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Existence and Hyers–Ulam Stability for Random Impulsive Stochastic Pantograph Equations with the Caputo Fractional Derivative

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Abstract: In this paper, we study the existence, uniqueness and Hyers–Ulam stability of a class of fractional stochastic pantograph equations with random impulses. Firstly, we establish sufficient conditions to ensure the existence of solutions for the considered equations by applying Schaefer's fixed point theorem under relaxed linear growth conditions. Secondly, we prove the solution for the considered equations is Hyers–Ulam stable via Gronwall's inequality. Moreover, the previous literature will be significantly generalized in our paper. Finally, an example is given to explain the efficiency of the obtained results.

Keywords: Hyers–Ulam stability; Caputo fractional derivative; random impulses; Schaefer's fixed point theorem

MSC: 60H10; 60H20; 37H30; 39B72

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1. Introduction

Pantograph equations were first proposed by Ockendona and Tayler [1] and used to describe various phenomena like the economy, electrodynamics, control, neutral networks and some other nonlinear dynamical systems [2–4]. Given the facts above, the existence, uniqueness and stability of different kinds of pantograph equations have been extensively investigated by many authors. In particular, the study of stochastic pantograph equations is still one of the hot spots that interests scholars. Some excellent and important articles have also emerged in this field. For example, Fan et al. [5] considered the existence and uniqueness of the solutions of stochastic pantograph equations under the Lipschitz condition and the linear growth condition. Fan et al. [6] investigated the αth moment asymptotical stability for the stochastic pantograph equation with the Razumikhin technique. Zhang et al. [7] studied the convergence and stability of nonlinear stochastic pantograph equations. Yang et al. [8] discussed mean-square stability analysis of nonlinear stochastic pantograph equations using the transformation approach. They all considered the following system:

$$\begin{cases} dx(t) = f(t, x(t), x(\theta t))dt + g(t, x(t), x(\theta t))dB(t), t > 0, \\ x_0 = x_0. \end{cases}$$

Mao et al. [9] considered a class of stochastic pantograph equations with Lévy jumps as follows:

$$\begin{cases} dx(t) = f(t, x(t), x(\theta t))dt + g(t, x(t), x(\theta t))dB(t), t \in [t_0, T], \\ B(t) = \int_{t_0}^{t} \int_U h(x(s), x(\theta s), u) N_{\bar{p}}(ds, du), \\ x(t) = \varphi(t),\ t \in [\theta t_0, t_0], \end{cases}$$

By applying the Burkholder–Davis–Gundy inequality and Kunita's inequality, the existence and uniqueness of solutions for the considered equation under the Lipschitz conditions and the local Lipschitz conditions are obtained. Meantime, they also established the pth exponential estimations and almost surely asymptotic estimations for the considered system. Recently, Hu et al. [10] discussed the following stochastic pantograph differential equations driven by G-Brownian motion:

$$\begin{cases} dx(t) = f(t, x(t), x(\theta t))dt + h(t, x(t), x(\theta t))d[B](t) + \sigma(t, x(t), x(\theta t))dB(t), t \geq 0, \\ x(0) = \xi \in R^n, \end{cases}$$

They proved the existence, uniqueness, asymptotic boundedness and exponential stability of the considered equations when the coefficients satisfy local Lipschitz and generalized Lyapunov conditions.

On the other hand, there exist instantaneous perturbations and abrupt changes at a certain time in different areas of the real world; we usually call these changes impulsive effects. The duration of impulses is very short in comparison with the whole duration and is negligible [11]. When the impulses exist at random, this affects the nature of the differential system. For more work on the study of random impulsive differential equations, we refer the reader to [12–18] and the references therein. In particular, Anguraj et al. [16] considered the existence and stability results for a class of random impulsive fractional pantograph equations, and Priyadharsini and Balasubramaniam [19] studied existence and uniqueness results for fuzzy fractional stochastic pantograph differential equations. Recently, Shu et al. [18] made the first step in that they investigated the existence and Hyers–Ulam stability of a class of random impulsive stochastic functional differential equations with infinite delays under Lipschitz conditions on a bounded and closed interval. Also, Luo et al. [20–22] discussed the Hyers–Ulam stability for some kinds of corresponding differential equations under some suitable conditions.

As far as we know, there are few papers that have discussed the existence, uniqueness and Hyers–Ulam stability of stochastic pantograph equations with random impulses and the Caputo fractional derivative. Based on this fact, we intend to study the existence, uniqueness and Hyers–Ulam stability of a class of random impulsive fractional stochastic pantograph equations as follows:

$$\begin{cases} {}^c\mathrm{D}_t^{\alpha} x(t) = f(t, x(t), x(\theta t)) + g(t, x(t), x(\theta t))dB(t), t \geq t_0, \\ x(\xi_k) = b_k(\tau_k) x(\xi_k^-),\ k = 1, 2, \ldots, \\ x_{t_0} = x_0, \end{cases} \tag{1}$$

where ${}^c\mathrm{D}_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$; $0 < \theta < 1$; τ_k is a random variable defined from Ω to $D_k \equiv (0, d_k)$ for all $k = 1, 2, \ldots$ and $0 < d_k < +\infty$; and Ω is a nonempty set. Then, assume that τ_i and τ_j are independent of each other as $i \neq j = 1, 2, \ldots$. The function $f : \mathcal{R}_\tau \times \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$, $g : \mathcal{R}_\tau \times \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$, where $\mathcal{R}_\tau = [\tau, b]$ and $\tau, b \in \mathcal{R}$. $b_k : D_k \to \mathcal{R}^n \times \mathcal{R}^n$, is a matrix-valued function. The impulsive moments ξ_k satisfy $t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_k < \infty$, and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \ldots$, $x(\xi_k^-) = \lim_{t \to \xi_k - 0} x(t)$. It is easy to see that $\{\xi_k\}$ is a process with independent increments. Let $\{G(t), t \geq 0\}$ be the simple counting process generated by $\{\xi_k\}$, and $B(t)$ be an n-dimensional Wiener process. Suppose that $(\Omega, \mathfrak{F}_t, P)$ is a probability space with a filtration $\{\mathfrak{F}_t\}_{t \geq 0}$ satisfying $\mathfrak{F}_t = \mathfrak{F}_t^1 \vee \mathfrak{F}_t^2$, where $\mathfrak{F}_t^1, \mathfrak{F}_t^2$ are the σ-algebra generated by $\{G(t), t \geq 0\}$ and $\{B(s), s \leq t\}$, respectively. Also, suppose $L^p(\Omega, \mathcal{R}^n)$ is the collection of all strongly measurable, pth integrable, $\mathfrak{F}_t$-measurable, $\mathcal{R}^n$)-valued random variables φ with the norm

$$\|\varphi\|^p = (\sup_{t \in [\tau, b]} \mathbb{E}\|\varphi(t)\|^p),$$

where $\varphi(t) \in L^p(\Omega, \mathcal{R}^n)$, $\mathbb{E}(\cdot)$ is the expectation with respect to the measure P, $t \in [\tau, b]$.

