mathematics

Article

The Non-Eigenvalue Form of Liouville's Formula and α-Matrix Exponential Solutions for Combined Matrix Dynamic Equations on Time Scales

Zhien Li [1], Chao Wang [1,*] and Ravi P. Agarwal [2,3]

1 Department of Mathematics, Yunnan University, Kunming 650091, China; lizhienyunnan@163.com
2 Department of Mathematics, Texas A&M University-Kingsville, 700 University Blvd., Kingsville, TX 78363-8202, USA; Ravi.Agarwal@tamuk.edu
3 Distinguished University Professor of Mathematics, Florida Institute of Technology, Melbourne, FL 32901, USA
* Correspondence: chaowang@ynu.edu.cn

Received: 27 August 2019; Accepted: 8 October 2019; Published: 13 October 2019

Abstract: In this paper, the non-eigenvalue forms of Liouville's formulas for delta, nabla and α-diamond matrix dynamic equations on time scales are given and proved. Meanwhile, a diamond matrix exponential function (or α-matrix exponential function) is introduced and some classes of homogenous linear diamond-α dynamic equations which possess the α-matrix exponential solutions is studied. The difference and relation of non-eigenvalue forms of Liouville's formulas among these representative types of dynamic equations is investigated. Moreover, we establish some sufficient conditions to guarantee transformational relation of Liouville's formulas and exponential solutions among these types of matrix dynamic equations. In addition, we provide several examples on various time scales to illustrate the effectiveness of our result.

Keywords: Liouville's formula; matrix dynamic equations; time scales

MSC: 34M25, 34N05, 26E70

1. Introduction

As an effective and powerful tool, time scale calculus is initiated to unify continuous and discrete analysis and is extensively applied to study dynamic equations [1–5]. Liouville's formula and Liouville's problems are very important topic in ordinary differential equations [6–10]. In Theorem 5.28 from [9], Liouville's formula of dynamic equations on time scales is given by Bohner and Peterson. The theorem is given as follows:

Theorem 1. *Let $A \in \mathcal{R}$ be a 2×2-matrix-valued function and assume that X is a solution of $X^{\Delta} = A(t)X$. Then X satisfies Liouville's formula*

$$\det X(t) = e_{trA+\mu \det A}(t, t_0) \det X(t_0), \ \text{for } t \in \mathbb{T}. \tag{1}$$

Please note that Liouville's formula given by the form (1) is very convenient to use due to the main reason that $trA + \mu \det A$ is independent of the eigenpolynomial and eigenvalue of A. The Liouville's formula for $n \times n$-matrix dynamic equations on time scales were studied in [11,12]. In [11,12], the authors provided the nice form of Liouville's formula by considering the eigenpolynomial and eigenvalue of the coefficient matrix of the dynamic equations. However, if A is a $n \times n$-matrix-valued function for n sufficiently large, the calculation of eigenpolynomial and eigenvalue of A becomes a complicated task and cannot be always achieved, so it will be a better way to provide a matrix form of Liouville's formula for the case of $A : \mathbb{T} \to \mathbb{R}^{n \times n}$ to avoid the calculation of eigenpolynomial and eigenvalue of A, similar to (1). Unfortunately, for $n \geq 3$, the matrix X that is a solution of dynamic equations $X^{\Delta} = A(t)X$ does not satisfy (1), i.e., the nice form (1) of Liouville's formula will not hold for $n \geq 3$.

On the other hand, the combined dynamic derivatives on time scales was proposed in [13], which can unify Δ-dynamic derivative ($\alpha = 1$) and ∇-dynamic derivative ($\alpha = 0$). Moreover, this combined type of derivative includes the hybrid dynamic derivatives between Δ and ∇ cases and was used to study various complex dynamic equations and inequalities on time scales [14–18].

However, since there is no Liouville's formula of diamond-α matrix dynamic equations, in this paper, we make the following contributions:

- (i) the combined matrix exponential function is introduced and studied;
- (ii) Liouville's formula of diamond-α matrix dynamic equations is obtained without considering the eigenpolynomial and eigenvalue;
- (iii) some classes of diamond-α matrix dynamic equations which have α-matrix exponential solutions are investigated;
- (iv) the obtained results are completely new even for Δ and ∇-matrix dynamic equations and several examples on various time scales are provided.

In particular, for the 2×2 coefficient matrix and $\alpha = 1$, Liouville's formula happens to be (1).

2. Liouville'S Formula for Δ-dynamic Equations

In this section, we will derive the non-eigenvalue form of Liouville's formula for Δ-matrix dynamic equations which will be used to study the combined dynamic equations on time scales.

Definition 1 ([9]). *Define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$; the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$; the graininess function $\mu : \mathbb{T} \to [0, +\infty)$ by $\mu(t) = \sigma(t) - t$ (denoted by μ); the $\nu(t)$ by $\nu(t) = t - \rho(t)$ (denoted by ν).*

Definition 2 ([9]). *Definite $e_A(t, t_0)$ by the unique matrix solution of the initial value problem:*

$$X^{\Delta}(t) = A(t)X(t), \;\; X_0 = I. \tag{2}$$

Lemma 1. *Let A be an upper triangular $n \times n$-matrix-valued function, then A is regressive iff each diagonal element of A is regressive.*

Proof. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

Hence

$$\det[I + \mu A] = \begin{vmatrix} 1 + \mu a_{11} & \mu a_{12} & \dots & \mu a_{1n} \\ 0 & 1 + \mu a_{22} & \dots & \mu a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 + \mu a_{nn} \end{vmatrix} = \prod_{i=1}^{n} (1 + \mu a_{ii}).$$

Therefore A is regressive iff each diagonal element of A is regressive. □

Remark 1. *Let $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ be an upper triangular matrix, we can obtain*

$$\det A = \prod_{i=1}^{n} a_{ii}, \ \ trA = \sum_{i=1}^{n} a_{ii}, \ \ \det\big(I + \mu A\big) = \prod_{i=1}^{n}\big(1 + \mu a_{ii}\big).$$

Hence for $n > 2$,

$$1 + \mu\big(trA + \mu \det A\big) = 1 + \mu\Big(\sum_{i=1}^{n} a_{ii} + \mu \prod_{i=1}^{n} a_{ii}\Big) \neq \prod_{i=1}^{n}\big(1 + \mu a_{ii}\big),$$

which implies that $I + \mu A$ is invertible does not be equivalent to $trA + \mu \det A$ is regressive, i.e., Liouville's Formula (1) is not suitable for $n \geq 3$.

It is easy to check the following Δ-derivative formula of determinant function by using the determinant algorithm and Theorem 1.20 from [9].

Lemma 2. *Let $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ be the following function matrix:*

$$A(t) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i} & a_{1(i+1)} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \cdots & a_{(i-1)i} & a_{(i-1)(i+1)} & \cdots & a_{(i-1)n} \\ a_{i1} & a_{i2} & \cdots & a_{ii} & a_{i(i+1)} & \cdots & a_{in} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)i} & a_{(i+1)(i+1)} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & a_{n(i+1)} & \cdots & a_{nn} \end{vmatrix},$$

then

$$A^{\Delta}(t) = \sum_{i=1}^{n} \begin{vmatrix} a^{\sigma}_{11} & a^{\sigma}_{12} & \cdots & a^{\sigma}_{1i} & a^{\sigma}_{1(i+1)} & \cdots & a^{\sigma}_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a^{\sigma}_{(i-1)1} & a^{\sigma}_{(i-1)2} & \cdots & a^{\sigma}_{(i-1)i} & a^{\sigma}_{(i-1)(i+1)} & \cdots & a^{\sigma}_{(i-1)n} \\ a^{\Delta}_{i1} & a^{\Delta}_{i2} & \cdots & a^{\Delta}_{ii} & a^{\Delta}_{i(i+1)} & \cdots & a^{\Delta}_{in} \\ a_{(i+1)1} & a_{(i+1)2} & \cdots & a_{(i+1)i} & a_{(i+1)(i+1)} & \cdots & a_{(i+1)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{ni} & a_{n(i+1)} & \cdots & a_{nn} \end{vmatrix}. \tag{3}$$

Remark 2. *In Lemma 2, notice that a_{ij} denotes the element that is located in the ith row and the jth column, i.e., $1 \leq i, j \leq n$. Hence $A^\Delta(t)$ equals to the following:*

$$
A^\Delta(t) = \begin{vmatrix} a_{11}^\Delta(t) & a_{12}^\Delta(t) & a_{13}^\Delta(t) & \dots & a_{1n}^\Delta(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) & \dots & a_{2n}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) & \dots & a_{3n}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1}(t) & a_{i2}(t) & a_{i3}(t) & \dots & a_{in}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & a_{n3}(t) & \dots & a_{nn}(t) \end{vmatrix} + \begin{vmatrix} a_{11}^\sigma(t) & a_{12}^\sigma(t) & a_{13}^\sigma(t) & \dots & a_{1n}^\sigma(t) \\ a_{21}^\Delta(t) & a_{22}^\Delta(t) & a_{23}^\Delta(t) & \dots & a_{2n}^\Delta(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) & \dots & a_{3n}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1}(t) & a_{i2}(t) & a_{i3}(t) & \dots & a_{in}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & a_{n3}(t) & \dots & a_{nn}(t) \end{vmatrix} + \dots
$$

$$
+ \begin{vmatrix} a_{11}^\sigma(t) & a_{12}^\sigma(t) & \dots & a_{1i}^\sigma(t) & a_{1(i+1)}^\sigma(t) & \dots & a_{1n}^\sigma(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(i-1)1}^\sigma(t) & a_{(i-1)2}^\sigma(t) & \dots & a_{(i-1)i}^\sigma(t) & a_{(i-1)(i+1)}^\sigma(t) & \dots & a_{(i-1)n}^\sigma(t) \\ a_{i1}^\Delta(t) & a_{i2}^\Delta(t) & \dots & a_{ii}^\Delta(t) & a_{i(i+1)}^\Delta(t) & \dots & a_{in}^\Delta(t) \\ a_{(i+1)1}(t) & a_{(i+1)2}(t) & \dots & a_{(i+1)i}(t) & a_{(i+1)(i+1)}(t) & \dots & a_{(i+1)n}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{ni}(t) & a_{n(i+1)}(t) & \dots & a_{nn}(t) \end{vmatrix} + \dots
$$

$$
+ \begin{vmatrix} a_{11}^\sigma(t) & a_{12}^\sigma(t) & \dots & a_{1i}^\sigma(t) & a_{1(i+1)}^\sigma(t) & \dots & a_{1n}^\sigma(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1}^\sigma(t) & a_{i2}^\sigma(t) & \dots & a_{ii}^\sigma(t) & a_{i(i+1)}^\sigma(t) & \dots & a_{in}^\sigma(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(n-1)1}^\sigma(t) & a_{(n-1)2}^\sigma(t) & \dots & a_{(n-1)i}^\sigma(t) & a_{(n-1)(i+1)}^\sigma(t) & \dots & a_{(n-1)n}^\sigma(t) \\ a_{n1}^\Delta(t) & a_{n2}^\Delta(t) & \dots & a_{ni}^\Delta(t) & a_{n(i+1)}^\Delta(t) & \dots & a_{nn}^\Delta(t) \end{vmatrix},
$$

where $3 \leq i \leq n-1$. For convenience, we denote the sum by (3). Similarly, the determinant symbol is with the same meaning for all theorems, lemmas and examples.

