $\beta(v_0, f\left(v_0\right)) \ge 1$;*

(iv) *f is continuous, or,*

(iv') *for all $v \in X$, with $\beta(v, f\left(v\right)) \ge 1$,*

$$\inf\left\{q\left(v, \nu\right) + q\left(v, f\left(v\right)\right)\right\} > 0,$$

for all $\nu \in X, \nu \neq f\left(\nu\right)$.

Then, f has a fixed point.

Proof. Like in the proof of Theorem 2, we construct a sequence $(v_n)_{n \in \mathbb{N}}$, defined by $v_n = f^n\left(v_0\right)$ for each $n \in \mathbb{N}$. Regarding the same arguments in Theorem 2, we presume that $v_n \neq v_{n-1}$ for any $n \in \mathbb{N}$.

Since f is triangular $\beta-$orbital admissible, on account of (ii) and Lemma 3, we find $\beta(v_{n-1}, v_n) \ge 1$.

Under the assumption (i), we have

$$\begin{aligned} 0 \le\ & \sigma(\beta(v,\nu)q(f\left(v\right), f\left(\nu\right)), \varepsilon) \\ & < \varepsilon - \beta(v,\nu)q(f\left(v\right), f\left(\nu\right)) \\ & < \mu\left(d\left(v,\nu\right)\right) - \beta(v,\nu)q(f\left(v\right), f\left(\nu\right)). \end{aligned}$$

From here, we have

$$\beta\left(v, \nu\right) q(f\left(v\right), f\left(\nu\right)) < \mu\left(q\left(v, \nu\right)\right) < q\left(v, \nu\right), \text{ for all distinct} v, \nu \in X \tag{11}$$

If we consider in (11) $v = v_{n-1}$ and $\nu = v_n$, we get

$$\beta(v_{n-1}, v_n) q\left(f\left(v_{n-1}\right), f\left(v_n\right)\right) \le \mu(q(v_{n-1}, v_n)) \tag{12}$$

In this way, we have

$$\begin{aligned} q\left(v_n, v_{n+1}\right) \le\ & \beta(v_{n-1}, v_n)\mu\left(q\left(f\left(v_{n-1}\right), f\left(v_n\right)\right)\right) \\ <\ & \mu(q(v_{n-1}, v_n)) < q(v_{n-1}, v_n) \end{aligned}$$

Hence, we obtain

$$q\left(v_n, v_{n+1}\right) < q(v_{n-1}, v_n).$$

It yields that $(q\left(v_n, v_{n+1}\right))_{n \in \mathbb{N}}$ is a decreasing sequence and it converges to $r \ge 0$.

We assert that $r = 0$. Suppose, on the contrary, that $r > 0$, and let $r_n = q\left(v_n, v_{n+1}\right)$. Thus, we have

$$0 < r < q\left(v_n, v_{n+1}\right) = r_n, \text{ for all } n \in \mathbb{N}. \tag{13}$$

Let $\varepsilon = r > 0$. Consequently, there is $\delta(\varepsilon) > 0$ so that (10) is fulfilled. Despite, due to definition of ε, there is $n_0 \in \mathbb{N}$ so that

$$\varepsilon < r_{n_0} = q(v_{n_0}, v_{n_0+1}) < \varepsilon + \delta.$$

Using (11), we have

$$\begin{aligned} q\left(v_{n_0+1}, v_{n_0+2}\right) \leq & \quad \beta(v_{n_0}, v_{n_0+1}) q\left(f\left(v_{n_0}\right), f\left(v_{n_0+1}\right)\right) \\ < & \quad \varepsilon = r. \end{aligned}$$

In this way, we reach a contradiction. Hence, $q\left(v_n, v_{n+1}\right) \to 0$, as $n \to \infty$.

Now, inspired by the proof from ([19]), we shall demonstrate that the sequence $(v_n)_{n\in\mathbb{N}}$ is Cauchy.

For a given $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$, such that (10) holds. Suppose that $\delta < \varepsilon$. Since $q\left(v_n, v_{n+1}\right) \to 0$, as $n \to \infty$, there exists $n_0 \in \mathbb{N}$, such that

$$q(v_{n-1}, v_n) < \delta, \text{ for all } n \geq n_0. \tag{14}$$

Just like in ([19]), we shall prove that, for any fixed $k \geq n_0$,

$$q(v_k, v_{k+l}) \leq \varepsilon, \text{ for all } l \in \mathbb{N}.$$

Suppose that (10) is satisfied for some $m \in \mathbb{N}$, and so (11), and let $l = m + 1$

$$\begin{aligned} q\left(v_{k-1}, v_{k+m}\right) \leq & \quad sq\left(v_{k-1}, v_k\right) + sq\left(v_k, v_{k+m}\right) \\ < & \quad s\left(\delta + \varepsilon\right). \end{aligned}$$

