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A Study on Fixed-Point Techniques under the α - F -Convex Contraction with an Application

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Abstract: In this paper, we consider several classes of mappings related to the class of α - F -contraction mappings by introducing a convexity condition and establish some fixed-point theorems for such mappings in complete metric spaces. The present result extends and generalizes the well-known results of Singh, Khan, and Kang (Mathematics, 2018, 6(6), 105), Istrăţescu (Liberta Math., 1981, 1, 151–163), and many others in the existing literature. An illustrative example is also provided to exhibit the utility of our main results. Finally, we derive the existence and uniqueness of a solution to an integral equation to support our main result and give a numerical example to validate the application of our obtained results.

Keywords: α -admissible; F -contraction; fixed point; α^* -admissible; α - F -convex contraction; integral equation



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1. Introduction and Preliminaries

In recent years, a great number of papers have presented generalizations of the well-known Banach–Picard contraction principle. Ćirić [1] investigated the generalized contraction and extension of Banach’s contraction to combine x , y , Tx , and Ty of all six possible values for all $x, y \in X$ and T a self-mapping on a metric space. In 1982, Istrăţescu [2] proposed a generalization of seven contraction principle values by introducing a “convexity” condition for the mapping iterates. He deduced that these conditions might be adapted for other classes of mappings to obtain some extensions of known fixed-point results. Alghamdi et al. [3] proved a generalization of the Banach contraction principle to the class of convex contractions in non-normal cone metric spaces. In 2015, Miculescu et al. [4] obtained a generalization of Istrăţescu’s fixed-point theorem concerning convex contractions. In 2017, Miculescu et al. [5] obtained a generalization of Matkowski’s fixed-point theorem and Istrăţescu’s fixed-point theorem of convex contraction of a comparison function ϕ such that $d(f^{[m]}(x), f^{[m]}(y)) \leq \phi(\max\{d(x, y), d(f(x), f(y)), \dots, d(f^{[m-1]}(x), f^{[m-1]}(y))\})$ for all $x, y \in X$. Latif et al. [6] introduced the new concepts of partial generalized convex contractions and partial generalized convex contractions of order two. Moreover, they established some approximate fixed-point results in a metric space endowed with an arbitrary binary relation and some approximate fixed-point results in a metric space endowed with a graph. In 2022, Latif et al. [7] established fixed points in the setting of metric spaces for generalized multivalued contractive mappings with respect to the w_b -distance. In 2013, Miandaragh et al. [8] expanded the concept of convex contractions to generalized convex contractions and generalized convex contractions of order two. In the same year, they proved some approximate fixed-point results in the setting of generalized α -convex contractive mapping

in [9]. Wardowski [10] introduced the F-contraction and proved a new fixed-point theorem concerning the F-contraction. Samet et al. [11] introduced a new concept of α - ψ -contractive-type mappings and established fixed-point theorems for such mappings in complete metric spaces. Asem and Singh proved some fixed-point theorems on a Meir–Keeler proximal contraction for non-self-mappings in [12]. Following that [13,14], Karapinar established some fixed-point theorems in different metric spaces for the concept of α -admissible mapping. Recently, Khan et al. [15] discussed the concepts of (α, p) -convex contraction and asymptotically T^2 -regular sequence and demonstrated that an (α, p) -convex contraction reduced to a two-sided convex contraction. Yildirim [16] introduced a definition of the F-Hardy–Rogers contraction of Nadler type and also proved some fixed-point theorems for such mappings using Mann’s iteration process in complete convex b -metric spaces. Singh et al. [17] discussed an α - F -convex contraction of six possible values (without rational type) in complete metric spaces. Eke et al. [18] introduced the convexity condition to some classes of contraction mappings, such as the Chatterjea contractive mapping and the Hardy and Rogers contractive mapping, and proved the fixed points of these maps in complete metric spaces. Following that, some works on the generalization of such classes of mappings in the setting of various spaces [19–34] appeared.

Singh et al. [17] discussed an α - F -convex contraction of six possible values (without rational type) and proved the fixed points of these maps in complete metric spaces. In this paper, we extend and generalize their main theorem into an α - F -convex contraction of seven possible values (with rational type) in the setting of complete metric spaces inspired and motivated by Singh et al. [17]. Examples and applications to integral equations are provided to illustrate the usability of our obtained results.