The highlights and main contributions of this paper are reflected in the subsequent key aspects:

- We investigate the existence, uniqueness and Hyers–Ulam stability of a class of random impulsive fractional stochastic pantograph equations under relaxed linear growth conditions. Compared with the previous literature [5–9,14,16], the corresponding conditions are required to satisfy the Lipschitz condition and the linear growth condition. However, in practical cases, the linear growth condition is usually violated. Therefore, the linear growth condition will be replaced by the relaxed linear growth conditions in our paper.
- We not only extend the stochastic pantograph equations to fractional order, but also consider the random impulsive disturbance. Also, some sufficient conditions to ensure the existence, uniqueness and Hyers–Ulam stability of the considered equations under the relaxed linear growth conditions are established by Schaefer's fixed point theorem, the Banach fixed point theorem and inequality skills. In other words, the previous models in [5–8,16] are special cases of our considered model. In fact, when $\alpha = 1$, the impulsive effects are eliminated and model (1) is reduced to the corresponding model in [5–8]. When we do not consider stochastic disturbance, model (1) is reduced to the corresponding model in [16]. Therefore, our results generalize the results of the previous literature [5–8,14,16] to a certain extent.

The rest of the paper includes the following sections. In Section 2, some definitions and lemmas will be provided. Section 3 aims to discuss the existence and uniqueness of system (1). In Section 4, we will deal with the Hyers–Ulam stability of the considered system (1). An example is established to illustrate the theoretical results in Section 5. Finally, we draw out the conclusion in Section 6.

2. Preliminaries

In this part, necessary definitions and lemmas are provided which will be used in later parts.

Definition 1 ([23])**.** *The fractional-order integral of order α for function $f \in L^1([a,b],\mathcal{R}^n)$ is denoted by*

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(s)}{(t-s)^{1-\alpha}}ds,$$

where $t \in [a,b]$ and $\alpha \in [0,1]$, $\Gamma(\cdot)$ is the gamma function.

Definition 2 ([24])**.** *The Caputo derivative of order α for function $f \in L^1([a,b],\mathcal{R}^n)$ is denoted by*

$$_cD_{a,t}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds,$$

where $n = [\alpha]+1$ and $[\alpha]$ is the integer part of α.

Lemma 1 ([25])**.** *For any $p \geq 1$ and for an arbitrary $L^p_{n\times n}[0,b]$-valued predictable process $\psi(s)$,*

$$\sup_{s\in[0,t]} \mathbb{E}\Big\|\int_0^s \psi(u)dB(u)\Big\|^p \leq \Big(\frac{p}{2}(p-1)\Big)^{\frac{p}{2}}\Big(\int_0^t (\mathbb{E}\|\psi(s)\|^p)^{\frac{2}{p}}ds\Big)^{\frac{p}{2}},\ t\in[0,b].$$

Definition 3. *For a given $b \in (\tau,+\infty)$, an $\mathcal{R}^n$-value $x(t)$ on $t \in [\tau,b]$ is called a solution to (1) in $(\Omega,\mathfrak{F}_t,P)$ if $\{x(t)_{\tau\leq t\leq b}\}$ is $\mathfrak{F}_t$-adapted and satisfies*

$$\begin{aligned} x(t) \quad &= \quad \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^k b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^k\prod_{j=1}^k b_i(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s,x(s),x(\theta s))ds \\ &+ \quad \frac{1}{\Gamma(\alpha)}\int_{\xi_k}^t (t-s)^{\alpha-1}f(s,x(s),x(\theta s))ds \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} g(s, x(s), x(\theta s)) dB(s) \\
&+ \quad \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} g(s, x(s), x(\theta s)) dB(s) \Big] I_{[\xi_k, \xi_{k+1})}(t),\ t \in [\tau, b],
\end{aligned}
$$

where $\prod_{j=i}^{k} b_i(\tau_j) = b_i(\tau_i) b_{i+1}(\tau_{i+1}) \cdots b_{k-1}(\tau_{k-1}) b_k(\tau_k)$ *and* $I_A(\cdot)$ *is the index function satisfying*

$$
I_A(t) = \begin{cases} 1,\ if\ t \in A, \\ 0,\ if\ t \notin A. \end{cases}
$$

Lemma 2 ([11] Schaefer's Fixed Point Theorem)**.** *Let X be a Banach space and* $\Phi : X \to X$ *be a completely continuous map. If the set*

$$
\Phi = \{x \in X : x = \lambda \Phi x \ for\ some\ \lambda \in (0,1)\}
$$

is bounded, then Φ *has a fixed point.*

3. Existence and Uniqueness

This section aims to discuss the existence and uniqueness of solutions for system (1). The following assumptions are also needed.

Hypothesis 1 (H1). *The function* $f : \mathcal{R}_\tau \times \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$ *satisfies the following:* f *is continuous with respect to* $t \in [\tau, b]$ *and measurable with respect to* $x(t) \in \mathcal{R}^n$*. There exists a constant* $\gamma \in (0, \alpha)$ *such that real-valued continuous functions* $m_1(t) \in L^{\frac{1}{\gamma}}$ *and there exists an* L^p *integrable and nondecreasing function* $\Psi : [0, \infty) \to [0, \infty)$ *such that*

$$
\mathbb{E}\|f(t, \delta_1, \delta_2)\|^p \leq m_1(t) \Psi(\mathbb{E}\|\delta_1\|^p + \mathbb{E}\|\delta_2\|^p). \tag{2}
$$

Hypothesis 2 (H2). *The function* $g : \mathcal{R}_\tau \times \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$ *satisfies the following:* g *is continuous with respect to* $t \in [\tau, b]$ *and measurable with respect to* $x(t) \in \mathcal{R}^n$*. There exists a constant* $\gamma \in (0, \alpha)$ *such that the real-valued continuous functions* $m_2(t) \in L^{\frac{1}{\gamma}}$ *and there exists an* L^p *integrable and nondecreasing function* $\Psi : [0, \infty) \to [0, \infty)$ *such that*

$$
\mathbb{E}\|g(t, \delta_1, \delta_2)\|^p \leq m_2(t) \Psi(\mathbb{E}\|\delta_1\|^p + \mathbb{E}\|\delta_2\|^p). \tag{3}
$$

Hypothesis 3 (H3). *There exists a positive constant M, for all* $\tau_j \in D_j (j = 1, 2, \ldots)$*, such that*

$$
\mathbb{E}\Big(\max_{i,k}\{\prod_{j=i}^{k} \|b_j(\tau_j)\|\}\Big)^p \leq M. \tag{4}
$$

Theorem 1. *Assume that conditions (H1)–(H3) hold, then system (1) has a solution* $x(t)$*, which is defined on* $[t_0, b]$*, provided that the following inequality holds*

$$
M^* \int_{t_0}^{b} [m_1(s) + m_2(s)] ds < \int_{\beta}^{\infty} \frac{ds}{\Psi(2s)}, \tag{5}
$$

where $M^* = 3^{p-1} \frac{\max\{1, M\}(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} (p(\alpha-1)+1 \neq 0)$, $\beta = 3^{p-1} M \mathbb{E}\|x_0\|^p$.

Proof. Define the operator $\Phi : C([t_0, b], L^p(\Omega, \mathcal{R}^n)) \to C([t_0, b], L^p(\Omega, \mathcal{R}^n))$ as follows:

$$
(\Phi x)(t) \quad = \quad \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i) x_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} f(s, x(s), x(\theta s)) ds
$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s,x(s),x(\theta s))ds\\
&+ \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k}b_i(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}g(s,x(s),x(\theta s))dB(s)\\
&+ \frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}g(s,x(s),x(\theta s))dB(s)\Big]I_{[\xi_k,\xi_{k+1})}(t),\ t\in[\tau,b],
\end{aligned}
$$

where $b \in (t_0, +\infty)$ satisfies (5). Naturally, it is easy to see that finding the solution of (1) is equivalent to getting the fixed point for the operator Φ.

Firstly, consider a bounded operator Φ.