Theorem 2 (Liouville's formula)**.** *Let $A \in \mathcal{R}$ be an upper triangular $n \times n$-matrix-valued function and assume that X is a solution of $X^\Delta = A(t)X$. Then X satisfies Liouville's formula*

$$
\det X(t) = e_{\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{i-1} (1+\mu a_{jj}) a_{ii}}(t, t_0) \det X_0, \quad for \;\; t \in \mathbb{T}.
$$

Proof. For the matrix A, by Lemma 1, we can obtain A is regressive iff each diagonal element of A is regressive. Let

$$
A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ 0 & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_{nn} \end{bmatrix}
$$

$$
\begin{aligned}
\big(\det X(t)\big)^{\Delta} &= \begin{vmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ 0 & x_{22} & x_{23} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_{nn} \end{vmatrix}^{\Delta} \\
&= \begin{vmatrix} x_{11}^{\Delta} & x_{12}^{\Delta} & x_{13}^{\Delta} & \dots & x_{1n}^{\Delta} \\ 0 & x_{22} & x_{23} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_{nn} \end{vmatrix} + \sum_{i=2}^{n-1} \begin{vmatrix} x_{11}^{\sigma} & x_{12}^{\sigma} & \dots & x_{1i}^{\sigma} & x_{1(i+1)}^{\sigma} & \dots & x_{1n}^{\sigma} \\ 0 & x_{22}^{\sigma} & \dots & x_{2i}^{\sigma} & x_{2(i+1)}^{\sigma} & \dots & x_{2n}^{\sigma} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & x_{ii}^{\Delta} & x_{i(i+1)}^{\Delta} & \dots & x_{in}^{\Delta} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & x_{nn} \end{vmatrix} \\
&= \sum_{i=1}^{n} \begin{vmatrix} x_{11}^{\sigma} & x_{12}^{\sigma} & \dots & x_{1i}^{\sigma} & x_{1(i+1)}^{\sigma} & \dots & x_{1n}^{\sigma} \\ 0 & x_{22}^{\sigma} & \dots & x_{2i}^{\sigma} & x_{2(i+1)}^{\sigma} & \dots & x_{2n}^{\sigma} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & x_{ii}^{\Delta} & x_{i(i+1)}^{\Delta} & \dots & x_{in}^{\Delta} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & x_{nn} \end{vmatrix} \\
&= \sum_{i=1}^{n} \prod_{j=1}^{i-1} x_{jj}^{\sigma} x_{ii}^{\Delta} \prod_{k=i+1}^{n} x_{kk} = \left[\sum_{i=1}^{n} \prod_{j=1}^{i-1} (1+\mu a_{jj}) a_{ii} \right] \det X.
\end{aligned}
$$

This completes the proof. □

Remark 3. *Let A be a 2×2-matrix-valued function. Then A is regressive iff the scalar-valued function $trA + \mu \det A$ is regressive (where trA denotes the trace of the matrix A, i.e., the sum of the diagonal elements of A). In fact, let*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

then we can obtain

$$
\begin{aligned}
\det(I+\mu A) &= \det \begin{bmatrix} 1+\mu a_{11} & \mu a_{12} \\ \mu a_{21} & 1+\mu a_{22} \end{bmatrix} \\
&= (1+\mu a_{11})(1+\mu a_{22}) - \mu a_{12}\mu a_{21} \\
&= 1+\mu(a_{11}+a_{22}) + \mu^2(a_{11}a_{22} - a_{12}a_{21}) \\
&= 1+\mu(trA + \mu \det A).
\end{aligned}
$$

Hence A is regressive iff the scalar-valued function $trA + \mu detA$ is regressive.

Remark 4. *Assume that for $n = k$, we can obtain similar characterizations of Remark 3. For $n = k+1$, let*

$$A_{k+1} = \begin{bmatrix} A_k & 0 \\ 0 & a_0 \end{bmatrix},$$

where A_{k+1} is $(k+1)\times(k+1)$-matrix-valued function, A_k is $k\times k$-matrix-valued function. Then

$$\begin{aligned}
\det(I+\mu A_{k+1}) &= \det\begin{bmatrix} I_k+\mu A_k & 0 \\ 0 & 1+\mu a_0 \end{bmatrix}, \\
&= (1+\mu a_0)\det(I+\mu A_k), \\
1+\mu(trA_{k+1}+\mu\det A_{k+1}) &= 1+\mu[trA_k+a_0+\mu a_0\det A_k] \\
&= 1+\mu trA_k+\mu a_0+\mu a_0\mu\det A_k.
\end{aligned}$$

Hence there exists A_{k+1} such that A_{k+1} is regressive and so is $trA_{k+1}+\mu\det A_{k+1}$.

Theorem 3 (Liouville's formula). *Let $A\in\mathcal{R}$ be $n\times n$-matrix-valued matrix function and assume that X is a solution of $X^\Delta = A(t)X$. Then X satisfies Liouville's formula*

$$\det X(t) = e_{\sum\limits_{i=1}^{n}\det A_i}(t,t_0)\det X_0, \quad for\ \ t\in\mathbb{T},$$

where

$$A_i = \begin{bmatrix}
1+\mu a_{11} & \mu a_{12} & \dots & \mu a_{1(i-1)} & \mu a_{1i} & 0 & \dots & 0 \\
\mu a_{21} & 1+\mu a_{22} & \dots & \mu a_{2(i-1)} & \mu a_{2i} & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mu a_{(i-1)1} & \mu a_{(i-1)2} & \dots & 1+\mu a_{(i-1)(i-1)} & \mu a_{(i-1)i} & 0 & 0 & 0 \\
a_{i1} & a_{i2} & \dots & a_{i(i-1)} & a_{ii} & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 1
\end{bmatrix}.$$

Proof. For $n=2$, by Remark 3, A is regressive implies $trA+\mu\det A$ is regressive. Let

$$A(t) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, X(t) = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}.$$

Then

$$\begin{aligned}
\big(\det X(t)\big)^\Delta &= \begin{vmatrix} x_{11}^\Delta & x_{12}^\Delta \\ x_{21} & x_{22} \end{vmatrix} + \begin{vmatrix} x_{11}^\sigma & x_{12}^\sigma \\ x_{21}^\Delta & x_{22}^\Delta \end{vmatrix} \\
&= \begin{vmatrix} a_{11}x_{11}+a_{12}x_{21} & a_{11}x_{12}+a_{12}x_{22} \\ x_{21} & x_{22} \end{vmatrix} + \begin{vmatrix} x_{11}+\mu x_{11}^\Delta & x_{12}+\mu x_{12}^\Delta \\ x_{21}^\Delta & x_{22}^\Delta \end{vmatrix} \\
&= a_{11}\det X(t) + \begin{vmatrix} x_{11}+(a_{11}x_{11}+a_{12}x_{21})\mu & x_{12}+(a_{11}x_{12}+a_{12}x_{22})\mu \\ a_{21}x_{11}+a_{22}x_{21} & a_{21}x_{12}+a_{22}x_{22} \end{vmatrix} \\
&= a_{11}\det X(t) + \begin{vmatrix} 1+\mu a_{11} & a_{12}\mu \\ a_{21} & a_{22} \end{vmatrix}\det X \\
&= \big(trA+\mu\det A\big)\det X \\
&= \left\{ \begin{vmatrix} a_{11} & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1+\mu a_{11} & a_{12}\mu \\ a_{21} & a_{22} \end{vmatrix} \right\}\det X.
\end{aligned}$$

For $n \geq 3$, let

$$A(t) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}, X(t) = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \dots & x_{nn} \end{bmatrix}.$$

Hence by Lemma 2, we have

$$\begin{aligned}
\left(X(t)\right)^{\Delta} &= \begin{vmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \dots & x_{nn} \end{vmatrix}^{\Delta} = \sum_{i=1}^{n} \begin{vmatrix} x^{\sigma}_{11} & x^{\sigma}_{12} & x^{\sigma}_{13} & \dots & x^{\sigma}_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x^{\sigma}_{(i-1)1} & x^{\sigma}_{(i-1)2} & x^{\sigma}_{(i-1)3} & \dots & x^{\sigma}_{(i-1)n} \\ x^{\Delta}_{i1} & x^{\Delta}_{i2} & x^{\Delta}_{i3} & \dots & x^{\Delta}_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \dots & x_{nn} \end{vmatrix} \\
&= \sum_{i=1}^{n} \begin{vmatrix} x_{11} + \mu x^{\Delta}_{11} & x_{12} + \mu x^{\Delta}_{12} & x_{13} + \mu x^{\Delta}_{13} & \dots & x_{1n} + \mu x^{\Delta}_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(i-1)1} + \mu x^{\Delta}_{(i-1)1} & x_{(i-1)2} + \mu x^{\Delta}_{(i-1)2} & x_{(i-1)3} + \mu x^{\Delta}_{(i-1)3} & \dots & x_{(i-1)n} + \mu x^{\Delta}_{(i-1)n} \\ x^{\Delta}_{i1} & x^{\Delta}_{i2} & x^{\Delta}_{i3} & \dots & x^{\Delta}_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \dots & x_{nn} \end{vmatrix} \\
&= \sum_{i=1}^{n} \begin{vmatrix} x_{11} + \mu \sum\limits_{j=1}^{n} a_{1j} x_{j1} & x_{12} + \mu \sum\limits_{j=1}^{n} a_{1j} x_{j2} & \dots & x_{1n} + \mu \sum\limits_{j=1}^{n} a_{1j} x_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ x_{(i-1)1} + \mu \sum\limits_{j=1}^{n} a_{(i-1)j} x_{j1} & x_{(i-1)2} + \mu \sum\limits_{j=1}^{n} a_{(i-1)j} x_{j2} & \dots & x_{(i-1)n} + \mu \sum\limits_{j=1}^{n} a_{(i-1)j} x_{jn} \\ \sum\limits_{j=1}^{n} a_{ij} x_{j1} & \sum\limits_{j=1}^{n} a_{ij} x_{j2} & \dots & \sum\limits_{j=1}^{n} a_{ij} x_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{vmatrix} \\
&= \sum_{i=1}^{n} \begin{vmatrix} x_{11} + \mu \sum\limits_{j=1}^{i} a_{1j} x_{j1} & x_{12} + \mu \sum\limits_{j=1}^{i} a_{1j} x_{j2} & \dots & x_{1n} + \mu \sum\limits_{j=1}^{i} a_{1j} x_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ x_{(i-1)1} + \mu \sum\limits_{j=1}^{i} a_{(i-1)j} x_{j1} & x_{(i-1)2} + \mu \sum\limits_{j=1}^{i} a_{(i-1)j} x_{j2} & \dots & x_{(i-1)n} + \mu \sum\limits_{j=1}^{i} a_{(i-1)j} x_{jn} \\ \sum\limits_{j=1}^{i} a_{ij} x_{j1} & \sum\limits_{j=1}^{i} a_{ij} x_{j2} & \dots & \sum\limits_{j=1}^{i} a_{ij} x_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{vmatrix} \\
&= \sum_{i=1}^{n} \det(A_i X) = \sum_{i=1}^{n} \det A_i \det X.
\end{aligned}$$

Then

$$X(t) = \left(e_{\sum\limits_{i=1}^{n} \det A_i}(t, t_0) \right) \det X_0.$$

This completes the proof. □

Theorem 4. *Let A be a $n \times n$-matrix-valued function. Then A is regressive iff the scalar-valued function $\sum_{i=1}^{n} \det A_i$ is regressive, where A_i is defined by Theorem 3.*

Proof. By Theorem 3, for $n = 2$ we can obtain

$$\det(I + \mu A) = 1 + \mu \left\{ \begin{vmatrix} a_{11} & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 + \mu a_{11} & a_{12}\mu \\ a_{21} & a_{22} \end{vmatrix} \right\}.$$