Throughout this paper, we use the following notations: \mathbb{R} represents $(-\infty, +\infty)$, \mathbb{R}_+ is $(0, +\infty)$, and \mathbb{R}_+^0 represents $[0, +\infty)$.

Definition 1. [11] Let $\Gamma : \Lambda \rightarrow \Lambda$ be a self-mapping on a nonvoid set Λ and $\alpha : \Lambda \times \Lambda \rightarrow [0, \infty)$ be a mapping. Then, Γ is said to be α -admissible if for all $\eta, m \in \Lambda, \alpha(\eta, m) \geq 1 \Rightarrow \alpha(\Gamma\eta, \Gamma m) \geq 1$.

Example 1. Let $\Lambda = (-\infty, \infty)$ and define $\Gamma : \Lambda \rightarrow \Lambda$ by

$$\Gamma\eta = \begin{cases} \ln|\eta|, & \text{if } \eta \neq 0; \\ 3, & \text{else.} \end{cases}$$

Define $\alpha : \Lambda \times \Lambda \rightarrow [0, \infty)$ by

$$\alpha(\eta, m) = \begin{cases} 3, & \text{if } \eta \geq m; \\ 0, & \text{else.} \end{cases}$$

Then, Γ is α -admissible as $\alpha(\eta, m) \geq 1 \Rightarrow \alpha(\Gamma\eta, \Gamma m) \geq 1$ for $\eta \geq m$ and $\alpha(\eta, m) = \alpha(m, \eta)$, for all $\eta = m$.

Definition 2. [13] Let $\Gamma : \Lambda \rightarrow \Lambda$ be a self-mapping and $\alpha : \Lambda \times \Lambda \rightarrow (-\infty, +\infty)$ be a mapping. Then, we say that Γ is triangular α -admissible if

- (Γ_1) $\alpha(\eta, m) \geq 1 \Rightarrow \alpha(\Gamma\eta, \Gamma m) \geq 1$, for all $\eta, m \in \Lambda$;
- (Γ_2) $\alpha(\eta, o) \geq 1$ and $\alpha(o, m) \geq 1$ imply $\alpha(\eta, m) \geq 1$, for all $\eta, m, o \in \Lambda$.

Example 2. Let $\Lambda = [0, +\infty)$, $\Gamma\eta = \eta^2 + e^\eta$ and

$$\alpha(\eta, m) = \begin{cases} 1, & \text{if } \eta, m \in [0, 1]; \\ 0, & \text{else.} \end{cases}$$

Hence, Γ is a triangular α -admissible mapping.

Definition 3. [14] Let $\Lambda \neq \emptyset$ and let Γ be an α -admissible mapping on Λ . Then, Λ has the property (H) if for each $\eta, m \in \text{Fix}(\Gamma)$, there exists $o \in \Lambda$ such that $\alpha(\eta, o) \geq 1$ and $\alpha(o, m) \geq 1$.

Definition 4. [17] An α -admissible mapping Γ is said to be α^* -admissible, if for each $\eta, m \in \text{Fix}(\Gamma) \neq \emptyset$, we get $\alpha(\eta, m) \geq 1$. If $\text{Fix}(\Gamma) = \emptyset$, then Γ is called vacuously α^* -admissible.

Example 3. Let $\Lambda = [0, \infty)$ and $\Gamma : \Lambda \rightarrow \Lambda$ by $\Gamma\eta = 3 + \eta$, for all $\eta \in \Lambda$. Define a mapping $\alpha : \Lambda \times \Lambda \rightarrow [0, \infty)$ by

$$\alpha(\eta, m) = \begin{cases} e^{(\eta-m)}, & \text{if } \eta \geq m; \\ 0, & \text{else.} \end{cases}$$

Then, Γ is α -admissible. Since Γ has no fixed point, we have $\text{Fix}(\Gamma) = \emptyset$, and Γ is vacuously α^* -admissible.