Let $\lambda \in (0,1)$. It follows that

$$
\begin{aligned}
x(t) &= \lambda\sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k}b_i(\tau_i)x_0+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k}b_i(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s,x(s),x(\theta s))ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s,x(s),x(\theta s))ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k}b_i(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}g(s,x(s),x(\theta s))dB(s)\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}g(s,x(s),x(\theta s))dB(s)\Big]I_{[\xi_k,\xi_{k+1})}(t),t\in[\tau,b],
\end{aligned}
$$

which yields

$$
\begin{aligned}
\|x(t)\|^p &\leq \Big\{\lambda\sum_{k=0}^{+\infty}\Big[\|\prod_{i=1}^{k}b_i(\tau_i)\|\|x_0\|+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\|\prod_{j=1}^{k}b_i(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}\|f(s,x(s),x(\theta s))\|ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}\|f(s,x(s),x(\theta s))\|ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\|\prod_{j=1}^{k}b_i(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}\|g(s,x(s),x(\theta s))\|dB(s)\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}\|g(s,x(s),x(\theta s))\|dB(s)\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big\}^p\\
&\leq 3^{p-1}\Big[\sum_{k=0}^{+\infty}\|\prod_{i=1}^{k}b_i(\tau_i)\|^p\|x_0\|^p\Big]+3^{p-1}\Big\{\sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}\|f(s,x(s),x(\theta s))\|ds\\
&\quad+\sum_{i=1}^{k}\|\prod_{j=1}^{k}b_i(\tau_j)\|\frac{1}{\Gamma(\alpha)}\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}\|f(s,x(s),x(\theta s))\|ds\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big\}^p\\
&\quad+3^{p-1}\Big\{\sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\|\prod_{j=1}^{k}b_i(\tau_j)\|\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}\|g(s,x(s),x(\theta s))\|dB(s)\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}\|g(s,x(s),x(\theta s))\|dB(s)\Big]I_{[\xi_k,\xi_{k+1})}(t)\Big\}^p\\
&\leq 3^{p-1}\max_{i,k}\Big[\|\prod_{i=1}^{k}b_i(\tau_i)\|^p\|x_0\|^p\Big]\\
&\quad+3^{p-1}\Big[\max_{i,k}\Big\{1,\|\prod_{i=1}^{k}b_i(\tau_i)\|\Big\}\Big]^p\Big[\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|f(s,x(s),x(\theta s))\|ds\Big]^p
\end{aligned}
$$

$$+3^{p-1}\Big[\max_{i,k}\Big\{1,\|\prod_{i=1}^{k}b_i(\tau_i)\|\Big\}\Big]^p\Big[\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\|g(s,x(s),x(\theta s))\|dB(s)\Big]^p.$$

Next, taking the mathematical expectation of the above inequality and letting $M \geq 3^{\frac{1}{p}-1}$ together with (H1)–(H3) and Lemma 1, we have

$$\begin{aligned}
&\mathbb{E}\|x(t)\|^p \leq 3^{p-1}M\mathbb{E}\|x_0\|^p \\
&+\frac{3^{p-1}\max\{1,M\}}{(\Gamma(\alpha))^p}\int_{t_0}^{t}(t-s)^{p(\alpha-1)}ds\times\int_{t_0}^{t}\mathbb{E}\|f(s,x(s),x(\theta s))\|^p ds \\
&+\frac{3^{p-1}\max\{1,M\}}{(\Gamma(\alpha))^p}\int_{t_0}^{t}(t-s)^{p(\alpha-1)}ds\times\int_{t_0}^{t}\mathbb{E}\|g(s,x(s),x(\theta s))\|^p dB(s) \\
\leq\ & 3^{p-1}M\mathbb{E}\|x_0\|^p+3^{p-1}\max\{1,M\}\frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p}\int_{t_0}^{t}\mathbb{E}\|f(s,x(s),x(\theta s))\|^p ds \\
&+3^{p-1}\max\{1,M\}\frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p}\Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}}\int_{t_0}^{t}\mathbb{E}\|g(s,x(s),x(\theta s))\|^p ds \\
\leq\ & 3^{p-1}M\mathbb{E}\|x_0\|^p+3^{p-1}\max\{1,M\}\frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p}\int_{t_0}^{t}m_1(s)\Psi(2\mathbb{E}\|x(s)\|^p)ds \\
&+3^{p-1}\max\{1,M\}\frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p}\Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}}\int_{t_0}^{t}m_2(s)\Psi(2\mathbb{E}\|x(s)\|^p)ds \\
\leq\ & 3^{p-1}M\mathbb{E}\|x_0\|^p+3^{p-1}\max\{1,M\}\frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p}\Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}} \\
&\times\int_{t_0}^{t}\Big[m_1(s)+m_2(s)\Big]\Psi(2\mathbb{E}\|x(s)\|^p)ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sup_{t_0\leq\zeta\leq t}\mathbb{E}\|x(\zeta)\|^p \leq\ & 3^{p-1}M\mathbb{E}\|x_0\|^p+3^{p-1}\max\{1,M\}\frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p}\Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}} \\
&\times\int_{t_0}^{t}\Big[m_1(s)+m_2(s)\Big]\Psi(2\sup_{t_0\leq\zeta\leq s}\mathbb{E}\|x(s)\|^p)ds.
\end{aligned}$$

Let

$$\Theta(t)=\sup_{t_0\leq\zeta\leq t}\mathbb{E}\|x(\zeta)\|^p,\ t\in[t_0,b], \tag{6}$$

We obtain

$$\Theta(t)\leq 3^{p-1}\Big[M\mathbb{E}\|x_0\|^p+\frac{\max\{1,M\}(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p}\Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}}\int_{t_0}^{t}\Big[m_1(s)+m_2(s)\Big]\Psi(2\Theta(s))ds\Big].$$

For convenience, let

$$\Upsilon(t)=3^{p-1}\Big[M\mathbb{E}\|x_0\|^p+\frac{\max\{1,M\}(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p}\Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}}\int_{t_0}^{t}\Big[m_1(s)+m_2(s)\Big]\Psi(2\Theta(s))ds\Big],$$

which impiles $\Theta(t)\leq\Upsilon(t),\ t\in[t_0,b]$ and $\Upsilon(t_0)=3^{p-1}M\mathbb{E}\|x_0\|^p=\beta$.

Taking the derivative of function $\Upsilon(t)$ with regard to t, we derive

$$\begin{aligned}
\Upsilon'(t) =\ & 3^{p-1}\max\{1,M\}\frac{(b-\tau)^{p(\alpha-1)+1}}{((p(\alpha-1)+1)\Gamma(\alpha))^p}\Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}}\Big[m_1(t)+m_2(t)\Big]\Psi(2\Theta(t)) \\
\leq\ & 3^{p-1}\max\{1,M\}\frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p}\Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}}\Big[m_1(t)+m_2(t)\Big]\Psi(2\Upsilon(t)),
\end{aligned}$$

which yields

$$\frac{\Upsilon'(t)}{\Psi(2\Upsilon(t))} \leq 3^{p-1} \max\{1, M\} \frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}} \Big[m_1(t) + m_2(t)\Big]. \quad (7)$$

Integrating (7) from t_0 to t and combining with (5), we have

$$\begin{aligned}
\int_{\Upsilon(t_0)}^{\Upsilon(t)} \frac{ds}{\Psi(2s)} &\leq 3^{p-1} \max\{1, M\} \frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}} \int_{t_0}^{t} \Big[m_1(s) + m_2(s)\Big] ds \\
&\leq 3^{p-1} \max\{1, M\} \frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}} \int_{t_0}^{b} \Big[m_1(s) + m_2(s)\Big] ds \\
&\leq \int_{\beta}^{\infty} \frac{ds}{\Psi(2s)},\ t \in [t_0, b],
\end{aligned}$$

which implies that there exists a constant ρ such that $\Upsilon(t) \leq \rho$. That is, $\Theta(t) \leq \rho$. Thus, $\sup_{t_0 \leq \zeta \leq T} \mathbb{E}\|x(\zeta)\|^p \leq \rho$, where ρ only depends on T and the functions $m_1(t), m_2(t)$ and $\Psi(t)$. Φ is bounded is proved.