Next, let

$$A(t) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} & & & & a_{1n} \\ & & A_{(0)} & & a_{2n} \\ & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix},$$

where $A_{(0)}$ is a $(n-1) \times (n-1)$-valued-matrix function. Assume that $(n-1) \times (n-1)$-matrix-value function is regressive iff the scalar-valued function $\sum_{i=1}^{n-1} \det A_i$ is regressive. That is

$$\det(I + \mu A_{(0)}) = 1 + \mu \sum_{i=1}^{n-1} \det A_i.$$

Hence

$$\begin{aligned} & 1 + \mu \left(\sum_{i=1}^{n-1} \det A_i + \det A_n \right) = 1 + \mu \sum_{i=1}^{n-1} \det A_i + \mu \det A_n = \det\left(I + \mu A_{(0)}\right) + \mu \det A_n \\ = \; & \det\left(I + \mu A_{(0)}\right) + \mu \begin{vmatrix} & & & & \mu a_{1n} \\ & & I + \mu A_{(0)} & & \mu a_{2n} \\ & & & & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} \\ = \; & \begin{vmatrix} & & & & \mu a_{1n} \\ & & I + \mu A_{(0)} & & \mu a_{2n} \\ & & & & \dots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix} + \begin{vmatrix} & & & & \mu a_{1n} \\ & & I + \mu A_{(0)} & & \mu a_{2n} \\ & & & & \dots \\ \mu a_{n1} & \mu a_{n2} & \mu a_{n3} & \dots & \mu a_{nn} \end{vmatrix} \\ = \; & \begin{vmatrix} & & & & \mu a_{1n} \\ & & I + \mu A_{(0)} & & \mu a_{2n} \\ & & & & \dots \\ \mu a_{n1} & \mu a_{n2} & \mu a_{n3} & \dots & 1 + \mu a_{nn} \end{vmatrix} = \det(I + \mu A). \end{aligned}$$

Therefore A is regressive iff the scalar-valued function $\sum_{i=1}^{n} \det A_i$ is regressive. This completes the proof. □

Example 1. *For (2), let $\mathbb{T} = \mathbb{R}$, Liouville's formula can be given as:*

$$\det X = \det X_0 e^{\int_{t_0}^{t} trA(\tau)\mathrm{d}\o}, \;\; trA(\tau) = \sum_{i=1}^{n} a_{ii}(\tau).$$

In fact, let $\mathbb{T}=\mathbb{R}$, then $\mu(t)=0$, $X^{\sigma}(t)=X(t)$ for any $t\in\mathbb{T}$, hence $X^{\Delta}(t)=X'$, therefore

$$\begin{aligned}
\big(\det X\big)' &= \begin{vmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \dots & x_{nn} \end{vmatrix}' = \sum_{i=1}^{n} \begin{vmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(i-1)1} & x_{(i-1)2} & x_{(i-1)3} & \dots & x_{(i-1)n} \\ x'_{i1} & x'_{i2} & x'_{i3} & \dots & x'_{in} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \dots & x_{nn} \end{vmatrix} \\
&= \sum_{i=1}^{n} \begin{vmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{(i-1)1} & x_{(i-1)2} & \dots & x_{(i-1)n} \\ \sum\limits_{j=1}^{n} a_{ij}x_{j1} & \sum\limits_{j=1}^{n} a_{ij}x_{j2} & \dots & \sum\limits_{j=1}^{n} a_{ij}x_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{vmatrix} \\
&= \sum_{i=1}^{n} \begin{vmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{(i-1)1} & x_{(i-1)2} & \dots & x_{(i-1)n} \\ a_{ii}x_{i1} & a_{ii}x_{i2} & \dots & a_{ii}x_{in} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{vmatrix} = \sum_{i=1}^{n} a_{ii}\big(\det X\big).
\end{aligned}$$

Hence

$$\det X = \det X_0 e^{\int_{t_0}^{t} trA(\tau)\mathrm{d}ø}.$$

Example 2. *For (2), let $\mathbb{T}=\mathbb{Z}h$, $h>0$, the solution of (2) can be given as:*

$$X(t) = \prod_{k=\frac{t-t_0}{h}-1}^{0} \big[I + hA(t_0+kh)\big] X_0.$$

Furthermore, by Theorem 3,

$$\det X(t) = \prod_{k=\frac{t-t_0}{h}-1}^{0} det\big[I+hA(t_0+kh)\big]\det X_0 = e_{\sum\limits_{i=1}^{n}\det A_i}(t,t_0)\det X_0, \quad for \ \ t\in\mathbb{T},$$

where

$$A_i = \begin{bmatrix} 1+ha_{11} & ha_{12} & \dots & ha_{1(i-1)} & ha_{1i} & 0 & \dots & 0 \\ ha_{21} & 1+ha_{22} & \dots & ha_{2(i-1)} & ha_{2i} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ ha_{(i-1)1} & ha_{(i-1)2} & \dots & 1+ha_{(i-1)(i-1)} & ha_{(i-1)i} & 0 & 0 & 0 \\ a_{i1} & a_{i2} & \dots & a_{i(i-1)} & a_{ii} & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

In fact, let $\mathbb{T} = \mathbb{Z}h$, $h > 0$, by $X^{\sigma}(t) - X(t) = \mu(t)A(t)X(t)$ and $\mu(t) = h$, we have $X(t+h) - X(t) = hA(t)X(t)$. Hence $X(x+h) = \big(I + hA(t)\big)X(t)$. Therefore

$$X(t) = \prod_{k=\frac{t-t_0}{h}-1}^{0} \big[I + hA(t_0 + kh)\big] X_0.$$

Example 3. *For (2), let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$, the solution of (2) can be given as:*

$$X(t) = \prod_{k=\frac{\ln t - \ln t_0}{\ln q}-1}^{0} \big[I + (q-1)q^k t_0 A(q^k t_0)\big] X_0.$$

Furthermore, by Theorem 3,

$$\begin{aligned} \det X(t) &= \prod_{k=\frac{\ln t - \ln t_0}{\ln q}-1}^{0} det\big[I + (q-1)q^k t_0 A(q^k t_0)\big] det X_0 \\ &= e_{\sum\limits_{i=1}^{n} \det A_i}(t, t_0) \det X_0, \ \ for \ \ t \in \mathbb{T}, \end{aligned}$$

where

$$A_i = \begin{bmatrix} 1+(q-1)ta_{11} & (q-1)ta_{12} & \dots & (q-1)ta_{1(i-1)} & (q-1)ta_{1i} & 0 & \dots & 0 \\ (q-1)ta_{21} & 1+(q-1)ta_{22} & \dots & (q-1)ta_{2(i-1)} & (q-1)ta_{2i} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (q-1)ta_{(i-1)1} & (q-1)ta_{(i-1)2} & \dots & 1+(q-1)ta_{(i-1)(i-1)} & (q-1)ta_{(i-1)i} & 0 & 0 & 0 \\ a_{i1} & a_{i2} & \dots & a_{i(i-1)} & a_{ii} & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

In fact, let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$, by $X^{\sigma}(t) - X(t) = \mu(t)A(t)X(t)$ and $\mu(t) = (q-1)t$, we have $X(qt) - X(t) = (q-1)tA(t)X(t)$. Hence $X(qx) = \big(I + (q-1)tA(t)\big)X(t)$. Therefore

$$X(t) = \prod_{k=\frac{\ln t - \ln t_0}{\ln q}-1}^{0} \big[I + (q-1)q^k t_0 A(q^k t_0)\big] X_0.$$

3. Liouville'S Formula for ∇-Dynamic Equations and Some Lemmas

In the following, we will obtain Liouville's formula for ∇-dynamic equations and some lemmas which will be used to discuss α-diamond dynamic equations are established.

Lemma 3. *By Definition 2, the matrix function $e_A(t, t_0)$ is nabla differentiable at t with*

$$e_A^{\nabla}(t, t_0) = A\big(\rho(t)\big)\big(I + \nu(t)A\big(\rho(t)\big)\big)^{-1} e_A(t, t_0)$$

Proof. For $e_A^{\Delta}(t, t_0) = \frac{e_A(\sigma(t), t_0) - e_A(t, t_0)}{\mu(t)} = A(t)e_A(t, t_0)$, we can obtain

$$e_A^{\nabla}(t, t_0) = \frac{e_A(t, t_0) - e_A(\rho(t), t_0)}{\nu(t)} = A(\rho(t))e_A(\rho(t), t_0).$$

On the other hand, by $f(t) - f(\rho(t)) = f^{\nabla}(t)\nu(t)$, we can obtain

$$e_A(t, t_0) - e_A(\rho(t), t_0) = \nu(t)A\big(\rho(t)\big)e_A(\rho(t), t_0),$$

that is $e_A(t, t_0) = \big(I + \nu(t)A\big(\rho(t)\big)\big)e_A(\rho(t), t_0)$. Hence

$$e_A^{\nabla}(t, t_0) = A\big(\rho(t)\big)\big(I + \nu(t)A\big(\rho(t)\big)\big)^{-1}e_A(t, t_0).$$

The proof is complete. □

Next, we consider the dynamic equations by nabla derivative.

Lemma 4. *Let A be an upper triangular $n \times n$-matrix-valued function, then A is regressive iff each diagonal element of A is regressive.*

Proof. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

Hence

$$\det[I - \nu A] = \begin{vmatrix} 1 - \nu a_{11} & -\nu a_{12} & \dots & -\nu a_{1n} \\ 0 & 1 - \nu a_{22} & \dots & -\nu a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 - \nu a_{nn} \end{vmatrix} = \prod_{i=1}^{n}\big(1 - \nu a_{ii}\big).$$

Therefore A is regressive iff each diagonal element of A is regressive. □

Remark 5. *Let $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ be an upper triangular matrix, we can obtain*

$$\det A = \prod_{i=1}^{n} a_{ii}, \ \ trA = \sum_{i=1}^{n} a_{ii}, \ \ \det\big(I - \nu A\big) = \prod_{i=1}^{n}\big(1 - \nu a_{ii}\big).$$

Hence for $n > 2$,

$$1 - \nu\big(trA - \nu detA\big) = 1 - \nu\Big(\sum_{i=1}^{n} a_{ii} - \nu\prod_{i=1}^{n} a_{ii}\Big) \neq \prod_{i=1}^{n}\big(1 - \nu a_{ii}\big),$$

which implies that $I + \nu A$ is invertible does not be equivalent to $trA + \nu \det A$ is regressive.

It is easy to check the following ∇-derivative formula of determinant function.

Lemma 5. *Let $A : \mathbb{T} \to \mathbb{R}^{n\times n}$ be the following function matrix:*

$$A(t) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1i} & a_{1(i+1)} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \dots & a_{(i-1)i} & a_{(i-1)(i+1)} & \dots & a_{(i-1)n} \\ a_{i1} & a_{i2} & \dots & a_{ii} & a_{i(i+1)} & \dots & a_{in} \\ a_{(i+1)1} & a_{(i+1)2} & \dots & a_{(i+1)i} & a_{(i+1)(i+1)} & \dots & a_{(i+1)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{ni} & a_{n(i+1)} & \dots & a_{nn} \end{vmatrix},$$

then

$$A^{\nabla}(t) = \sum_{i=1}^{n} \begin{vmatrix} a^{\rho}_{11} & a^{\rho}_{12} & \dots & a^{\rho}_{1i} & a^{\rho}_{1(i+1)} & \dots & a^{\rho}_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a^{\rho}_{(i-1)1} & a^{\rho}_{(i-1)2} & \dots & a^{\rho}_{(i-1)i} & a^{\rho}_{(i-1)(i+1)} & \dots & a^{\rho}_{(i-1)n} \\ a^{\nabla}_{i1} & a^{\nabla}_{i2} & \dots & a^{\nabla}_{ii} & a^{\nabla}_{i(i+1)} & \dots & a^{\nabla}_{in} \\ a_{(i+1)1} & a_{(i+1)2} & \dots & a_{(i+1)i} & a_{(i+1)(i+1)} & \dots & a_{(i+1)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{ni} & a_{n(i+1)} & \dots & a_{nn} \end{vmatrix}.$$