Example 4. Let $\Lambda = [0, \infty)$ and $\Gamma : \Lambda \rightarrow \Lambda$ by $\Gamma\eta = \frac{\eta^3}{9}$, for all $\eta \in \Lambda$. Define $\alpha : \Lambda \times \Lambda \rightarrow [0, \infty)$ by

$$\alpha(\eta, m) = \begin{cases} 1, & \text{if } \eta, m \in [0, 3]; \\ 0, & \text{else.} \end{cases}$$

Clearly, Γ is α -admissible and $\text{Fix}(\Gamma) = \{0, 3\}$. Then, Γ is α^* -admissible.

Example 5. Let $\Lambda = [0, \infty)$ and define $\Gamma : \Lambda \rightarrow \Lambda$ by $\Gamma\eta = \sqrt{\frac{\eta(\eta^2+6)}{5}}$, for all $\eta \in \Lambda$. Define $\alpha : \Lambda \times \Lambda \rightarrow [0, \infty)$ by

$$\alpha(\eta, m) = \begin{cases} 1, & \text{if } \eta, m \in [0, 2]; \\ 0, & \text{else.} \end{cases}$$

Clearly, Γ is α -admissible and $\text{Fix}(\Gamma) = \{0, 2, 3\}$. Note that Γ is not α^* -admissible, since $\alpha(\eta, 3) = 0$ for $\eta \in \{0, 2\}$.

Definition 5. [10] For a nonvoid set Λ , a function $\mathcal{Q} : \Lambda \times \Lambda \rightarrow \mathbb{R}_+^0$ is said to be metric if it satisfies the following conditions:

1. $\mathcal{Q}(\eta, m) \geq 0$ and $\mathcal{Q}(\eta, m) = 0$ if and only if $\eta = m$.
2. $\mathcal{Q}(\eta, m) = \mathcal{Q}(m, \eta)$, for all $\eta, m \in \Lambda$.
3. $\mathcal{Q}(\eta, m) \leq \mathcal{Q}(\eta, o) + \mathcal{Q}(o, m)$, for all $\eta, m, o \in \Lambda$.

In addition, the pair (Λ, \mathcal{Q}) is called a metric space.

Definition 6. [10] Let $F \in \mathfrak{S}$ be the set of all mappings $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfying the stipulations:

- (F₁) F is strictly nondecreasing, i.e., for all $\delta, \epsilon \in \mathbb{R}_+$ such that $\delta < \epsilon \implies F(\delta) < F(\epsilon)$;
- (F₂) For each sequence $\{\delta_\beta\}_{\beta \in \mathbb{N}}$, $\lim_{\beta \rightarrow \infty} \delta_\beta = 0 \iff \lim_{\beta \rightarrow \infty} F(\delta_\beta) = -\infty$;
- (F₃) There exists $k \in (0, 1)$ such that $\lim_{\delta \rightarrow 0^+} \delta^k F(\delta) = 0$.

Definition 7. [10] A mapping $\Gamma : \Lambda \rightarrow \Lambda$ is said to be an F -contraction on a metric space (Λ, \mathcal{Q}) if there exists $F \in \mathfrak{S}$ and $\mu > 0$ such that for all $\eta, m \in \Lambda$,

$$\mathcal{Q}(\Gamma\eta, \Gamma m) > 0 \implies \mu + F(\mathcal{Q}(\Gamma\eta, \Gamma m)) \leq F(\mathcal{Q}(\eta, m)). \tag{1}$$

Example 6. [10] The following functions $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ are in \mathfrak{S} :

- (i) $F(\delta) = \ln\delta$;
- (ii) $F(\delta) = \ln\delta + \delta$;

- (iii) $F(\delta) = \frac{-1}{\sqrt{\delta}}$;
- (iv) $F(\delta) = \ln(\delta^2 + \delta)$.

The following theorem was proved by Wardowski [10].

Theorem 1. [10] Let (Λ, \mathcal{Q}) be a complete metric space and $\Gamma : \Lambda \rightarrow \Lambda$ be an F -contraction. Then, we get

- (i) $\mathfrak{o} \in \Lambda$ is the unique fixed point of Γ ;
- (ii) For all $\eta \in \Lambda$, the sequence $\{\Gamma^{\beta}\eta\}$ is convergent to $\mathfrak{o} \in \Lambda$.