Next, we prove Φ is completely continuous. We will divide the proof into the following steps.

- Φ is continuous.

  Let $\{x^n\} \in C([t_0, b], L^p(\Omega, \mathcal{R}^n))$ such that $x^n \to x$ as $n \to \infty$. For any $t \in [t_0, b]$, we get

$$\begin{aligned}
& \mathbb{E}\|(\Phi x_n)(t) - (\Phi x)(t)\|^p \\
= \ & 4^{p-1}\Big\{ \sum_{k=0}^{+\infty} \Big[\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} \mathbb{E}\|b_i(\tau_j)\| \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} \mathbb{E}\|f(s, x_n(s), x_n(\theta s)) - f(s, x(s), x(\theta s))\| ds \\
& + 4^{p-1} \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} \mathbb{E}\|f(s, x_n(s), x_n(\theta s)) - f(s, x(s), x(\theta s))\| ds \\
& + 4^{p-1} \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} \mathbb{E}\|b_i(\tau_j)\| \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} \mathbb{E}\|g(s, x_n(s), x_n(\theta s)) - g(s, x(s), x(\theta s))\| dB(s) \\
& + 4^{p-1} \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} \mathbb{E}\|g(s, x_n(s), x_n(\theta s)) - g(s, x(s), x(\theta s))\| dB(s) \Big] I_{[\xi_k, \xi_{k+1})}(t) \Big\}^p \\
\leq \ & 4^{p-1} \max\{1, M\} \Big[\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \mathbb{E}\|f(s, x_n(s), x_n(\theta s)) - f(s, x(s), x(\theta s))\| ds \Big]^p \\
& + 4^{p-1} \max\{1, M\} \Big[\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \mathbb{E}\|g(s, x_n(s), x_n(\theta s)) - g(s, x(s), x(\theta s))\| dB(s) \Big]^p . \\
\leq \ & 4^{p-1} \max\{1, M\} \frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}} \Big[\int_{t_0}^{t} \mathbb{E}\|f(s, x_n(s), x_n(\theta s)) \\
& - f(s, x(s), x(\theta s))\|^p ds + \int_{t_0}^{t} (t-s)^{\alpha-1} \mathbb{E}\|g(s, x_n(s), x_n(\theta s)) - g(s, x(s), x(\theta s))\|^p ds \Big] \\
& \to 0 \ as \ n \to \infty,
\end{aligned}$$

which implies that Φ is continuous.

Let $\chi = C([t_0, b], L^p(\Omega, \mathcal{R}^n))$ and $\|x\|_\chi^p = \sup_{t \in [\tau, b]} \mathbb{E}\|\varphi(t)\|^p$. We denote

$$B_r = \{x \in \chi \mid \|x\|_\chi^p \leq r\}.$$

- Φ maps B_r into an equicontinuous set.

  Suppose that $x \in B_r$, $t_1, t_2 \in [t_0, b] \cap (\xi_{k-1}, \xi_k)$. From (H1)–(H3), we obtain

$$
\begin{aligned}
& \mathbb{E}\|(\Phi x)(t_2) - (\Phi x)(t_1)\|^p \\
\leq\; & 4^{p-1}\mathbb{E}\Big\| \sum_{k=0}^{+\infty} \Big[\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t_2 - s)^{\alpha-1} f(s, x(s), x(\theta s)) ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t_2} (t_2 - s)^{\alpha-1} f(s, x(s), x(\theta s)) ds \Big(I_{[\xi_k, \xi_{k+1})}(t_2) - I_{[\xi_k, \xi_{k+1})}(t_1) \Big) \Big] \Big\|^p \\
& + 4^{p-1}\mathbb{E}\Big\| \sum_{k=0}^{+\infty} \Big[\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] f(s, x(s), x(\theta s)) ds \\
& + \frac{1}{\Gamma(\alpha)} [\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, x(s), x(\theta s)) ds \\
& + \int_{\xi_k}^{t_1} \Big((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \Big) f(s, x(s), x(\theta s)) ds] (I_{[\xi_k, \xi_{k+1})}(t_1) \Big] \Big\|^p \\
& + 4^{p-1}\mathbb{E}\Big\| \sum_{k=0}^{+\infty} \Big[\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t_2 - s)^{\alpha-1} g(s, x(s), x(\theta s)) dB(s) \\
& + \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t_2} (t_2 - s)^{\alpha-1} g(s, x(s), x(\theta s)) dB(s) \Big(I_{[\xi_k, \xi_{k+1})}(t_2) - I_{[\xi_k, \xi_{k+1})}(t_1) \Big) \Big] \Big\|^p \\
& + 4^{p-1}\mathbb{E}\Big\| \sum_{k=0}^{+\infty} \Big[\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] g(s, x(s), x(\theta s)) dB(s) \\
& + \frac{1}{\Gamma(\alpha)} [\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} g(s, x(s), x(\theta s)) dB(s) \\
& + \int_{\xi_k}^{t_1} \Big((t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \Big) g(s, x(s), x(\theta s)) dB(s)] (I_{[\xi_k, \xi_{k+1})}(t_1) \Big] \Big\|^p \\
=\; & 4^{p-1}\mathbb{E} \sum_{i=1}^{4} \|G_i\|^p.
\end{aligned}
$$

Next, we estimate each term of the above inequality,

$$
\begin{aligned}
\mathbb{E}\|G_1\|^p \;=\; & \mathbb{E}\Big\| \sum_{k=0}^{+\infty} \Big[\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t_2 - s)^{\alpha-1} f(s, x(s), x(\theta s)) ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t_2} (t_2 - s)^{\alpha-1} f(s, x(s), x(\theta s)) ds \Big(I_{[\xi_k, \xi_{k+1})}(t_2) - I_{[\xi_k, \xi_{k+1})}(t_1) \Big) \Big] \Big\|^p \\
\leq\; & \max\{1, M\} \frac{1}{(\Gamma(\alpha))^p} \int_{t_0}^{t_1} (t_2 - s)^{p(\alpha-1)} \mathbb{E}\|f(s, x(s), x(\theta s))\|^p ds \\
& \times \mathbb{E}\Big\| \Big(I_{[\xi_k, \xi_{k+1})}(t_2) - I_{[\xi_k, \xi_{k+1})}(t_1) \Big) \Big\|^p \\
\leq\; & \max\{1, M\} \frac{1}{(\Gamma(\alpha))^p} \int_{t_0}^{t_1} (t_2 - s)^{p(\alpha-1)} m_1(s) \Psi(2r) ds \\
& \times \mathbb{E}\Big\| \Big(I_{[\xi_k, \xi_{k+1})}(t_2) - I_{[\xi_k, \xi_{k+1})}(t_1) \Big) \Big\|^p \\
\to\; & 0, \; as \; t_2 \to t_1,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\|G_2\|^p \;=\; & \mathbb{E}\Big\| \sum_{k=0}^{+\infty} \Big[\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] f(s, x(s), x(\theta s)) ds \\
& + \frac{1}{\Gamma(\alpha)} [\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s, x(s), x(\theta s)) ds
\end{aligned}
$$