Theorem 5 (Liouville's formula)**.** *Let $A \in \mathcal{R}$ be an upper triangular $n \times n$-matrix-valued function and assume that X is a solution of $X^{\nabla} = A(t)X$. Then X satisfies Liouville's formula*

$$\det X(t) = e_{\sum\limits_{i=1}^{n} \prod\limits_{j=1}^{i-1} \left(1 - \nu a_{jj}\right) a_{ii}}(t, t_0) \det X_0, \quad for \;\; t \in \mathbb{T}.$$

Proof. For the matrix A, by Lemma 4, we can obtain A is regressive iff each diagonal element of A is regressive. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} \quad X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ 0 & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x_{nn} \end{bmatrix}$$

$$
\begin{aligned}
\big(\det X(t)\big)^{\nabla} &= \begin{vmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ 0 & x_{22} & x_{23} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_{nn} \end{vmatrix}^{\nabla} \\
&= \begin{vmatrix} x_{11}^{\nabla} & x_{12}^{\nabla} & x_{13}^{\nabla} & \dots & x_{1n}^{\nabla} \\ 0 & x_{22} & x_{23} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x_{nn} \end{vmatrix} + \sum_{i=2}^{n-1} \begin{vmatrix} x_{11}^{\rho} & x_{12}^{\rho} & \dots & x_{1i}^{\rho} & x_{1(i+1)}^{\rho} & \dots & x_{1n}^{\rho} \\ 0 & x_{22}^{\rho} & \dots & x_{2i}^{\rho} & x_{2(i+1)}^{\rho} & \dots & x_{2n}^{\rho} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & x_{ii}^{\nabla} & x_{i(i+1)}^{\nabla} & \dots & x_{in}^{\nabla} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & x_{nn} \end{vmatrix} \\
&= \sum_{i=1}^{n} \begin{vmatrix} x_{11}^{\rho} & x_{12}^{\rho} & \dots & x_{1i}^{\rho} & x_{1(i+1)}^{\rho} & \dots & x_{1n}^{\rho} \\ 0 & x_{22}^{\rho} & \dots & x_{2i}^{\rho} & x_{2(i+1)}^{\rho} & \dots & x_{2n}^{\rho} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & \dots & \dots & x_{ii}^{\nabla} & x_{i(i+1)}^{\nabla} & \dots & x_{in}^{\nabla} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & x_{nn} \end{vmatrix} \\
&= \sum_{i=1}^{n} \prod_{j=1}^{i-1} x_{jj}^{\rho} x_{ii}^{\nabla} \prod_{k=i+1}^{n} x_{kk} = \left[\sum_{i=1}^{n} \prod_{j=1}^{i-1} (1 - \nu a_{jj}) a_{ii} \right] \det X.
\end{aligned}
$$

This completes the proof. □

Remark 6. *Let A be a 2×2-matrix-valued function. Then A is regressive iff the scalar-valued function $trA - \nu \det A$ is regressive (where trA denotes the trace of the matrix A, i.e., the sum of diagonal elements of A). In fact, let*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

then we can obtain

$$
\begin{aligned}
\det(I - \nu A) &= \det \begin{bmatrix} 1 - \nu a_{11} & -\nu a_{12} \\ -\nu a_{21} & 1 - \nu a_{22} \end{bmatrix} \\
&= (1 - \nu a_{11})(1 - \nu a_{22}) - \nu a_{12} \nu a_{21} \\
&= 1 - \nu(a_{11} + a_{22}) + \nu^2 (a_{11} a_{22} - a_{12} a_{21}) \\
&= 1 - \nu(trA - \nu \det A).
\end{aligned}
$$

Hence A is regressive iff the scalar-valued function $trA - \nu \det A$ is regressive.

Remark 7. *Assume that for $n = k$, we can obtain similar characterizations of Remark 6. For $n = k + 1$, let*

$$A_{k+1} = \begin{bmatrix} A_k & 0 \\ 0 & a_0 \end{bmatrix},$$

where A_{k+1} is $(k+1)\times(k+1)$-matrix-valued function, A_k is $k\times k$-matrix-valued function. Then

$$\begin{aligned}\det(I-\nu A_{k+1}) &= \det\begin{bmatrix} I_k-\nu A_k & 0 \\ 0 & 1-\nu a_0\end{bmatrix} \\ &= (1-\nu a_0)\det(I-\nu A_k), \\ 1-\nu(trA_{k+1}-\nu\det A_{k+1}) &= 1-\nu[trA_k+a_0-\nu a_0\det A_k] \\ &= 1-\nu trA_k-\nu a_0+\nu a_0\nu\det A_k.\end{aligned}$$

Hence there exists A_{k+1} such that A_{k+1} is regressive and so is $trA_{k+1}-\nu\det A_{k+1}$.

Theorem 6 (Liouville's formula)**.** *Let $A\in\mathcal{R}$ be $n\times n$-matrix-valued matrix function and assume that X is a solution of $X^{\nabla}=A(t)X$. Then X satisfies Liouville's formula*

$$\det X(t)=e_{\sum\limits_{i=1}^{n}\det\tilde{A}_i}(t,t_0)\det X_0,\quad for\ \ t\in\mathbb{T},$$

where

$$\tilde{A}_i=\begin{bmatrix} 1-\nu a_{11} & -\nu a_{12} & \dots & -\nu a_{1(i-1)} & -\nu a_{1i} & 0 & \dots & 0 \\ -\nu a_{21} & 1-\nu a_{22} & \dots & -\nu a_{2(i-1)} & -\nu a_{2i} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\nu a_{(i-1)1} & -\nu a_{(i-1)2} & \dots & 1-\nu a_{(i-1)(i-1)} & -\nu a_{(i-1)i} & 0 & 0 & 0 \\ a_{i1} & a_{i2} & \dots & a_{i(i-1)} & a_{ii} & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

Proof. For $n=2$, by Remark 4, A is regressive implies $trA-\nu\det A$ is regressive. Let

$$A(t)=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}, X(t)=\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22}\end{bmatrix}.$$

Then

$$\begin{aligned}\big(\det X(t)\big)^{\nabla} &= \begin{vmatrix} x_{11}^{\nabla} & x_{12}^{\nabla} \\ x_{21} & x_{22}\end{vmatrix}+\begin{vmatrix} x_{11}^{\rho} & x_{12}^{\rho} \\ x_{21}^{\nabla} & x_{22}^{\nabla}\end{vmatrix} \\ &= \begin{vmatrix} a_{11}x_{11}+a_{12}x_{21} & a_{11}x_{12}+a_{12}x_{22} \\ x_{21} & x_{22}\end{vmatrix}+\begin{vmatrix} x_{11}+\mu x_{11}^{\nabla} & x_{12}+\mu x_{12}^{\nabla} \\ x_{21}^{\nabla} & x_{22}^{\nabla}\end{vmatrix} \\ &= a_{11}\det X(t)+\begin{vmatrix} x_{11}+(a_{11}x_{11}+a_{12}x_{21})(-\nu) & x_{12}+(a_{11}x_{12}+a_{12}x_{22})(-\nu) \\ a_{21}x_{11}+a_{22}x_{21} & a_{21}x_{12}+a_{22}x_{22}\end{vmatrix} \\ &= a_{11}\det X(t)+\begin{vmatrix} 1-\nu a_{11} & a_{12}(-\nu) \\ a_{21} & a_{22}\end{vmatrix}\det X \\ &= \big(trA-\nu\det A\big)\det X \\ &= \left\{\begin{vmatrix} a_{11} & 0 \\ 0 & 1\end{vmatrix}+\begin{vmatrix} 1-\nu a_{11} & a_{12}(-\nu) \\ a_{21} & a_{22}\end{vmatrix}\right\}\det X.\end{aligned}$$

For $n \geq 3$, let

$$A(t) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}, X(t) = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \dots & x_{nn} \end{bmatrix}.$$

Hence by Lemma 5, we have

$$\big(X(t)\big)^{\nabla} = \begin{vmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \dots & x_{nn} \end{vmatrix}^{\nabla} = \sum_{i=1}^{n} \begin{vmatrix} x_{11}^{\rho} & x_{12}^{\rho} & x_{13}^{\rho} & \dots & x_{1n}^{\rho} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(i-1)1}^{\rho} & x_{(i-1)2}^{\rho} & x_{(i-1)3}^{\rho} & \dots & x_{(i-1)n}^{\rho} \\ x_{i1}^{\nabla} & x_{i2}^{\nabla} & x_{i3}^{\nabla} & \dots & x_{in}^{\nabla} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \dots & x_{nn} \end{vmatrix}$$

$$= \sum_{i=1}^{n} \begin{vmatrix} x_{11} - \nu x_{11}^{\nabla} & x_{12} - \nu x_{12}^{\nabla} & x_{13} - \nu x_{13}^{\nabla} & \dots & x_{1n} - \nu x_{1n}^{\nabla} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(i-1)1} - \nu x_{(i-1)1}^{\nabla} & x_{(i-1)2} - \nu x_{(i-1)2}^{\nabla} & x_{(i-1)3} - \nu x_{(i-1)3}^{\nabla} & \dots & x_{(i-1)n} - \nu x_{(i-1)n}^{\nabla} \\ x_{i1}^{\nabla} & x_{i2}^{\nabla} & x_{i3}^{\nabla} & \dots & x_{in}^{\nabla} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & \dots & x_{nn} \end{vmatrix}$$

$$= \sum_{i=1}^{n} \begin{vmatrix} x_{11} - \nu \sum\limits_{j=1}^{n} a_{1j} x_{j1} & x_{12} - \nu \sum\limits_{j=1}^{n} a_{1j} x_{j2} & \dots & x_{1n} - \nu \sum\limits_{j=1}^{n} a_{1j} x_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ x_{(i-1)1} - \nu \sum\limits_{j=1}^{n} a_{(i-1)j} x_{j1} & x_{(i-1)2} - \nu \sum\limits_{j=1}^{n} a_{(i-1)j} x_{j2} & \dots & x_{(i-1)n} - \nu \sum\limits_{j=1}^{n} a_{(i-1)j} x_{jn} \\ \sum\limits_{j=1}^{n} a_{ij} x_{j1} & \sum\limits_{j=1}^{n} a_{ij} x_{j2} & \dots & \sum\limits_{j=1}^{n} a_{ij} x_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{vmatrix}$$

$$= \sum_{i=1}^{n} \begin{vmatrix} x_{11} - \nu \sum\limits_{j=1}^{i} a_{1j} x_{j1} & x_{12} - \nu \sum\limits_{j=1}^{i} a_{1j} x_{j2} & \dots & x_{1n} - \nu \sum\limits_{j=1}^{i} a_{1j} x_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ x_{(i-1)1} - \nu \sum\limits_{j=1}^{i} a_{(i-1)j} x_{j1} & x_{(i-1)2} - \nu \sum\limits_{j=1}^{i} a_{(i-1)j} x_{j2} & \dots & x_{(i-1)n} - \nu \sum\limits_{j=1}^{i} a_{(i-1)j} x_{jn} \\ \sum\limits_{j=1}^{i} a_{ij} x_{j1} & \sum\limits_{j=1}^{i} a_{ij} x_{j2} & \dots & \sum\limits_{j=1}^{i} a_{ij} x_{jn} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{vmatrix}$$

$$= \sum_{i=1}^{n} \det(\tilde{A}_i X) = \sum_{i=1}^{n} \det \tilde{A}_i \det X.$$

Then

$$X(t) = \left(e_{\sum\limits_{i=1}^{n} \det \tilde{A}_i}(t, t_0) \right) \det X_0.$$

This completes the proof. $\square$

Theorem 7. *Let A be a $n \times n$-matrix-valued function. Then A is regressive iff the scalar-valued function $\sum_{i=1}^{n} \det \tilde{A}_i$ is regressive, where $\tilde{A}_i$ is defined by Theorem 6.*

Proof. By Theorem 6, for $n = 2$ we can obtain

$$\det(I - \nu A) = 1 - \nu \left\{ \begin{vmatrix} a_{11} & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 - \nu a_{11} & a_{12}(-\nu) \\ a_{21} & a_{22} \end{vmatrix} \right\}.$$