Definition 8. [1] Let Γ be a self-mapping on a metric space (Λ, \mathcal{Q}) . Then, we say that Γ is orbitally continuous on Λ if $\lim_{\mathfrak{k} \rightarrow \infty} \Gamma^{\beta\mathfrak{k}}\eta = \mathfrak{o}$ implies that $\lim_{\mathfrak{k} \rightarrow \infty} \Gamma^{\beta\mathfrak{k}}\eta = \Gamma\mathfrak{o}$.

Let $\Gamma : \Lambda \rightarrow \Lambda$ be a self-mapping on a nonvoid set Λ . Define $\text{Fix}(\Gamma) = \{\eta : \Gamma\eta = \eta, \text{ for all } \eta \in \Lambda\}$.

We establish some fixed-point theorems on an α - F -convex contraction of possible seven values included rational type with an application to integral equations, inspired by Singh et al. [17].

2. Main Results

First, we introduce the concept of “ α - F -convex contraction” with examples.

Definition 9. A self-mapping Γ on Λ is said to be an α - F -convex contraction, if there exist two functions $\alpha : \Lambda \times \Lambda \rightarrow [0, \infty)$ and $F \in \mathfrak{S}$ such that for all $\eta, \mathfrak{m} \in \Lambda$,

$$\mathcal{Q}^b(\Gamma^2\eta, \Gamma^2\mathfrak{m}) > 0 \implies \mu + F(\alpha(\eta, \mathfrak{m})\mathcal{Q}^b(\Gamma^2\eta, \Gamma^2\mathfrak{m})) \leq F(\mathfrak{p}^b(\eta, \mathfrak{m})), \tag{2}$$

where $b \in [1, \infty)$, $\mu > 0$ and

$$\mathfrak{p}^b(\eta, \mathfrak{m}) = \max\left\{ \mathcal{Q}^b(\eta, \mathfrak{m}), \mathcal{Q}^b(\eta, \Gamma\eta), \mathcal{Q}^b(\Gamma\eta, \Gamma^2\eta), \mathcal{Q}^b(\mathfrak{m}, \Gamma\mathfrak{m}), \mathcal{Q}^b(\Gamma\mathfrak{m}, \Gamma^2\mathfrak{m}), \frac{\mathcal{Q}^b(\eta, \Gamma\mathfrak{m}) + \mathcal{Q}^b(\mathfrak{m}, \Gamma\eta)}{2}, \frac{\mathcal{Q}^b(\Gamma\eta, \Gamma^2\mathfrak{m}) + \mathcal{Q}^b(\Gamma\mathfrak{m}, \Gamma^2\eta)}{2} \right\}. \tag{3}$$

Example 7. Let $F(\mathfrak{x}) = \ln(\mathfrak{x}), \mathfrak{x} > 0$. Obviously, $F \in \mathfrak{S}$. Let Γ be a self-mapping on a metric space (Λ, \mathcal{Q}) . We postulate that the convex contraction of type 2 ([2]) putting $\alpha(\eta, \mathfrak{m}) = 1$, for all $\eta, \mathfrak{m} \in \Lambda, \mathbf{e}^{-\mu} = \mathbf{k} = \sum_{\mathfrak{k}=1}^7 \alpha_{\mathfrak{k}} < 1$ and $\alpha_{\mathfrak{k}} \geq 0$ for all $\mathfrak{k} = 1, 2, \dots, 7$.

$$\begin{aligned} \mathcal{Q}(\Gamma^2\eta, \Gamma^2\mathfrak{m}) &\leq \alpha_1\mathcal{Q}(\eta, \mathfrak{m}) + \alpha_2\mathcal{Q}(\eta, \Gamma\eta) + \alpha_3\mathcal{Q}(\Gamma\eta, \Gamma^2\eta) \\ &\quad + \alpha_4\mathcal{Q}(\mathfrak{m}, \Gamma\mathfrak{m}) + \alpha_5\mathcal{Q}(\Gamma\mathfrak{m}, \Gamma^2\mathfrak{m}) + \alpha_6\left(\frac{\mathcal{Q}(\eta, \Gamma\mathfrak{m}) + \mathcal{Q}(\mathfrak{m}, \Gamma\eta)}{2}\right) \\ &\quad + \alpha_7\left(\frac{\mathcal{Q}(\Gamma\eta, \Gamma^2\mathfrak{m}) + \mathcal{Q}(\Gamma\mathfrak{m}, \Gamma^2\eta)}{2}\right), \end{aligned}$$