$$
\begin{aligned}
& + \int_{\xi_k}^{t_1} \Big((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \Big) f(s,x(s),x(\theta s))ds] (I_{[\xi_k,\xi_{k+1})}(t_1) \Big] \Big\|^p \\
\leq\ & \Big[2^{p-1} \max\{1,M\} \frac{1}{(\Gamma(\alpha))^p} \int_{t_0}^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{p(\alpha-1)}] \mathbb{E}\|f(s,x(s),x(\theta s))\|^p ds \\
& + 2^{p-1} \frac{1}{(\Gamma(\alpha))^p} [\int_{t_1}^{t_2} (t_2-s)^{p(\alpha-1)} \mathbb{E}\|f(s,x(s),x(\theta s))\|^p ds \Big] \mathbb{E} \Big\| \Big(I_{[\xi_k,\xi_{k+1})}(t_1) \Big) \Big\|^p \\
\leq\ & \Big[2^{p-1} \max\{1,M\} \frac{1}{(\Gamma(\alpha))^p} \int_{t_0}^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{p(\alpha-1)}] m_1(s) \Psi(2r) ds \\
& + 2^{p-1} \frac{1}{(\Gamma(\alpha))^p} [\int_{t_1}^{t_2} (t_2-s)^{p(\alpha-1)} m_1(s) \Psi(2r) ds \Big] \mathbb{E} \Big\| \Big(I_{[\xi_k,\xi_{k+1})}(t_1) \Big) \Big\|^p \\
\to\ & 0, \ as \ t_2 \to t_1,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\|G_3\|^p =\ & \mathbb{E} \Big\| \sum_{k=0}^{+\infty} \Big[\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t_2-s)^{\alpha-1} g(s,x(s),x(\theta s)) dB(s) \\
& + \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t_2} (t_2-s)^{\alpha-1} g(s,x(s),x(\theta s)) dB(s) \Big(I_{[\xi_k,\xi_{k+1})}(t_2) - I_{[\xi_k,\xi_{k+1})}(t_1) \Big) \Big] \Big\|^p \\
\leq\ & \max\{1,M\} \frac{1}{(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2} \Big)^{\frac{p}{2}} \int_{t_0}^{t_1} (t_2-s)^{p(\alpha-1)} \mathbb{E}\|g(s,x(s),x(\theta s))\|^p ds \\
& \times \mathbb{E} \Big\| \Big(I_{[\xi_k,\xi_{k+1})}(t_2) - I_{[\xi_k,\xi_{k+1})}(t_1) \Big) \Big\|^p \\
\leq\ & \max\{1,M\} \frac{1}{(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2} \Big)^{\frac{p}{2}} \int_{t_0}^{t_1} (t_2-s)^{p(\alpha-1)} m_2(s) \Psi(2r) ds \\
& \times \mathbb{E} \Big\| \Big(I_{[\xi_k,\xi_{k+1})}(t_2) - I_{[\xi_k,\xi_{k+1})}(t_1) \Big) \Big\|^p \\
\to\ & 0, \ as \ t_2 \to t_1,
\end{aligned}
$$

$$
\begin{aligned}
& \mathbb{E}\|G_4\|^p = \mathbb{E} \Big\| \sum_{k=0}^{+\infty} \Big[\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] g(s,x(s),x(\theta s)) dB(s) \\
& + \frac{1}{\Gamma(\alpha)} [\int_{t_1}^{t_2} (t_2-s)^{\alpha-1} g(s,x(s),x(\theta s)) dB(s) \\
& + \int_{\xi_k}^{t_1} \Big((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \Big) g(s,x(s),x(\theta s)) dB(s)] (I_{[\xi_k,\xi_{k+1})}(t_1) \Big] \Big\|^p \\
\leq\ & \Big[2^{p-1} \frac{\max\{1,M\}}{(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2} \Big)^{\frac{p}{2}} \int_{t_0}^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{p(\alpha-1)}] \mathbb{E}\|g(s,x(s),x(\theta s))\|^p ds \\
& + 2^{p-1} \frac{1}{(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2} \Big)^{\frac{p}{2}} [\int_{t_1}^{t_2} (t_2-s)^{p(\alpha-1)} \mathbb{E}\|g(s,x(s),x(\theta s))\|^p ds \Big] \mathbb{E} \Big\| \Big(I_{[\xi_k,\xi_{k+1})}(t_1) \Big) \Big\|^p \\
\leq\ & \Big[2^{p-1} \frac{\max\{1,M\}}{(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2} \Big)^{\frac{p}{2}} \int_{t_0}^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{p(\alpha-1)}] m_2(s) \Psi(2r) ds \\
& + 2^{p-1} \frac{1}{(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2} \Big)^{\frac{p}{2}} [\int_{t_1}^{t_2} (t_2-s)^{p(\alpha-1)} m_2(s) \Psi(2r) ds \Big] \mathbb{E} \Big\| \Big(I_{[\xi_k,\xi_{k+1})}(t_1) \Big) \Big\|^p \\
\to\ & 0, \ as \ t_2 \to t_1.
\end{aligned}
$$

Thus, $\mathbb{E}\|(\Phi x)(t_2) - (\Phi x)(t_1)\|^p \to 0$, as $t_2 \to t_1$, which implies that Φ maps B_r into an equicontinuous set.

- $\Phi(B_r)$ is uniformly bounded.

  From (H1)–(H3) and (5), we have

$$
\begin{aligned}
&\mathbb{E}\|(\Phi x)(t)\|^p = \mathbb{E}\Big\| \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k}\prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} f(s,x(s),x(\theta s))ds \\
&+\frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} f(s,x(s),x(\theta s))ds \\
&+\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k}\prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} g(s,x(s),x(\theta s))dB(s) \\
&+\frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} g(s,x(s),x(\theta s))dB(s) \Big] I_{[\xi_k,\xi_{k+1})}(t) \Big\|^p \\
\leq\ & 3^{p-1} M\mathbb{E}\|x_0\|^p + 3^{p-1}\max\{1,M\} \frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \int_{t_0}^{t} \mathbb{E}\|f(s,x(s),x(\theta s))\|^p ds \\
&+3^{p-1}\max\{1,M\} \frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}} \int_{t_0}^{t} \mathbb{E}\|g(s,x(s),x(\theta s))\|^p ds \\
\leq\ & 3^{p-1} M\mathbb{E}\|x_0\|^p + 3^{p-1}\max\{1,M\} \frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \int_{t_0}^{t} m_1(s)\Psi(2\mathbb{E}\|x(s)\|^p) ds \\
&+3^{p-1}\max\{1,M\} \frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}} \int_{t_0}^{t} m_2(s)\Psi(2\mathbb{E}\|x(s)\|^p) ds \\
\leq\ & 3^{p-1} M\mathbb{E}\|x_0\|^p + 3^{p-1}\max\{1,M\} \frac{(b-\tau)^{p(\alpha-1)+2}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}} \\
&\times\Psi(2\|x\|_{L^p})(\|m_1(t)\|L^{\frac{1}{\gamma}} + \|m_2(t)\|L^{\frac{1}{\gamma}}),
\end{aligned}
$$

which yields that $\Phi(B_r)$ is uniformly bounded.

- Φ maps B_r into a precompact set.