Next, let

$$A(t) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} & & A_{(0)} & & a_{1n} \\ & & & & a_{2n} \\ & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix},$$

where $A_{(0)}$ is a $(n-1) \times (n-1)$-valued-matrix function. Assume that $(n-1) \times (n-1)$-matrix-value function is regressive iff the scalar-valued function $\sum_{i=1}^{n-1} \det \tilde{A}_i$ is regressive. That is

$$\det(I - \nu A_{(0)}) = 1 - \nu \sum_{i=1}^{n-1} \det \tilde{A}_i.$$

Hence

$$\begin{aligned} & 1 - \nu \left(\sum_{i=1}^{n-1} \det A_i + \det A_n \right) = 1 - \nu \sum_{i=1}^{n-1} \det A_i - \nu \det A_n = \det\left(I - \nu A_{(0)}\right) - \nu \det A_n \\ = {} & \det\left(I - \nu A_{(0)}\right) - \nu \begin{vmatrix} & & I - \nu A_{(0)} & & -\nu a_{1n} \\ & & & & -\nu a_{2n} \\ & & & & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} \\ = {} & \begin{vmatrix} & & I - \nu A_{(0)} & & -\nu a_{1n} \\ & & & & -\nu a_{2n} \\ & & & & \dots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix} + \begin{vmatrix} & & I - \nu A_{(0)} & & -\nu a_{1n} \\ & & & & -\nu a_{2n} \\ & & & & \dots \\ -\nu a_{n1} & -\nu a_{n2} & -\nu a_{n3} & \dots & -\nu a_{nn} \end{vmatrix} \\ = {} & \begin{vmatrix} & & I - \nu A_{(0)} & & -\nu a_{1n} \\ & & & & \mu a_{2n} \\ & & & & \dots \\ \mu a_{n1} & -\nu a_{n2} & -\nu a_{n3} & \dots & 1 - \nu a_{nn} \end{vmatrix} = \det(I - \nu A). \end{aligned}$$

Therefore A is regressive iff the scalar-valued function $\sum_{i=1}^{n} \det \tilde{A}_i$ is regressive. This completes the proof. □

Definition 3 ([19])**.** *Define $\hat{e}_A(t, t_0)$ by the unique matrix solution of the initial value problem:*

$$X^{\nabla}(t) = A(t)X(t), \ \ X_0 = I. \tag{4}$$

Lemma 6. *By Definition 3, the matrix function $\hat{e}_A(t, t_0)$ is delta differentiable at t with*

$$\hat{e}_A^{\Delta}(t, t_0) = A\big(\sigma(t)\big)\big(I - \mu(t)A(\sigma(t))\big)^{-1}\hat{e}_A(t, t_0).$$

Proof. For $\hat{e}_A^{\nabla}(t,t_0) = \frac{\hat{e}_A(t,t_0) - \hat{e}_A(\rho(t),t_0)}{\nu(t)} = A(t)e_A(t,t_0)$, we can obtain

$$\hat{e}_A^{\Delta}(t,t_0) = \frac{\hat{e}_A(\sigma(t),t_0) - \hat{e}_A(t,t_0)}{\mu(t)} = A\big(\sigma(t)\big)\hat{e}_A(\sigma(t),t_0).$$

On the other hand, by $f(\sigma(t)) - f(t) = f^{\Delta}(t)\mu(t)$, we can obtain

$$\hat{e}_A(\sigma(t),t_0) - \hat{e}_A(t,t_0) = \mu(t)A\big(\sigma(t)\big)\hat{e}_A(\sigma(t),t_0),$$

that is $\hat{e}_A(t,t_0) = \big(I - \mu(t)A\big(\sigma(t)\big)\big)\hat{e}_A(\sigma(t),t_0)$. Hence

$$\hat{e}_A^{\Delta}(t,t_0) = A\big(\sigma(t)\big)\big(I - \mu(t)A\big(\sigma(t)\big)\big)^{-1}\hat{e}_A(t,t_0).$$

The proof is complete. □

Example 4. *For (4), let $\mathbb{T} = \mathbb{R}$, Liouville's formula can be given as:*

$$detX = detX_0 e^{\int_{t_0}^{t} trA(\tau)\mathrm{d}\o}, \quad trA(\tau) = \sum_{i=1}^{n} a_{ii}(\tau).$$

In fact, let $\mathbb{T} = \mathbb{R}$, we can obtain $\nu(t) = 0$ for any $t \in \mathbb{T}$, hence $X^{\nabla}(t) = X'$. The result is obvious.

Example 5. *For (4), let $\mathbb{T} = \mathbb{Z}h$, $h > 0$, the solution of (4) can be given as:*

$$X(t) = \prod_{k=\frac{t-t_0}{h}}^{1} \big[I - hA(t_0 + kh)\big]^{-1} X_0.$$

Furthermore, by Theorem 6,

$$\begin{aligned} \det X(t) &= \prod_{k=\frac{t-t_0}{h}}^{1} det\big[I - hA(t_0+kh)\big]^{-1} detX_0 \\ &= \tilde{e}_{\sum\limits_{i=1}^{n} \det \tilde{A}_i}(t,t_0)\det X_0, \quad for \ \ t \in \mathbb{T}, \end{aligned}$$

where

$$\tilde{A}_i = \begin{bmatrix} 1-ha_{11} & -ha_{12} & \dots & -ha_{1(i-1)} & -ha_{1i} & 0 & \dots & 0 \\ -ha_{21} & 1-ha_{22} & \dots & -ha_{2(i-1)} & -ha_{2i} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -ha_{(i-1)1} & -ha_{(i-1)2} & \dots & 1-ha_{(i-1)(i-1)} & -ha_{(i-1)i} & 0 & 0 & 0 \\ a_{i1} & a_{i2} & \dots & a_{i(i-1)} & a_{ii} & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

In fact, let $\mathbb{T} = \mathbb{Z}h$, $h > 0$, by $X(t) - X^{\rho}(t) = \nu(t)X^{\nabla}(t) = \nu(t)A(t)X(t)$ and $\nu(t) = h$, we have $X(t+h) - X(t) = hA(t+h)X(t+h)$. Hence $X(x+h) = \big(I - hA(t+h)\big)^{-1}X(t)$. Therefore

$$X(t) = \prod_{k=\frac{t-t_0}{h}}^{1} \big[I - hA(t_0 + kh)\big]^{-1} X_0.$$

Example 6. *For (4), let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$, the solution of (4) can be given as:*

$$X(t) = \prod_{k=\frac{\ln t - \ln t_0}{\ln q}}^{1} \left[I - (q-1)q^k t_0 A(q^k t_0)\right]^{-1} X_0.$$

Furthermore, by Theorem 6,

$$\begin{aligned} \det X(t) &= \prod_{k=\frac{\ln t - \ln t_0}{\ln q}}^{1} det\left[I - (q-1)q^k t_0 A(q^k t_0)\right]^{-1} det X_0 \\ &= e_{\sum_{i=1}^{n} \det \tilde{A}_i}(t, t_0) \det X_0, \ \ for \ \ t \in \mathbb{T}, \end{aligned}$$

where

$$A_i = \begin{bmatrix} 1-(1-\frac{1}{q})ta_{11} & -(1-\frac{1}{q})ta_{12} & \dots & -(1-\frac{1}{q})ta_{1(i-1)} & -(1-\frac{1}{q})ta_{1i} & 0 & \dots & 0 \\ -(1-\frac{1}{q})ta_{21} & 1-(1-\frac{1}{q})ta_{22} & \dots & -(1-\frac{1}{q})ta_{2(i-1)} & -(1-\frac{1}{q})ta_{2i} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -(1-\frac{1}{q})ta_{(i-1)1} & -(1-\frac{1}{q})ta_{(i-1)2} & \dots & 1-(1-\frac{1}{q})ta_{(i-1)(i-1)} & -(1-\frac{1}{q})ta_{(i-1)i} & 0 & 0 & 0 \\ a_{i1} & a_{i2} & \dots & a_{i(i-1)} & a_{ii} & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

In fact, let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$, by $X(t) - X^{\rho}(t) = \nu(t)A(t)X(t)$ and $\nu(t) = (1-\frac{1}{q})t$, we have $X(qt) - X(t) = (q-1)qtA(qt)X(qt)$. Hence $X(qt) = \big(I - (q-1)qtA(qt)\big)^{-1}X(t)$. Therefore

$$X(t) = \prod_{k=\frac{\ln t - \ln t_0}{\ln q}}^{1} \left[I - (q-1)q^k t_0 A(q^k t_0)\right]^{-1} X_0.$$

4. Liouville'S Formula of Diamond-α Dynamic Equations

In the sequel, we will introduce the α-matrix exponential function and obtain Liouville's formula of diamond-α dynamic equations. Several examples are provided on various time scales.

Definition 4 ([13])**.** *Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ sense. For $t \in \mathbb{T}$ we define the diamond-α dynamic derivative $f^{\diamondsuit_\alpha}(t)$ by*

$$f^{\diamondsuit_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha) f^{\nabla}(t), \ \ 0 \le \alpha \le 1.$$

Thus f is diamond-α differentiable if and only if f is Δ and ∇ differentiable.

Definition 5 ([14])**.** *Let $f(\cdot) : \mathbb{T} \to \mathbb{R}$, we define the generalized diamond exponential function by*

$$\ddot{e}_f(t, t_0) = \alpha e_f(t, t_0) + (1-\alpha)\hat{e}_f(t, t_0), \ \ 0 \le \alpha \le 1.$$

Theorem 8. *The generalized diamond exponential function is the solution of the diamond α-dynamic equation as follows:*

$$x^{\diamondsuit_\alpha} = \left(\alpha f(t) + (1-\alpha)\frac{f^{\rho}(t)}{1+\nu(t) f^{\rho}(t)}\right)\alpha e_f(t, t_0) + \left(\alpha \frac{f^{\sigma}(t)}{1-\mu f^{\sigma}(t)} + (1-\alpha) f(t)\right)(1-\alpha)\hat{e}_f(t, t_0), \quad (5)$$

where $0 \le \alpha \le 1$ with the initial condition $x(t_0) = 1$. When $\alpha = 1$, the dynamic equation is Δ-dynamic equation, and when $\alpha = 0$, the dynamic equation is ∇-dynamic equation.