where $\eta, \mathfrak{m} \in \Lambda$ with $\eta \neq \mathfrak{m}$. Then, we obtain

$$\begin{aligned} \alpha(\eta, \mathfrak{m})\mathcal{Q}(\Gamma^2\eta, \Gamma^2\mathfrak{m}) &= \mathcal{Q}(\Gamma^2\eta, \Gamma^2\mathfrak{m}) \\ &\leq \sum_{\mathfrak{k}=1}^7 \alpha_{\mathfrak{k}} \max\left\{ \mathcal{Q}(\eta, \mathfrak{m}), \mathcal{Q}(\eta, \Gamma\eta), \mathcal{Q}(\Gamma\eta, \Gamma^2\eta), \mathcal{Q}(\mathfrak{m}, \Gamma\mathfrak{m}), \mathcal{Q}(\Gamma\mathfrak{m}, \Gamma^2\mathfrak{m}), \right. \\ &\quad \left. \frac{\mathcal{Q}(\eta, \Gamma\mathfrak{m}) + \mathcal{Q}(\mathfrak{m}, \Gamma\eta)}{2}, \frac{\mathcal{Q}(\Gamma\eta, \Gamma^2\mathfrak{m}) + \mathcal{Q}(\Gamma\mathfrak{m}, \Gamma^2\eta)}{2} \right\}, \end{aligned}$$

which implies that

$$\alpha(\eta, m) \mathcal{Q}(\Gamma^2 \eta, \Gamma^2 m) \leq \mathbf{kp}^1(\eta, m) = e^{-\mu} \mathbf{p}^1(\eta, m).$$

Applying the natural logarithm on both sides, we get

$$\mu + F(\alpha(\eta, m) \mathcal{Q}(\Gamma^2 \eta, \Gamma^2 m)) \leq F(\mathbf{p}^1(\eta, m)).$$

Therefore,

$$\mathcal{Q}(\Gamma^2 \eta, \Gamma^2 m) > 0 \implies \mu + F(\alpha(\eta, m) \mathcal{Q}(\Gamma^2 \eta, \Gamma^2 m)) \leq F(\mathbf{p}^1(\eta, m)),$$

for all $\eta, m \in \Lambda$. We conclude that Γ is an α - F -convex contraction with $b = 1$.

Example 8. Let $\Lambda = [0, 1]$ with $\mathcal{Q}(\eta, m) = |\eta - m|$. Define a mapping $\Gamma : \Lambda \rightarrow \Lambda$ by $\Gamma \eta = \frac{\eta^2}{2} + \frac{5}{16}$, for all $\eta \in \Lambda$ with $\alpha(\eta, m) = 1$, for all $\eta, m \in \Lambda$. Then, Γ is α -admissible. Now, we get Γ is nonexpansive, since we obtain

$$|\Gamma \eta - \Gamma m| = \frac{1}{2} |\eta^2 - m^2| \leq |\eta - m|, \text{ for all } \eta, m \in \Lambda.$$

Setting $F \in \mathfrak{S}$ such that $F(x) = \ln x, x > 0$. Then, for all $\eta, m \in \Lambda$ with $\eta \neq m$, we obtain

$$\begin{aligned} \alpha(\eta, m) |\Gamma^2 \eta, \Gamma^2 m| &= |\Gamma^2 \eta, \Gamma^2 m| \\ &= \frac{1}{512} (|(64\eta^4 + 80\eta^2 - 64m^4 - 80m^2)|) \\ &\leq \frac{1}{512} (64|\eta^4 - m^4| + 80|\eta^2 - m^2|) \\ &\leq \frac{1}{2} |\Gamma \eta - \Gamma m| + \frac{5}{16} |\eta - m| \\ &\leq \frac{13}{16} \max\{|\Gamma \eta - \Gamma m|, |\eta - m|\} \\ &\leq e^{-\mu} \mathbf{p}^1(\eta, m), \end{aligned}$$

where $-\mu = \ln(\frac{13}{16})$. Applying the logarithm on both sides, we have

$$\mu + F(\alpha(\eta, m) \mathcal{Q}(\Gamma^2 \eta, \Gamma^2 m)) \leq F(\mathbf{p}^1(\eta, m)).$$

We conclude that Γ is an α - F -convex contraction with $b = 1$.