  For a given fixed $t \in [t_0, b]$, let ε be a positive constant satisfying $0 < \varepsilon < t - t_0$. For any $x \in B_r$, we define

$$
\begin{aligned}
(\Phi_\varepsilon x)(t) &= \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i)x_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k}\prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} f(s,x(s),x(\theta s))ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t-\varepsilon} (t-s)^{\alpha-1} f(s,x(s),x(\theta s))ds \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k}\prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} g(s,x(s),x(\theta s))dB(s) \\
&+ \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t-\varepsilon} (t-s)^{\alpha-1} g(s,x(s),x(\theta s))dB(s) \Big] I_{[\xi_k,\xi_{k+1})}(t),\ t \in (t_0, t-\varepsilon).
\end{aligned}
$$

Then, the set $Q_\varepsilon(t) = \{(\Phi_\varepsilon x)(t) : x \in B_r\}$ is relatively compact in χ for each $\varepsilon \in (0, t-t_0)$. By (H1)–(H3), we obtain

$$
\begin{aligned}
\mathbb{E}\|(\Phi x) - (\Phi_\varepsilon x)\|^p &\leq 2^{p-1}\mathbb{E}\Big\| \sum_{k=0}^{+\infty} \frac{1}{\Gamma(\alpha)} \int_{t-\varepsilon}^{t} f(s,x(s),x(\theta s))ds I_{[\xi_k,\xi_{k+1})}(t) \Big\|^p \\
&\quad +2^{p-1}\mathbb{E}\Big\| \sum_{k=0}^{+\infty} \frac{1}{\Gamma(\alpha)} \int_{t-\varepsilon}^{t} g(s,x(s),x(\theta s))dB(s) I_{[\xi_k,\xi_{k+1})}(t) \Big\|^p \\
&\leq 2^{p-1} \frac{\varepsilon^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \int_{t-\varepsilon}^{t} m_1(s)\Psi(2r)ds \\
&\quad +2^{p-1} \frac{\varepsilon^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2}\Big)^{\frac{p}{2}} \int_{t-\varepsilon}^{t} m_2(s)\Psi(2r)ds
\end{aligned}
$$

$$\rightarrow \quad 0, \ as \ \varepsilon \rightarrow 0,$$

which implies that Φ maps B_r into a precompact set. That is to say, for any precompact set which is closed to set $Q(t) = \{(\Phi x)(t) : x \in B_r\}$, $Q(t)$ is relatively compact in χ. Then, Φ is compact by the Arzela–Ascoli theorem. Thus, Φ is completely continuous. Therefore, system (1) must have a solution which is the fixed point of Φ by Lemma 2. □

Next, we intend to verify the uniqueness of the solutions for system (1) by using the Banach fixed point theorem.

Hypothesis 4 (H4). *For any continuous functions $f : \mathcal{R}_\tau \times \mathcal{R}^n \times \mathcal{R}^n \rightarrow \mathcal{R}^n$ and $g : \mathcal{R}_\tau \times \mathcal{R}^n \times \mathcal{R}^n \rightarrow \mathcal{R}^n$, there exist two positive constants L_1, L_2 such that*

$$\mathbb{E}|f(t, \delta_1, \delta_2) - f(t, \delta_{1*}, \delta_{2*})\|^p \leq L_1[\mathbb{E}\|\delta_1 - \delta_{1*}\|^p + \mathbb{E}\|\delta_2 - \delta_{2*}\|^p],$$

and

$$\mathbb{E}|g(t, \delta_1, \delta_2) - g(t, \delta_{1*}, \delta_{2*})\|^p \leq L_2[\mathbb{E}\|\delta_1 - \delta_{1*}\|^p + \mathbb{E}\|\delta_2 - \delta_{2*}\|^p],$$

where $\delta_1, \delta_2, \delta_{1*}, \delta_{2*} \in \mathcal{R}^n$ and $t \in [\tau, b]$.

Theorem 2. *Assume that conditions (H3) and (H4) hold; then, system (1) has a unique solution $x(t)$, which is defined on $[t_0, b]$.*

Proof. Let $b \in (\tau, +\infty)$ define the following operator $N : \chi \rightarrow \chi$,

$$\begin{aligned}
(Nx)(t) \quad = \quad & \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i) x_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} f(s, x(s), x(\theta s)) ds \\
+ \quad & \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} f(s, x(s), x(\theta s)) ds \\
+ \quad & \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} g(s, x(s), x(\theta s)) dB(s) \\
+ \quad & \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} g(s, x(s), x(\theta s)) dB(s) \Big] I_{[\xi_k, \xi_{k+1})}(t), \ t \in [\tau, b].
\end{aligned}$$

Then,

$$\begin{aligned}
& \mathbb{E}\Big\|(Nx)(t) - (Ny)(t)\Big\|^p \\
= \quad & \mathbb{E}\Big\| \sum_{k=0}^{+\infty} \Big[\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} [f(s, x(s), x(\theta s)) f(s, y(s), y(\theta s))] ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} [f(s, x(s), x(\theta s)) - f(s, y(s), y(\theta s))] ds \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} [g(s, x(s), x(\theta s)) - g(s, y(s), y(\theta s))] dB(s) \\
& + \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} [g(s, x(s), x(\theta s)) - g(s, y(s), y(\theta s))] dB(s) \Big] I_{[\xi_k, \xi_{k+1})}(t) \Big\|^p \\
\leq \quad & 4^{p-1} \frac{\max\{1, M\} (b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2} \Big)^{\frac{p}{2}} \Big[\int_{t_0}^{t} \mathbb{E}\|f(s, x(s), x(\theta s)) - f(s, y(s), y(\theta s))\|^p ds \\
& + \int_{t_0}^{t} \mathbb{E}\|g(s, x(s), x(\theta s)) - g(s, y(s), y(\theta s))\|^p ds \Big] \\
\leq \quad & 4^{p-1} \max\{1, M\} \frac{(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2} \Big)^{\frac{p}{2}} (L_1 + L_2)
\end{aligned}$$

$$\times \int_{t_0}^{t} \mathbb{E}[\|x(s) - y(s)\|^p + \|x(\theta s) - y(\theta s)\|^p] ds.$$

Taking the supremum over t, we have

$$\|Nx - Ny\|_{\mathcal{X}}^p \leq \Delta(b) \|x - y\|_{\mathcal{X}}^p,$$

where $\Delta(b) = 4^{p-1} \max\{1, M\} \frac{2(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} (L_1 + L_2)$. If we choose a suitable b_1 with $0 < b_1 < b$ such that $\Delta(b_1) < 1$, we immediately obtain that N is a contraction map. That is, the uniqueness of the solutions of system (1) is proven by using the Banach fixed point theorem. □

Remark 1. *The existence and uniqueness of the random impulsive fractional stochastic pantograph equations in our paper under relaxed linear growth conditions have been obtained, which will be generalized in the context of the previous literature [5–9,14,16].*

4. Hyers–Ulam Stability

This section is devoted to proving the Hyers–Ulam stability of the solutions of system (1). We recall the following definition of Hyers–Ulam stability.

Definition 4 ([26]). *System (1) is said to be Hyers–Ulam stable if there exists a real number $\Delta > 0$ such that $\forall \varepsilon > 0$, if each $\mathcal{R}^n$-value stochastic process $y(t)$ satisfies*

$$\begin{aligned}
&\mathbb{E}\Big\| y(t) - \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i) y_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} f(s, y(s), y(\theta s)) ds \\
+ \quad & \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} f(s, y(s), y(\theta s)) ds + \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} g(s, y(s), y(\theta s)) dB(s) \\
+ \quad & \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} g(s, y(s), y(\theta s)) dB(s) \Big] I_{[\xi_k, \xi_{k+1})}(t) \Big\|^p \leq \varepsilon, \ \forall t \in [\tau, b],
\end{aligned}$$

and if there exists a solution $x(t)$ of system (1) with initial value $x_0 = y_0$ satisfying $\mathbb{E}\|y(t) - x(t)\|^p \leq \Delta\varepsilon, \forall t \in [\tau, b]$.