Proof. By Definitions 4 and 5, we can obtain

$$\begin{aligned}
(\ddot{e}_f(t,t_0))^{\diamondsuit_\alpha}(t) &= \alpha\Big(\alpha e_f^{\Delta}(t,t_0) + (1-\alpha)\hat{e}_f^{\Delta}(t,t_0)\Big) + (1-\alpha)\Big(\alpha e_f^{\nabla}(t,t_0) + (1-\alpha)\hat{e}_f^{\nabla}(t,t_0)\Big) \\
&= \alpha\left(\alpha f(t)e_f(t,t_0) + (1-\alpha)\frac{f^{\rho}(t)}{1+\nu(t)f^{\rho}(t)}\hat{e}_f(t,t_0)\right) \\
&\quad +(1-\alpha)\left(\alpha\frac{f^{\sigma}(t)}{1-\mu(t)f^{\sigma}(t)}e_f(t,t_0) + (1-\alpha)f(t)\hat{e}_f(t,t_0)\right) \\
&= \left(\alpha f(t) + (1-\alpha)\frac{f^{\rho}(t)}{1+\nu(t)f^{\rho}(t)}\right)\alpha e_f(t,t_0) \\
&\quad +\left(\alpha\frac{f^{\sigma}(t)}{1-\mu f^{\sigma}(t)} + (1-\alpha)f(t)\right)(1-\alpha)\hat{e}_f(t,t_0).
\end{aligned}$$

Hence the generalized diamond exponential function $\ddot{e}_f(t,t_0)$ is the solution of (5). Further when $\alpha = 1$, we obtain the dynamic Equation (5) as follows:

$$x^{\diamondsuit_1}(t) = x^{\Delta}(t) = f(t)e_f(t,t_0), \tag{6}$$

for the solution of (6) is $x = e_f(t,t_0)$, hence (6) can be denoted by $x^{\Delta}(t) = f(t)x(t)$. Similarly, we can also obtain the case for $\alpha = 0$. The proof is complete. □

Theorem 9. *Let $f(\cdot) : \mathbb{T} \to \mathbb{T}$, then $f(t)$ is a solution of the following equation*

$$f(t) = e^{\int_{t_0}^{t} \frac{\log\left(1-\mu(\tau)f^{\sigma}(\tau)\right)}{\mu(\tau)}\Delta\tau}, \tag{7}$$

iff (5) can be reduced to

$$x^{\diamondsuit_\alpha}(t) = f(t)x(t), \quad 0 \le \alpha \le 1. \tag{8}$$

Proof. For $f(t) = e^{\int_{t_0}^{t} \frac{1-\mu(\tau)f^{\sigma}(\tau)}{\mu(\tau)}\Delta\tau}$, we can obtain

$$\begin{aligned}
f^{\Delta}(t) &= \frac{e^{\int_{t_0}^{\sigma(t)} \frac{\log\left(1-\mu(\tau)f^{\sigma}(\tau)\right)}{\mu(\tau)}\Delta\tau} - e^{\int_{t_0}^{t} \frac{\log\left(1-\mu(\tau)f^{\sigma}(\tau)\right)}{\mu(\tau)}\Delta\tau}}{\mu(t)} \\
&= \frac{e^{\int_{t}^{\sigma(t)} \frac{\log\left(1-\mu(\tau)f^{\sigma}(\tau)\right)}{\mu(\tau)}\Delta\tau} - 1}{\mu(t)} e^{\int_{t_0}^{t} \frac{\log\left(1-\mu(\tau)f^{\sigma}(\tau)\right)}{\mu(\tau)}\Delta\tau} \\
&= \frac{1-\mu(t)f^{\sigma}(t) - 1}{\mu(t)} e^{\int_{t_0}^{t} \frac{\log\left(1-\mu(\tau)f^{\sigma}(\tau)\right)}{\mu(\tau)}\Delta\tau} = -f^{\sigma}(t)f(t),
\end{aligned} \tag{9}$$

$$
\begin{aligned}
f^{\nabla}(t) &= \frac{e^{\int_{t_0}^{t}\frac{\log\left(1-\mu(\tau)f^{\sigma}(\tau)\right)}{\mu(\tau)}\Delta\tau} - e^{\int_{t_0}^{\rho(t)}\frac{\log\left(1-\mu(\tau)f^{\sigma}(\tau)\right)}{\mu(\tau)}\Delta\tau}}{\nu(t)} \\
&= \frac{e^{\int_{\rho(t)}^{t}\frac{\log\left(1-\mu(\tau)f^{\sigma}(\tau)\right)}{\mu(\tau)}\Delta\tau} - 1}{\nu(t)} e^{\int_{t_0}^{\rho(t)}\frac{\log\left(1-\mu(\tau)f^{\sigma}(\tau)\right)}{\mu(\tau)}\Delta\tau} \\
&= \frac{1-\nu(t)f(t)-1}{\nu(t)} e^{\int_{t_0}^{\rho(t)}\frac{\log\left(1-\mu(\tau)f^{\sigma}(\tau)\right)}{\mu(\tau)}\Delta\tau} = -f(t)f^{\rho}(t),
\end{aligned} \tag{10}
$$

hence we have the solution of (9) and (10) with the initial value $f(t_0) = 1$ is equivalent to the solution of (7).

On the other hand, by (5), we can obtain

$$
\begin{aligned}
\alpha f(t) + (1-\alpha)\frac{f^{\rho}(t)}{1+\nu(t)f^{\rho}(t)} &= f(t) + (1-\alpha)\frac{f^{\rho}(t) - f(t) - \nu(t)f^{\rho}(t)f(t)}{1+\nu(t)f^{\rho}(t)} \\
&= f(t) + (1-\alpha)\frac{-\nu(t)f^{\nabla}(t) - \nu(t)f^{\rho}(t)f(t)}{1+\nu(t)f^{\rho}(t)} = f(t) \\
\alpha\frac{f^{\sigma}(t)}{1-\mu f^{\sigma}(t)} + (1-\alpha)f(t) &= \alpha\frac{f^{\sigma}(t) - f(t) + \mu(t)f(t)f^{\sigma}(t)}{1-\mu f^{\sigma}(t)} + f(t) \\
&= \alpha\frac{\mu(t)f^{\Delta}(t) + \mu(t)f(t)f^{\sigma}(t)}{1-\mu f^{\sigma}(t)} + f(t) = f(t).
\end{aligned}
$$

Therefore the dynamic Equation (5) can be reduced to

$$x^{\diamondsuit_\alpha} = f(t)x(t).$$

Conversely, if $x^{\diamondsuit_\alpha} = f(t)x(t)$ with $x(t_0) = 1$, then

$$\alpha f(t) + (1-\alpha)\frac{f^{\rho}(t)}{1+\nu(t)f^{\rho}(t)} = \alpha\frac{f^{\sigma}(t)}{1-\mu f^{\sigma}(t)} + (1-\alpha)f(t),$$

that is

$$-\nu(t)(1-\alpha)\frac{f^{\nabla}(t) + f^{\rho}(t)f(t)}{1+\nu(t)f^{\rho}(t)} = \mu(t)\alpha\frac{f^{\Delta}(t) + f(t)f^{\sigma}(t)}{1-\mu f^{\sigma}(t)},$$

for any $0 \le \alpha \le 1$, thus $f^{\nabla}(t) + f^{\rho}(t)f(t) = f^{\Delta}(t) + f(t)f^{\sigma}(t) = 0$, so we have

$$f^{\nabla}(t) = -f^{\rho}(t)f(t), \quad f^{\Delta}(t) = -f(t)f^{\sigma}(t)$$

i.e., $f(t)$ with $f(t_0) = 1$ is a solution of (7). The proof is complete. □

Example 7. *For* (7)*, let* $\mathbb{T} = \mathbb{Z}h$*,* $h > 0$*, by Theorem 9, then the solution of* (7) *can be given by*

$$
f(t) = \begin{cases}
\dfrac{\prod_{k=0}^{\frac{t-t_0}{h}-2}\left(1-hf^{\sigma}(t_0+kh)\right)}{1+h\prod_{k=0}^{\frac{t-t_0}{h}-2}\left(1-hf^{\sigma}(t_0+kh)\right)}, & t > t_0, \\
\prod_{k=0}^{\frac{t-t_0}{h}+1}\left(1-hf(t_0+kh)\right)^{-1}, & t < t_0.
\end{cases}
$$

In fact, by $f(t)=e^{\int_{t_0}^{t}\frac{\log(1-\mu(\tau)f^{\sigma}(\tau))}{\mu(\tau)}\Delta\tau}$, *we can obtain*

$$\begin{aligned} f(t) &= e^{\int_{t_0}^{t}\frac{\log(1-\mu(\tau)f^{\sigma}(\tau))}{\mu(\tau)}\Delta\tau} = e^{\sum\limits_{k=0}^{\frac{t-t_0}{h}-1}\int_{t_0+kh}^{t_0+(k+1)h}\frac{\log(1-\mu(\tau)f^{\sigma}(\tau))}{\mu(\tau)}\Delta\tau} \\ &= \prod_{k=0}^{\frac{t-t_0}{h}-1} e^{\int_{t_0+kh}^{\sigma(t_0+kh)}\frac{\log(1-\mu(\tau)f^{\sigma}(\tau))}{\mu(\tau)}\Delta\tau} = \prod_{k=0}^{\frac{t-t_0}{h}-1}\big(1-hf^{\sigma}(t_0+kh)\big) \\ &= \big(1-hf(t)\big)\prod_{k=0}^{\frac{t-t_0}{h}-2}\big(1-hf^{\sigma}(t_0+kh)\big), \end{aligned}$$

that is

$$f(t)=\frac{\prod\limits_{k=0}^{\frac{t-t_0}{h}-2}\big(1-hf^{\sigma}(t_0+kh)\big)}{1+h\prod\limits_{k=0}^{\frac{t-t_0}{h}-2}\big(1-hf^{\sigma}(t_0+kh)\big)}.$$

For $t\in\mathbb{T}$, $t<t_0$, *we can obtain*

$$\begin{aligned} f(t) &= e^{\int_{t_0}^{t}\frac{\log(1-\mu(\tau)f^{\sigma}(\tau))}{\mu(\tau)}\Delta\tau} = e^{-\sum\limits_{k=0}^{\frac{t-t_0}{h}+1}\int_{t_0+(k-1)h}^{t_0+kh}\frac{\log(1-\mu(\tau)f^{\sigma}(\tau))}{\mu(\tau)}\Delta\tau} \\ &= \prod_{k=0}^{\frac{t-t_0}{h}+1} e^{-\int_{t_0+(k-1)h}^{\sigma(t_0+(k-1)h)}\frac{\log(1-\mu(\tau)f^{\sigma}(\tau))}{\mu(\tau)}\Delta\tau} = \prod_{k=0}^{\frac{t-t_0}{h}+1}\big(1-hf^{\sigma}(t_0+(k-1)h)\big)^{-1} \\ &= \prod_{k=0}^{\frac{t-t_0}{h}+1}\big(1-hf(t_0+kh)\big)^{-1}, \end{aligned}$$

for any $0\le k\le\frac{t-t_0}{h}+1$, $t_0+kh>t$. *Therefore*

$$f(t)=\prod_{k=0}^{\frac{t-t_0}{h}+1}\big(1-hf(t_0+kh)\big)^{-1}.$$

Example 8. *For* (7), *let* $\mathbb{T}=\overline{q^{\mathbb{Z}}}$, $q>1$, *by Theorem 9, then the solution of* (7) *can be given by*

$$f(t)=\begin{cases} \dfrac{\prod\limits_{k=0}^{\frac{\ln t-\ln t_0}{\ln q}-2}\big(1-(q-1)t_0q^kf^{\sigma}(t_0q^k)\big)}{1+(1-\frac{1}{q})t\prod\limits_{k=0}^{\frac{\ln t-\ln t_0}{\ln q}-2}\big(1-(q-1)t_0q^kf^{\sigma}(t_0q^k)\big)}, & t>t_0, \\ \prod\limits_{k=0}^{\frac{\ln t-\ln t_0}{lnq}+1}\big(1-(q-1)t_0q^{k-1}f(t_0q^k)\big)^{-1}, & t<t_0. \end{cases}$$

In fact, by $f(t) = e^{\int_{t_0}^{t} \frac{\log(1-\mu(\tau) f^{\sigma}(\tau))}{\mu(\tau)} \Delta\tau}$*, we can obtain*

$$
\begin{aligned}
f(t) &= e^{\int_{t_0}^{t} \frac{\log(1-\mu(\tau) f^{\sigma}(\tau))}{\mu(\tau)} \Delta\tau} = e^{\sum_{k=0}^{\frac{t-t_0}{h}-1} \int_{t_0 q^k}^{t_0 q^{k+1}} \frac{\log(1-\mu(\tau) f^{\sigma}(\tau))}{\mu(\tau)} \Delta\tau} \\
&= \prod_{k=0}^{\frac{\ln t - \ln t_0}{\ln q}-1} e^{\int_{t_0 q^k}^{\sigma(t_0 q^k)} \frac{\log(1-\mu(\tau) f^{\sigma}(\tau))}{\mu(\tau)} \Delta\tau} = \prod_{k=0}^{\frac{\ln t - \ln t_0}{\ln q}-1} \left(1-(q-1)t_0 q^k f^{\sigma}(t_0 q^k)\right) \\
&= \left(1 - \left(1 - \frac{1}{q}\right) t f(t)\right) \prod_{k=0}^{\frac{\ln t - \ln t_0}{\ln q}-2} \left(1-(q-1)t_0 q^k f^{\sigma}(t_0 q^k)\right),
\end{aligned}
$$

that is

$$
f(t) = \frac{\prod_{k=0}^{\frac{\ln t - \ln t_0}{\ln q}-2} \left(1-(q-1)t_0 q^k f^{\sigma}(t_0 q^k)\right)}{1 + (1-\frac{1}{q}) t \prod_{k=0}^{\frac{\ln t - \ln t_0}{\ln q}-2} \left(1-(q-1)t_0 q^k f^{\sigma}(t_0 q^k)\right)}.
$$