Example 9. Define $\Gamma : [0, 1] \rightarrow [0, 1]$ by $\Gamma \eta = \frac{1-\eta^2}{2}$, for all $\eta \in [0, 1]$ and $\alpha(\eta, m) = 1$, for all $\eta, m \in [0, 1]$, with usual metric $\mathcal{Q}(\eta, m) = |\eta - m|$. Then, Γ is α -admissible. Setting $F \in \mathfrak{S}$ such that $F(x) = \ln x, x > 0$. Then, for all $\eta, m \in [0, 1]$ with $\eta \neq m$, we obtain

$$|\Gamma \eta, \Gamma m| = \frac{1}{2} |\eta^2 - m^2| \leq |\eta - m|$$

and

$$\begin{aligned} \alpha(\eta, m)|\Gamma^2\eta, \Gamma^2m| &= |\Gamma^2\eta, \Gamma^2m| \\ &= \frac{1}{8}(|2\eta^2 - \eta^4 - 2m^2 + m^4|) \\ &= \frac{1}{8}(2|\eta^2 - m^2| + |\eta^4 - m^4|) \\ &\leq \frac{1}{2}|\eta - m| + \frac{1}{4}|\eta^2 - m^2| \\ &= \frac{1}{2}|\eta - m| + \frac{1}{2}|\Gamma\eta - \Gamma m| \\ &\leq 1. \max\{|\eta - m|, |\Gamma\eta - \Gamma m|\} \\ &\leq e^{-\mu}p^1(\eta, m). \end{aligned}$$

where $-\mu = \ln(1)$. Applying the logarithm on both sides, we get

$$\mu + F(\alpha(\eta, m)\mathcal{Q}(\Gamma^2\eta, \Gamma^2m)) \leq F(p^1(\eta, m)).$$

However, by $\mu = -\ln(1)$, there does not exist any $\mu > 0$ such that

$$\mu + F(\alpha(\eta, m)\mathcal{Q}(\Gamma^2\eta, \Gamma^2m)) \leq F(p^1(\eta, m)).$$

Therefore, Γ is not an α - F -convex contraction with $b = 1$. Now, we see

$$|\Gamma\eta - \Gamma m|^2 = \frac{1}{4}|\eta^2 - m^2|^2 \leq |\eta - m|^2$$

and

$$\begin{aligned} \alpha(\eta, m)(|\Gamma^2\eta, \Gamma^2m|^2) &= \frac{1}{64}|2\eta^2 - \eta^4 - 2m^2 + m^4|^2 \\ &\leq \frac{1}{64}(4|\eta^2 - m^2|^2 + |\eta^4 - m^4|^2) \\ &= \frac{1}{16}|\eta^2 - m^2|^2 + \frac{1}{64}|\eta^4 - m^4|^2 \\ &\leq \frac{1}{4}|\eta - m|^2 + \frac{1}{16}|\eta^2 - m^2|^2 \\ &\leq \frac{5}{16} \max\{|\eta - m|^2, |\Gamma\eta - \Gamma m|^2\} \\ &\leq \frac{5}{16}p^2(\eta, m) \\ &= e^{-\mu}p^2(\eta, m). \end{aligned}$$

Applying the logarithm on both sides, we get

$$\mu + F(\alpha(\eta, m)\mathcal{Q}(\Gamma^2\eta, \Gamma^2m)) \leq F(p^2(\eta, m)),$$

where $-\mu = \ln \frac{5}{16}$. Therefore, Γ is an α - F -convex contraction with $b = 2$.

Now, first, we prove the Lemma through an α - F -convex contraction.