Theorem 3. *Assume that conditions (H1)–(H4) hold; then, system (1) is Hyers–Ulam stable.*

Proof. From Definition 3, the solution of system (1) has the following form:

$$\begin{aligned}
x(t) \quad = \quad & \sum_{k=0}^{+\infty} \Big[\prod_{i=1}^{k} b_i(\tau_i) x_0 + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} f(s, x(s), x(\theta s)) ds \\
+ \quad & \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} f(s, x(s), x(\theta s)) ds \\
+ \quad & \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \prod_{j=1}^{k} b_i(\tau_j) \int_{\xi_{i-1}}^{\xi_i} (t-s)^{\alpha-1} g(s, x(s), x(\theta s)) dB(s) \\
+ \quad & \frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} g(s, x(s), x(\theta s)) dB(s) \Big] I_{[\xi_k, \xi_{k+1})}(t).
\end{aligned}$$

Assume that there exists another solution $y(t)$ of system (1) with the initial value $y_0 = x_0$. Then, we have

$$\begin{aligned}
&\mathbb{E}\Big\| y(t) - \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)y_0 + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_i(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s,y(s),y(\theta s))ds \\
+\;& \frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s,y(s),y(\theta s))ds + \frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}g(s,y(s),y(\theta s))dB(s) \\
+\;& \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_i(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}g(s,y(s),y(\theta s))dB(s)\Big] I_{[\xi_k,\xi_{k+1})}(t)\Big\|^p \le \varepsilon.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\Big\| y(t) - x(t)\Big\|^p \\
\le\;& 2^{p-1}\mathbb{E}\Big\| y(t) - \sum_{k=0}^{+\infty}\Big[\prod_{i=1}^{k} b_i(\tau_i)y_0 + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_i(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}f(s,y(s),y(\theta s))ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}f(s,y(s),y(\theta s))ds + \frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}g(s,y(s),y(\theta s))dB(s) \\
&+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_i(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}g(s,y(s),y(\theta s))dB(s)\Big] I_{[\xi_k,\xi_{k+1})}(t)\Big\|^p \\
&+2^{p-1}\mathbb{E}\Big\| \sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_i(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}\Big(f(s,y(s),y(\theta s)) - f(s,x(s),x(\theta s))\Big)ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}\Big(f(s,y(s),y(\theta s)) - f(s,x(s),x(\theta s))\Big)ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}\Big(g(s,y(s),y(\theta s)) - g(s,x(s),x(\theta s))\Big)dB(s) \\
&+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_i(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}\Big(g(s,y(s),y(\theta s)) - g(s,x(s),x(\theta s))\Big)dB(s)\Big] I_{[\xi_k,\xi_{k+1})}(t)\Big\|^p \\
\le\;& 2^{p-1}\varepsilon + 2^{p-1}\Lambda,
\end{aligned}$$

where

$$\begin{aligned}
\Lambda \;=\;& \mathbb{E}\Big\| \sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_i(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}\Big(f(s,y(s),y(\theta s)) - f(s,x(s),x(\theta s))\Big)ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}\Big(f(s,y(s),y(\theta s)) - f(s,x(s),x(\theta s))\Big)ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}\Big(g(s,y(s),y(\theta s)) - g(s,x(s),x(\theta s))\Big)dB(s) \\
&+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_i(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}\Big(g(s,y(s),y(\theta s)) - g(s,x(s),x(\theta s))\Big)dB(s)\Big] I_{[\xi_k,\xi_{k+1})}(t)\Big\|^p \\
\le\;& 2^{p-1}\mathbb{E}\Big\| \sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_i(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}\Big(f(s,y(s),y(\theta s)) - f(s,x(s),x(\theta s))\Big)ds \\
&+\frac{1}{\Gamma(\alpha)}\int_{\xi_k}^{t}(t-s)^{\alpha-1}\Big(f(s,y(s),y(\theta s)) - f(s,x(s),x(\theta s))\Big)ds\Big] I_{[\xi_k,\xi_{k+1})}(t)\Big\|^p \\
&+2^{p-1}\mathbb{E}\Big\| \sum_{k=0}^{+\infty}\Big[\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\prod_{j=1}^{k} b_i(\tau_j)\int_{\xi_{i-1}}^{\xi_i}(t-s)^{\alpha-1}\Big(g(s,y(s),y(\theta s)) - g(s,x(s),x(\theta s))\Big)dB(s)
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{\Gamma(\alpha)} \int_{\xi_k}^{t} (t-s)^{\alpha-1} \Big(g(s, y(s), y(\theta s)) - g(s, x(s), x(\theta s)) \Big) dB(s) \Big] I_{[\xi_k, \xi_{k+1})}(t) \Big\|^p \\
\leq \;& \frac{2^{p-1} \max\{1, M\} (b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2} \Big)^{\frac{p}{2}} \Big[\int_{t_0}^{t} \mathbb{E}\| f(s, y(s), y(\theta s)) - f(s, x(s), x(\theta s)) \|^p ds \\
& + \int_{t_0}^{t} \mathbb{E}\| g(s, y(s), y(\theta s)) - g(s, x(s), x(\theta s)) \|^p ds \Big] \\
\leq \;& 2^{p-1} \max\{1, M\} \frac{2(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2} \Big)^{\frac{p}{2}} (L_1 + L_2) \int_{t_0}^{t} \mathbb{E}\| y(s) - x(s) \|^p ds \\
= \;& \Lambda^* \int_{t_0}^{t} \mathbb{E}\| y(s) - x(s) \|^p ds,
\end{aligned}
$$

where $\Lambda^* = 2^{p-1} \max\{1, M\} \frac{2(b-\tau)^{p(\alpha-1)+1}}{(p(\alpha-1)+1)(\Gamma(\alpha))^p} \Big(\frac{p(p-1)}{2} \Big)^{\frac{p}{2}} (L_1 + L_2)$. Thus,

$$
\mathbb{E}\Big\| y(t) - x(t) \Big\|^p \leq 2^{p-1}\varepsilon + 2^{p-1}\Lambda^* \int_{t_0}^{t} \mathbb{E}\| y(s) - x(s) \|^p ds.
$$

By applying Gronwall's inequality, we obtain

$$
\mathbb{E}\Big\| y(t) - x(t) \Big\|^p \leq 2^{p-1}\varepsilon \cdot e^{2^{p-1}\Lambda^*},
$$

which implies that there exists a constant $\mathcal{M} = 2^{p-1} \cdot e^{2^{p-1}\Lambda^*}$ such that

$$
\mathbb{E}\Big\| y(t) - x(t) \Big\|^p \leq \mathcal{M}\varepsilon.
$$

Therefore, we deduce that system (1) is Hyers–Ulam stabile via Definition 4. The proof is complete. □

5. Example

In this section, an abstract example is given in the following to explain the obtained results.