For $t \in \mathbb{T}$, $t < t_0$*, we can obtain*

$$
\begin{aligned}
f(t) &= e^{\int_{t_0}^{t} \frac{\log(1-\mu(\tau) f^{\sigma}(\tau))}{\mu(\tau)} \Delta\tau} = e^{-\sum_{k=0}^{\frac{\ln t - \ln t_0}{\ln q}+1} \int_{t_0 q^{k-1}}^{t_0 q^k} \frac{\log(1-\mu(\tau) f^{\sigma}(\tau))}{\mu(\tau)} \Delta\tau} \\
&= \prod_{k=0}^{\frac{\ln t - \ln t_0}{\ln q}+1} e^{-\int_{t_0 q^{k-1}}^{\sigma(t_0 q^{k-1})} \frac{\log(1-\mu(\tau) f^{\sigma}(\tau))}{\mu(\tau)} \Delta\tau} = \prod_{k=0}^{\frac{\ln t - \ln t_0}{\ln q}+1} \left(1-(q-1)t_0 q^{k-1} f^{\sigma}(t_0 q^{k-1})\right)^{-1} \\
&= \prod_{k=0}^{\frac{\ln t - \ln t_0}{\ln q}+1} \left(1-(q-1)t_0 q^{k-1} f(t_0 q^k)\right)^{-1},
\end{aligned}
$$

for any $0 \le k \le \frac{\ln t - \ln t_0}{\ln q} + 1$, $t_0 q^k > t$*. Therefore*

$$
f(t) = \prod_{k=0}^{\frac{\ln t - \ln t_0}{\ln q}+1} \left(1-(q-1)t_0 q^{k-1} f(t_0 q^k)\right)^{-1}.
$$

Next we consider the diamond α-dynamic equations on time scale as follows:

$$
\begin{aligned}
X^{\diamond_\alpha} &= \left(\alpha A(t) + (1-\alpha) A^{\rho}(t) \left(I + \nu(t) A^{\rho}(t)\right)^{-1}\right) \alpha e_A(t, t_0) \\
&\quad + \left(\alpha A^{\sigma}(t) \left(1 - \mu(t) A^{\sigma}(t)\right)^{-1} + (1-\alpha) A(t)\right)(1-\alpha) \hat{e}_A(t, t_0), \;\; 0 \le \alpha \le 1,
\end{aligned} \tag{11}
$$

where $A : \mathbb{T} \to \mathbb{R}^{n \times n}$, $e_A(t, t_0)$ is solution of (2), $\hat{e}_A(t, t_0)$ is solution of (3) with the initial value

$$
X(t_0) = X_0 \in \mathbb{R}^{n \times n}.
$$

Definition 6. *Let* $\mathbb{T}$ *be a time scale and* $A(\cdot) : \mathbb{T} \to \mathbb{R}^{n \times n}$ *be* Δ *and* ∇*-differentiable on* $\mathbb{T}$*. For* $t \in \mathbb{T}$ *we define the diamond-*α *dynamic derivative* $A^{\diamond_\alpha}(t)$ *as*

$$
A^{\diamond_\alpha}(t) = \alpha A^{\Delta}(t) + (1-\alpha) A^{\nabla}(t), \;\; 0 \le \alpha \le 1.
$$

Obviously, $A(t)$ *is diamond-*α *differentiable if and only if* $A(t)$ *is* Δ *and* ∇ *differentiable.*

Definition 7. *Let $A(\cdot) : \mathbb{T} \to \mathbb{R}^{n\times n}$, we define the generalized diamond matrix exponential function (i.e., α-matrix exponential function) by*

$$\ddot{e}_A(t,t_0) = \alpha e_A(t,t_0) + (1-\alpha)\hat{e}_A(t,t_0), \;\; 0 \le \alpha \le 1.$$

Remark 8. *For the diamond α-dynamic Equation* (11)*, if $\alpha = 1$, then* (11) *turns into* (2)*; if $\alpha = 0$, then* (11) *turns into* (3)*.*

Theorem 10. *The generalized α-matrix exponential function $\ddot{e}_A(t,t_0)$ is the solution of* (11)*.*

Proof. Since

$$\begin{aligned}\big(\ddot{e}_A(t,t_0)\big)^{\diamondsuit_\alpha} \;\; &= \;\; \alpha\big(\alpha e_A^{\Delta}(t,t_0) + (1-\alpha)\hat{e}_A^{\Delta}(t,t_0)\big) \\ &\quad +(1-\alpha)\big(\alpha e_A^{\nabla}(t,t_0) + (1-\alpha)\hat{e}_A^{\nabla}(t,t_0)\big)\end{aligned}$$

by Lemma 3 and Lemma 6, we can obtain

$$\begin{aligned}\big(\ddot{e}_A(t,t_0)\big)^{\diamondsuit_\alpha} \;\; &= \;\; \alpha\big[\alpha A(t)e_A(t,t_0) + (1-\alpha)A\big(\sigma(t)\big)\big(I - \mu(t)A\big(\sigma(t)\big)\big)^{-1}\hat{e}_A(t,t_0)\big] \\ &\quad +(1-\alpha)\big[A\big(\rho(t)\big)\big(I + \nu(t)A\big(\rho(t)\big)\big)^{-1}e_A(t,t_0) + (1-\alpha)A\hat{e}_A(t,t_0)\big] \\ &= \;\; \big[\alpha A(t) + (1-\alpha)A\big(\rho(t)\big)\big(I + \nu(t)A\big(\rho(t)\big)\big)^{-1}\big]\alpha e_A(t,t_0) \\ &\quad +\big[\alpha A\big(\sigma(t)\big)\big(I - \mu(t)A\big(\sigma(t)\big)\big)^{-1} + (1-\alpha)A(t)\big](1-\alpha)\hat{e}_A(t,t_0).\end{aligned}$$

The proof is complete. □

Theorem 11. *Let $A(\cdot) : \mathbb{T} \to \mathbb{R}^{n\times n}$, then the diamond α-dynamic Equation* (11) *can be written as:*

$$X^{\diamondsuit_\alpha} = A(t)X(t), \;\; 0 \le \alpha \le 1,$$

iff $A(t)$ satisfies

$$A(t) = e_{-A^\sigma}(t,t_0). \tag{12}$$

Proof. If $A(t)$ is solution of the equation $A(t) = e_{-A^\sigma}(t, t_0)$, then

$$
\begin{aligned}
& \alpha A(t) + (1-\alpha)A\big(\rho(t)\big)\big(I + \nu(t)A\big(\rho(t)\big)\big)^{-1} \\
= \ & A(t) + (1-\alpha)\big(A\big(\rho(t)\big) - A(t) - \nu(t)A(t)A\big(\rho(t)\big)\big)\big(I + \nu(t)A\big(\rho(t)\big)\big)^{-1} \\
= \ & A(t) - \nu(t)(1-\alpha)\big(A^\nabla(t) + A(t)A\big(\rho(t)\big)\big)\big(I + \nu(t)A\big(\rho(t)\big)\big)^{-1}, \\
& \alpha A\big(\sigma(t)\big)\big(I - \mu(t)A\big(\sigma(t)\big)\big)^{-1} + (1-\alpha)A(t) \\
= \ & \alpha(-\mu(t))^{-1}(-\mu(t))A^\sigma(t)\big(I - \mu(t)A^\sigma(t)\big)^{-1} + (1-\alpha)A(t) \\
= \ & \alpha(-\mu(t))^{-1}\big(-I + I - \mu(t)A^\sigma(t)\big)\big(I - \mu(t)A^\sigma(t)\big)^{-1} + (1-\alpha)A(t) \\
= \ & \alpha(-\mu(t))^{-1}\big[-I\big(I - \mu(t)A^\sigma(t)\big)^{-1} + \big(I - \mu(t)A^\sigma(t)\big)\big(I - \mu(t)A^\sigma(t)\big)^{-1}\big] \\
& +(1-\alpha)A(t) \\
= \ & \alpha(-\mu(t))^{-1}\big[\big(I - \mu(t)A^\sigma(t)\big)^{-1}(-I) + \big(I - \mu(t)A^\sigma(t)\big)^{-1}\big(I - \mu(t)A^\sigma(t)\big)\big] \\
& +(1-\alpha)A(t) \\
= \ & \alpha(-\mu(t))^{-1}\big(I - \mu(t)A^\sigma(t)\big)^{-1}(-\mu(t))A^\sigma(t) + (1-\alpha)A(t) \\
= \ & \alpha\big(I - \mu(t)A^\sigma(t)\big)^{-1}\big(A^\sigma(t) - A(t) + \mu(t)A^\sigma(t)A(t)\big) + A(t) \\
= \ & \alpha\mu(t)\big(I - \mu(t)A^\sigma(t)\big)^{-1}\big(A^\Delta(t) + A^\sigma(t)A(t)\big) + A(t).
\end{aligned}
$$

On the other hand, for $A(t) = e_{-A^\sigma}(t, t_0)$, we can obtain

$$
\begin{aligned}
A^\Delta(t) &= -A^\sigma(t)e_{-A^\sigma}(t, t_0) = -A^\sigma(t)A(t), \\
A^\nabla(t) &= -A^\sigma(\rho(t))e_{-A^\sigma}(\rho(t), t_0) = -A(t)A^\rho(t),
\end{aligned}
$$

hence the dynamic Equation (11) can turn into:

$$X^{\diamond_\alpha} = A(t)X(t), \;\; 0 \le \alpha \le 1.$$

If the diamond α-dynamic Equation (11) can be given as:

$$X^{\diamond_\alpha} = A(t)X(t), \;\; 0 \le \alpha \le 1,$$

then

$$\alpha A(t) + (1-\alpha)A^\rho(t)\big(I + \nu(t)A^\rho(t)\big)^{-1} = \alpha A^\sigma(t)\big(I - \mu(t)A^\sigma(t)\big)^{-1} + (1-\alpha)A(t),$$

hence

$$(1-\alpha)\nu(t)\big(A^\nabla(t) + A(t)A^\rho(t)\big)\big(I + \nu(t)A^\rho(t)\big)^{-1} = \alpha\mu(t)\big(A^\Delta + A^\sigma(t)A(t)\big)\big(I - \mu(t)A^\sigma(t)\big)^{-1},$$

for any $0 \le \alpha \le 1$. Therefore

$$A^\nabla(t) + A(t)A^\rho(t) = A^\Delta + A^\sigma(t)A(t) = 0,$$

thus $A(t) = e_{-A^\sigma}(t, t_0)$. The proof is complete. □

Example 9. *For (12), let $\mathbb{T} = \mathbb{Z}h$, $h > 0$, by Theorem 11, then the solution of (12) can be given by*

$$
A(t) = \begin{cases}
\displaystyle\prod_{k=\frac{t-t_0}{h}-2}^{0} \Big(I - hA^\sigma(t_0 + kh)\Big)\Big[I + h \prod_{k=\frac{t-t_0}{h}-2}^{0} \Big(I - hA^\sigma(t_0 + kh)\Big)\Big], & t > t_0, \\
\displaystyle\prod_{k=\frac{t-t_0}{h}+1}^{0} \Big(I - hA(t_0 + kh)\Big)^{-1}, & t < t_0.
\end{cases}
$$