Lemma 1. Let (Λ, \mathcal{Q}) be a complete metric space, $\Gamma : \Lambda \rightarrow \Lambda$ a given map, and let $\alpha : \Lambda \times \Lambda \rightarrow [0, \infty)$ be a mapping. Suppose that the following affirmations hold:

(i) There exists $b \in [1, \infty)$ and $\mu > 0$ such that for all $\eta, m \in \Lambda$,

$$\mathcal{Q}^b(\Gamma^2\eta, \Gamma^2m) > 0 \implies \mu + F(\alpha(\eta, m)\mathcal{Q}^b(\Gamma^2\eta, \Gamma^2m)) \leq F(p^b(\eta, m)),$$

where

$$p^b(\eta, m) = \max\left\{Q^b(\eta, m), Q^b(\eta, \Gamma\eta), Q^b(\Gamma\eta, \Gamma^2\eta), Q^b(m, \Gamma m), Q^b(\Gamma m, \Gamma^2 m), \frac{Q^b(\eta, \Gamma m) + Q^b(m, \Gamma\eta)}{2}, \frac{Q^b(\Gamma\eta, \Gamma^2 m) + Q^b(\Gamma m, \Gamma^2\eta)}{2}\right\}.$$

- (ii) Γ is α -admissible;
- (iii) There exists $\eta_0 \in \Lambda$ such that $\alpha(\eta_0, \Gamma\eta_0) \geq 1$.

Define a sequence $\{\eta_\beta\}$ in Λ by $\eta_{\beta+1} = \Gamma\eta_\beta = \Gamma^{\beta+1}\eta_0$, for all $\beta \geq 0$, then $\{Q^b(\eta_\beta, \eta_{\beta+1})\}$ is a strictly decreasing sequence in Λ .

Proof. Let $\eta_0 \in \Lambda$ be such that $\alpha(\eta_0, \Gamma\eta_0) \geq 1$ and define a sequence $\{\eta_\beta\}$ by $\eta_{\beta+1} = \Gamma\eta_\beta$, for all $\beta \in \mathbb{N} \cup \{0\}$. By (ii), we have

$$\alpha(\eta_0, \eta_1) = \alpha(\eta_0, \Gamma\eta_0) \geq 1 \Rightarrow \alpha(\eta_2, \eta_3) = \alpha(\Gamma\eta_1, \Gamma^2\eta_0) \geq 1.$$

Inductively, we obtain $\alpha(\eta_\beta, \eta_{\beta+1}) \geq 1$, for all $\beta \geq 0$. Postulating that $\eta_\beta \neq \eta_{\beta+1}$ for all $\beta \geq 0$, then $Q^b(\eta_\beta, \eta_{\beta+1}) > 0$, for all $\beta \geq 0$. Let $v = \max\{Q^b(\eta_0, \eta_1), Q^b(\eta_1, \eta_2)\}$. From (3), taking $\eta = \eta_0$ and $m = \eta_1$, we obtain

$$\begin{aligned} p^b(\eta_0, \eta_1) &= \max\left\{Q^b(\eta_0, \eta_1), Q^b(\eta_0, \Gamma\eta_0), Q^b(\Gamma\eta_0, \Gamma^2\eta_0), Q^b(\eta_1, \Gamma\eta_1), Q^b(\Gamma\eta_1, \Gamma^2\eta_1), \frac{Q^b(\eta_0, \Gamma\eta_1) + Q^b(\eta_1, \Gamma\eta_0)}{2}, \frac{Q^b(\Gamma\eta_0, \Gamma^2\eta_1) + Q^b(\Gamma\eta_1, \Gamma^2\eta_0)}{2}\right\} \\ &= \max\left\{Q^b(\eta_0, \eta_1), Q^b(\eta_0, \eta_1), Q^b(\eta_1, \eta_2), Q^b(\eta_1, \eta_2), Q^b(\eta_2, \eta_3), \frac{Q^b(\eta_0, \eta_2) + Q^b(\eta_1, \eta_1)}{2}, \frac{Q^b(\eta_1, \eta_3) + Q^b(\eta_2, \eta_2)}{2}\right\} \\ &= \max\left\{Q^b(\eta_0, \eta_1), Q^b(\eta_1, \eta_2), Q^b(\eta_2, \eta_3), \frac{Q^b(\eta_0, \eta_2)}{2}, \frac{Q^b(\eta_1, \eta_3)}{2}\right\} \\ &= \max\{Q^b(\eta_0, \eta_1), Q^b(\eta_1, \eta_2), Q^b(\eta_2, \eta_3)\}. \end{aligned} \tag{4}$$