Example 1. *Consider the following abstract system:*

$$
\begin{cases}
{}^c D_t^{\alpha} x(t) = \Big[\int_{-\tau}^{0} \tau_1(\theta) x(\theta t) d\theta \Big] dt + \Big[\int_{-\tau}^{0} \tau_2(\theta) x(\theta t) d\theta \Big] dB(t), t \geq t_0,\ t \neq \xi_k, \\
x(\xi_k) = b(k)\tau_k x(\xi_k^-),\ k = 1, 2, \ldots, \\
x_0 = \xi_k = \{\xi_\theta : -\tau \leq \theta \leq 0\},
\end{cases} \tag{8}
$$

where ${}^c D_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$. x is an $\mathcal{R}$-valued stochastic process and $0 < \theta < 1$. τ_k is a random variable defined from Ω to $D_k \equiv (0, d_k)$ for all $k = 1, 2, \ldots$ and $0 < d_k < +\infty$, and Ω is a nonempty set. Assume that τ_i and τ_j are independent of each other as $i \neq j = 1, 2, \ldots$. b is a function of k and $\tau_1, \tau_2 : [-\tau, 0] \to \mathcal{R}$ are continuous functions. The impulsive moments ξ_k satisfy $t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_k < \infty$, and $\xi_k = \xi_{k-1} + \tau_k$ for $k = 1, 2, \ldots$, $x(\xi_k^-) = \lim_{t \to \xi_k - 0} x(t)$. $B(t)$ is an n-dimensional Wiener process.

Assume the following conditions hold:

(i) $\max_{i,k}\{\prod_{j=i}^{k} \|b(j)(\tau_j)\|\}^2 < \infty$;

(ii) $\int_{-\tau}^{0} \tau_1^2(\theta) d\theta \leq \infty$, $\int_{-\tau}^{0} \tau_2^2(\theta) d\theta < \infty$.

In view of assumptions (i) and (ii), we easily obtain that the assumptions (H1)–(H4) hold. Thus, system (8) has a unique solution $x(t)$, which is Hyers–Ulam stable.

6. Conclusions

In this article, the existence, uniqueness and Hyers–Ulam stability of a class of fractional stochastic pantograph equations with random impulses are considered. Utilizing Schaefer's fixed point theorem and the Banach fixed point theorem, we obtain the criteria of existence and uniqueness for a solution of the considered system under relaxed linear growth conditions. Then, Hyers–Ulam stability is also derived for the considered equation using Gronwall's inequality. Moreover, some known existing equations are significantly generalized in our paper. In future work, we intend to study the corresponding exponential stability results for a class of fractional neutral stochastic pantograph equations with random impulses.

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References

1. Ockendon, J.; Taylor, A. The dynamics of a current collection system for an electric locomotive. *Proc. R. Soc. Lond.* **1971**, *322*, 447–468.
2. Iserles, A. Exact and discretized stability of the pantograph equations. *Appl. Numer. Math.* **1997**, *24*, 295–308. [CrossRef]
3. Guan, K.; Luo, Z. Stability results for impulsive pantograph equations. *Appl. Math. Lett.* **2013**, *26*, 1169–1174. [CrossRef]
4. Iserles, A. On the generalized pantograph functional-differential equation. *Eur. J. Appl. Math.* **1993**, *4*, 1–38. [CrossRef]
5. Fan, Z.; Liu, M.; Cao, W. Existence and uniqueness of the solutions and convergence of semi-implicit Euler methods for stochastic pantograph equations. *J. Math. Anal. Appl.* **2007**, *325*, 1142–1159. [CrossRef]
6. Fan, Z.; Song, M.; Liu, M. The αth moment stability for the stochastic pantograph equation. *J. Comput. Appl. Math.* **2009**, *233*, 109–120. [CrossRef]
7. Zhang, H.; Xiao, Y.; Guo, F. Convergence and stability of a numerical method for nonlinear stochastic pantograph equations. *J. Frankl. Inst.* **2014**, *351*, 3089–3103. [CrossRef]
8. Yang, H.; Yang, Z.; Wang, P.; Han, D. Mean-square stability analysis for nonlinear stochastic pantograph equations by transformation approach. *J. Math. Anal. Appl.* **2019**, *479*, 977–986. [CrossRef]
9. Mao, W.; Hu, L.; Mao, X. The existence and asymptotic estimations of solutions to stochastic pantograph equations with diffusion and Lévy jumps. *Appl. Math. Comput.* **2015**, *268*, 883–896. [CrossRef]
10. Hu, L.; Ren, Y.; He, Q. Pantograph stochastic differential equations driven by G-Brownian motion. *J. Math. Anal. Appl.* **2019**, *480*, 123381. [CrossRef]
11. Samoilenko, A.; Perestyuk, N. *Impulsive Differential Equations*; World Scientific: Singapore, 1995.
12. Wu, S. The Euler scheme for random impulsive differential equations. *Appl. Math. Comput.* **2007**, *191*, 164–175. [CrossRef]
13. Wu, S.; Guo, X.; Zhou, Y. *P*-moment stability of functional differential equations with random impulses. *Comput. Math. Appl.* **2006**, *52*, 1683–1694. [CrossRef]
14. Gowrisankar, M.; Mohankumar, P.; Vinodkumar, A. Stability results of random impulsive semilinear differential equations. *Acta Math. Sci.* **2014**, *4*, 1055–1071. [CrossRef]
15. Vinodkumar, A.; Gowrisankar, M.; Mohankumar, P. Existence, uniqueness and stability of random impulsive neutral partial differential equations. *J. Egypt. Math. Soc.* **2015**, *23*, 31–36. [CrossRef]
16. Anguraj, A.; Vinodkumar, A.; Malar, K. Existence and stability results for random impulsive fractional pantograph equations. *Filomat* **2016**, *14*, 3839–3854. [CrossRef]
17. Andres, J. Application of the randomized sharkovsky-type theorems to random impulsive differential equations and inclusions. *J. Dyn. Differ. Equ.* **2019**, *31*, 2127–2144. [CrossRef]
18. Li, S.; Shu, L.; Shu, X.; Xu, F. Existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays. *Stochastics* **2019**, *91*, 857–872. [CrossRef]
19. Priyadharsini, J.; Balasubramaniam, P. Solvability of fuzzy fractional stochastic Pantograph differential system. *Iran. J. Fuzzy Syst.* **2022**, *19*, 47–60.

20. Luo, D.; Luo, Z. Existence and Hyers-Ulam stability results for a class of fractional order delay differential equations with non-instantaneous impulses. *Math. Slovaca* **2020**, *70*, 1231–1248. [CrossRef]
21. Wang, X.; Luo, D.; Zhu, Q. Ulam-Hyers stability of Caputo type fuzzy fractional differential equatons with time-delays. *Chaos Solitons Fractals* **2022**, *156*, 111822. [CrossRef]
22. Luo, D.; Abdeljawad, T.; Luo, Z. Ulam-Hyers stability results for a novel nonlinear nable Caputo fractional variable-order difference system. *Turk. J. Math.* **2021**, *45*, 456–470. [CrossRef]
23. Kilbas, A.; Srivastava, M.; Trujillo, J. *Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies*; Elsevier Science B.V.: Amsterdam, The Netherland, 2006.
24. Samko, S.; Kilbas, A.; Marichev, O. *Fractional Integrals and Derivatives-Theory and Applications*; Gordon and Breach Science Publishers: New York, NY, USA, 1987.
25. Prato, G.; Zabczyk, J. *Stochastic Equations in Infinite Dimensions*; Cambridge University Press: Cambridge, UK, 1992.
26. Ahmadova, A.; Mahmudov, N. Ulam-Hyers stability of Caputo type fractional stochastic neutral differential equations. *Stat. Probab. Lett.* **2021**, *168*, 108949. [CrossRef]