In fact, by $A(t) = e_{-A^\sigma}(t, t_0)$, we can obtain

$$A(t) = \prod_{k=\frac{t-t_0}{h}-1}^{0} \Big(I - hA^\sigma(t_0+kh)\Big) = \Big(I - hA(t)\Big) \prod_{k=\frac{t-t_0}{h}-2}^{0} \Big(I - hA^\sigma(t_0+kh)\Big),$$

that is

$$A(t) = \prod_{k=\frac{t-t_0}{h}-2}^{0} \Big(I - hA^\sigma(t_0+kh)\Big) \Big[I + h \prod_{k=\frac{t-t_0}{h}-2}^{0} \Big(I - hA^\sigma(t_0+kh)\Big)\Big].$$

For $t \in \mathbb{T}$, $t < t_0$, we can obtain

$$A(t) = \prod_{k=\frac{t-t_0}{h}+1}^{0} \Big(I - hA^\sigma(t_0+(k-1)h)\Big)^{-1} = \prod_{k=\frac{t-t_0}{h}+1}^{0} \Big(1 - hA(t_0+kh)\Big)^{-1},$$

for any $0 \le k \le \frac{t-t_0}{h} + 1$, $t_0 + kh > t$. Therefore

$$A(t) = \prod_{k=\frac{t-t_0}{h}+1}^{0} \Big(I - hA(t_0+kh)\Big)^{-1}.$$

Example 10. *For (12), let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$, by Theorem 11, then the solution of (12) can be given by*

$$A(t) = \begin{cases} \prod\limits_{k=\frac{\ln t - \ln t_0}{\ln q}-2}^{0} \Big(I - (q-1)t_0 q^k A^\sigma(t_0 q^k)\Big) \\ \times \Big[I + (1-\frac{1}{q})t \prod\limits_{k=\frac{\ln t - \ln t_0}{\ln q}-2}^{0} \Big(I - (q-1)t_0 q^k A^\sigma(t_0 q^k)\Big)\Big]^{-1}, \ t > t_0, \\ \prod\limits_{k=\frac{\ln t - \ln t_0}{\ln q}+1}^{0} \Big(I - (q-1)t_0 q^{k-1} A(t_0 q^k)\Big)^{-1}, \ t < t_0. \end{cases}$$

In fact, by $A(t) = e_{-A^\sigma}(t, t_0)$, we can obtain

$$\begin{aligned} A(t) &= \prod_{k=\frac{\ln t - \ln t_0}{\ln q}-1}^{0} \Big(I - (q-1)t_0 q^k A^\sigma(t_0 q^k)\Big) \\ &= \Big(I - (1-\frac{1}{q})tA(t)\Big) \prod_{k=\frac{\ln t - \ln t_0}{\ln q}-2}^{0} \Big(I - (q-1)t_0 q^k A^\sigma(t_0 q^k)\Big) \end{aligned}$$

that is

$$A(t) = \prod_{k=\frac{\ln t - \ln t_0}{\ln q}-2}^{0} \Big(I - (q-1)t_0 q^k A^\sigma(t_0 q^k)\Big) \Big[I + (1-\frac{1}{q})t \prod_{k=\frac{\ln t - \ln t_0}{\ln q}-2}^{0} \Big(I - (q-1)t_0 q^k A^\sigma(t_0 q^k)\Big)\Big]^{-1}.$$

For $t \in \mathbb{T}$, $t < t_0$, we can obtain

$$\begin{aligned} A(t) &= \prod_{k=\frac{\ln t - \ln t_0}{\ln q}+1}^{0} \left(I - (q-1) t_0 q^{k-1} A^{\sigma}(t_0 q^{k-1}) \right)^{-1} \\ &= \prod_{k=\frac{\ln t - \ln t_0}{\ln q}+1}^{0} \left(I - (q-1) t_0 q^{k-1} A(t_0 q^{k}) \right)^{-1}, \end{aligned}$$

for any $0 \le k \le \frac{\ln t - \ln t_0}{\ln q} + 1$, $t_0 q^k > t$. Therefore

$$A(t) = \prod_{k=\frac{\ln t - \ln t_0}{\ln q}+1}^{0} \left(I - (q-1) t_0 q^{k-1} A(t_0 q^{k}) \right)^{-1}.$$

Theorem 12 (Liouville's formula)**.** *Let $A(\cdot) : \mathbb{T} \to \mathbb{R}^{n\times n}$, X is a solution of (11), then X satisfies Liouville's formula*

$$\det X(t) = \det \big(\alpha X_1 + (1-a) X_2\big) \det X_0, \ \ for \ \ t \in \mathbb{T},$$

where X_1 is solution of $X^{\Delta} = A(t)X(t)$, X_2 is solution of $X^{\nabla} = A(t)X(t)$.

Proof. By Theorem 10, we can obtain $\alpha X_1 + (1-a)X_2$ is a solution of (11), hence

$$\det X(t) = \det \big(\alpha X_1 + (1-a) X_2\big) \det X_0, \ \ for \ \ t \in \mathbb{T}.$$

The proof is complete. □

Example 11. *For (11), let $\mathbb{T} = \mathbb{R}$, any $t \in \mathbb{T}$ are dense points, according to Theorem 12, we have (11) can be written as*

$$X^{\diamond_\alpha} = X' = A(t)X(t), \ \ 0 \le \alpha \le 1.$$

Furthermore, Liouville's formula can be given by

$$\det X = \det X_0 e^{\int_{t_0}^{t} trA(\tau) d\o}, \ \ trA(\tau) = \sum_{i=1}^{n} a_{ii}(\tau).$$

In fact, $\mu(t) = \nu(t) = 0$ for any $t \in \mathbb{T}$, hence $X^{\Delta}(t) = X^{\nabla}(t) = X'$, therefore $X^{\diamond_\alpha} = X'$ for any $0 \le \alpha \le 1$. By Examples 1 and 4 we can obtain desired results.

Example 12. *For (11), let $\mathbb{T} = \mathbb{Z}h$, $h > 0$, by Theorem 12, the solution of (11) can be given as:*

$$X(t) = \left\{ \alpha \prod_{k=\frac{t-t_0}{h}-1}^{0} [I + hA(t_0 + kh)] + (1-\alpha) \prod_{k=\frac{t-t_0}{h}}^{1} [I - hA(t_0+kh)]^{-1} \right\} X_0, \ 0 \le \alpha \le 1.$$

In fact, by Theorem 10, we can obtain $X(t) = \alpha e_A(t,t_0) + (1-\alpha)\hat{e}_A(t,t_0)$. On the other hand, by Examples 2 and 5, we can obtain

$$e_A(t,t_0) = \prod_{k=\frac{t-t_0}{h}-1}^{0} [I + hA(t_0+kh), \ \ \hat{e}_A(t,t_0) = \prod_{k=\frac{t-t_0}{h}}^{1} [I - hA(t_0+kh)]^{-1}.$$

Example 13. *For* (11)*, let* $\mathbb{T} = q^{\mathbb{Z}}$*,* $q > 1$*, by Theorem 12, the solution of* (11) *can be given as:*

$$X(t) = \left\{ \alpha \prod_{k=\frac{\ln t - \ln t_0}{\ln q} - 1}^{0} \left[I + (q-1) q^k t_0 A(q^k t_0) \right] + (1-\alpha) \prod_{k=\frac{\ln t - \ln t_0}{\ln q}}^{1} \left[I - (q-1) q^k t_0 A(q^k t_0) \right]^{-1} \right\} X_0,$$

where $0 \le \alpha \le 1$*. In fact, by Theorem 10, we can obtain* $X(t) = \alpha e_A(t, t_0) + (1-\alpha) \hat{e}_A(t, t_0)$*. On the other hand, by Examples 3 and 6, we can obtain*

$$e_A(t, t_0) = \prod_{k=\frac{\ln t - \ln t_0}{\ln q} - 1}^{0} \left[I + (q-1) q^k t_0 A(q^k t_0) \right], \ \hat{e}_A(t, t_0) = \prod_{k=\frac{\ln t - \ln t_0}{\ln q}}^{1} \left[I - (q-1) q^k t_0 A(q^k t_0) \right]^{-1}.$$

Author Contributions: All authors have equal contribution in this work. All authors jointly worked on the results and they read and approved the final manuscript.

Funding: This work is supported by Youth Fund of NSFC (No. 11601470), Dong Lu youth excellent teachers development program of Yunnan University (No. wx069051), IRTSTYN and Joint Key Project of Yunnan Provincial Science and Technology Department of Yunnan University (No. 2018FY001(-014)).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Anderson, D.R. Taylor polynomials for nabla dynamic equations on time scales. *PanAm. Math. J.* **2002**, *12*, 17–27.
2. Anderson, D.R.; Bullock, J.; Erbe, L.; Peterson, A.; Tran, H. Nabla dynamic equations on time scales. *PanAm. Math. J.* **2003**, *13*, 1–47.
3. Atici, F.M.; Biles, D.C.; Lebedinsky, A. An application of time scales to economics. *Math. Comput. Model.* **2006**, *43*, 718–726.
4. Atici, F.M.; Biles, D.C. Further development of stochastic calculus on time scales. *PanAm. Math. J.* **2015**, *25*, 13–24.
5. Wang, C.; Agarwal, R.P. Almost periodic solution for a new type of neutral impulsive stochastic Lasota-Wazewska timescale model. *Appl. Math. Lett.* **2017**, *70*, 58–65.
6. Agarwal, R.P.; Bohner, M.; Wong, P.J.Y. Sturm-Liouville eigenvalue problems on time scales. *Appl. Math. Comput.* **1999**, *99*, 153–166.
7. Agarwal, R.P.; O'Regan, D. *An Introduction to Ordinary Differential Equations*; Springer: New York, NY, USA, 2008.
8. Agarwal, R.P.; Bohner, M.; O'Regan, D.; Peterson, A. Dynamic equations on time scales: A survey. *J. Comput. Appl. Math.* **2002**, *141*, 1–26.
9. Bohner, M.; Peterson, A. *Dynamic Equations on Time Scales, An Introduction with Applications*; Birkhäuser: Boston, MA, USA, 2001.
10. Hartman, P. *Ordinary Differential Equations*; Wiley: New York, NY, USA, 1964.
11. Adamec, L. A remark on Liouville's formula on small time scales. *J. Math. Anal. Appl.* **2005**, *304*, 504–510.
12. Cormani, V. Liouville's formula on time scales. *Dyn. Systems Appl.* **2003**, *12*, 79–86.
13. Sheng, Q.; Fadag, M.; Henderson, J.; Davis, J.M. An exploration of combined dynamic dervatives on times cales and their applications. *Nonlinear Anal. RWA* **2006**, *7*, 395–413.
14. Mozyrska, D.; Torres, D.F.M. A study of diamond-alpha dynamic equations on regular time scales. *arXiv* **2009**, arXiv:0902.1380.
15. Malinowska, A.B.; Torres, D.F.M. On the diamond-alpha Riemann integral and mean value theorems on time scales. *Dyn. Syst. Appl.* **2009**, *18*, 469–482.
16. Rogers, J.W., Jr.; Sheng, Q. Notes on the diamond-α dynamic derivative on time scales. *J. Math. Anal. Appl.* **2007**, *326*, 228–241.

17. Sheng, Q. Hybrid approximations via second order combined dynamic derivatives on time scales. *Electron. J. Qual. Theory Differ. Equ.* **2007**, *17*, 1–13.
18. Ammi, M.R.S.; Ferreira, R.A.C.; Torres, D.F.M. Diamond-α Jensen's inequality on time scales. *J. Inequal. Appl.* **2008**, *2008*, 576876.
19. Bohner, M.; Peterson, A. *Advances in Dynamic Equations on Time Scales*; Birkhäuser: Boston, MA, USA, 2003.