From here, we have

$$\mu\left(q\left(v_{k-1}, v_{k+m}\right)\right) < \mu\left(s\left(\delta + \varepsilon\right)\right) < \delta + \varepsilon.$$

Now, if $\mu\left(q\left(v_{k-1}, v_{k+m}\right)\right) \geq \varepsilon$, then using (10)

$$q(v_k, v_{k+m+1}) \leq \beta(v_{k-1}, v_{k+m}) q(f\left(v_{k-1}\right), f\left(v_{k+m}\right)) < \varepsilon$$

If $\mu\left(q\left(v_{k-1}, v_{k+m}\right)\right) < \varepsilon$, then, by (11),

$$\begin{aligned} q(v_k, v_{k+m+1}) \leq & \quad \beta(v_{k-1}, v_{k+m}) q(f\left(v_{k-1}\right), f\left(v_{k+m}\right)) \\ & \leq \mu\left(q\left(v_{k-1}, v_{k+m}\right)\right) < \varepsilon \end{aligned}$$

In this way, we have

$$q(v_n, v_m) < \varepsilon, \text{ for all } m \geq n \geq n_0.$$

It easy to see that there exists a sequence $(\beta_n)_{n\in\mathbb{N}}$, which converges to 0, such that

$$q(v_n, v_m) < \beta_n, \text{ for all } m \geq n \geq n_0 \tag{15}$$

Hence, $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since the considered space is complete, there exists $v^* \in X$ so that $v_n \to v^*$

In case when f is continuous, we find

$$v^* = \lim_{n\to\infty} f^{n+1}\left(v_0\right) = f\left(\lim_{n\to\infty} f^n\left(v_0\right)\right) = f\left(v^*\right).$$

Otherwise, since $q(v_n, v_m) \to 0$, as $n, m \to \infty$, then for each $\varepsilon > 0$, there exists $k_\varepsilon \in \mathbb{N}$, such that, if $k > k_\varepsilon$, then

$$q(v_{k_\varepsilon}, v_k) < \varepsilon.$$

Since $q(\upsilon, \cdot)$ is $s-$ lower semicontinuous, we have

$$q(\upsilon_{k_\varepsilon}, \upsilon^*) \quad \leq \liminf_{k\to\infty} sq\left(\upsilon_{k_\varepsilon}, \upsilon_k\right) \leq s\varepsilon. \tag{16}$$

Choosing $\varepsilon = \frac{1}{l}$ and $n_\varepsilon = n_l$, we have from (16)

$$\lim_{l\to\infty} q(\upsilon_{n_l}, \upsilon^*) = 0. \tag{17}$$

We shall prove that υ^* is a fixed point of f. Suppose that $f(\upsilon^*) \neq \upsilon^*$. We have

$$0 < \inf\left\{q\left(\upsilon_{n_l}, \upsilon^*\right) + q\left(\upsilon_{n_l}, f\left(\upsilon_{nl}\right)\right)\right\} \to 0, \text{ as } l \to \infty,$$

which is a contradiction. Therefore, $f(\upsilon^*) = \upsilon^*$. □

Theorem 5. *Suppose that the condition from Theorem 4 holds. If we suppose that for $\upsilon^*, \nu^* \in Fix(f)$, we have:*
(i) $\beta(\upsilon^*, \nu^*) \geq 1$;
(ii) For $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \mu\left(d\left(\upsilon^*, \nu^*\right)\right) < \varepsilon + \delta,$$

then $\upsilon^* = \nu^*$.

Proof. Let $\upsilon^*, \nu^* \in Fix(f)$ with $\beta(\upsilon^*, \nu^*) \geq 1$. Suppose $\upsilon^* \neq \nu^*$. From (ii), using (11), we have

$$\begin{aligned} q\left(\upsilon^*, \nu^*\right) \leq & \quad \beta(\upsilon^*, \nu^*) q\left(f\left(\upsilon^*\right), f\left(\nu^*\right)\right) < \mu(q(\upsilon^*, \nu^*)) \\ < & \quad q(\upsilon^*, \nu^*), \end{aligned}$$

a contradiction. Consequently, f admits a unique fixed point. □

4. Conclusions

In this paper, we have considered two new contractions in the setting of wt-distance over b-metric space. We discuss and investigate the necessary conditions to guarantee both the existence and uniqueness of a fixed point. Our main results cover several published results in the literature. Indeed, by letting β function in a proper way, we get some new results. For example, for $\beta(u, v) = 1$, we get the standard fixed point results in the setting of wt-distance over b-metric space. It is known that b-metric space itself is a b-metric and, furthermore, b-metric turns out to be a standard metric for $s = 1$. As it is seen in [14,15,17,20], for different choices of $\sigma \in \Sigma$, we shall get more different consequences.

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