By (F_1) and $\alpha(\eta_0, \eta_1) \geq 1$, by (2) and (4), we obtain

$$\begin{aligned} F(Q^b(\eta_2, \eta_3)) &= F(Q^b(\Gamma^2\eta_0, \Gamma^2\eta_1)) \\ &\leq F(\alpha(\eta_0, \eta_1)Q^b(\Gamma^2\eta_0, \Gamma^2\eta_1)) \\ &\leq F(p^b(\eta_0, \eta_1)) - \mu \\ &= F(\max\{Q^b(\eta_0, \eta_1), Q^b(\eta_1, \eta_2), Q^b(\eta_2, \eta_3)\}) - \mu \\ &\leq F(\max\{v, Q^b(\eta_2, \eta_3)\}) - \mu. \end{aligned} \tag{5}$$

If $\max\{v, Q^b(\eta_2, \eta_3)\} = Q^b(\eta_2, \eta_3)$, then (5) gives

$$F(Q^b(\eta_2, \eta_3)) \leq F(Q^b(\eta_2, \eta_3)) - \mu < F(Q^b(\eta_2, \eta_3)).$$

This is a contradiction. It follows that

$$F(Q^b(\eta_2, \eta_3)) \leq F(v) - \mu < F(v).$$

Since $\mu > 0$ and by (F_1) , we have

$$Q^b(\eta_2, \eta_3) < v = \max\{Q^b(\eta_0, \eta_1), Q^b(\eta_1, \eta_2)\}.$$

Again, by (3) with $\eta = \eta_1$ and $m = \eta_2$, we get

$$\begin{aligned}
 p^b(\eta_1, \eta_2) &= \max\{Q^b(\eta_1, \eta_2), Q^b(\eta_1, \Gamma\eta_1), Q^b(\Gamma\eta_1, \Gamma^2\eta_1), Q^b(\eta_2, \Gamma\eta_2), Q^b(\Gamma\eta_2, \Gamma^2\eta_2), \\
 &\quad \frac{Q^b(\eta_1, \Gamma\eta_2) + Q^b(\eta_2, \Gamma\eta_1)}{2}, \frac{Q^b(\Gamma\eta_1, \Gamma^2\eta_2) + Q^b(\Gamma\eta_2, \Gamma^2\eta_1)}{2}\} \\
 &= \max\{Q^b(\eta_1, \eta_2), Q^b(\eta_1, \eta_2), Q^b(\eta_2, \eta_3), Q^b(\eta_2, \eta_3), Q^b(\eta_3, \eta_4), \\
 &\quad \frac{Q^b(\eta_1, \eta_3) + Q^b(\eta_2, \eta_2)}{2}, \frac{Q^b(\eta_2, \eta_4) + Q^b(\eta_3, \eta_3)}{2}\} \\
 &= \max\{Q^b(\eta_1, \eta_2), Q^b(\eta_2, \eta_3), Q^b(\eta_3, \eta_4), \frac{Q^b(\eta_1, \eta_3)}{2}, \frac{Q^b(\eta_2, \eta_4)}{2}\} \\
 &= \max\{Q^b(\eta_1, \eta_2), Q^b(\eta_2, \eta_3), Q^b(\eta_3, \eta_4)\}.
 \end{aligned} \tag{6}$$

By (2) and (6), we obtain

$$\begin{aligned}
 F(Q^b(\eta_3, \eta_4)) &= F(Q^b(\Gamma^2\eta_1, \Gamma^2\eta_2)) \\
 &\leq F(\alpha(\eta_1, \eta_2)Q^b(\Gamma^2\eta_1, \Gamma^2\eta_2)) \\
 &\leq F(p^b(\eta_1, \eta_2)) - \mu \\
 &= F(\max\{Q^b(\eta_1, \eta_2), Q^b(\eta_2, \eta_3), Q^b(\eta_3, \eta_4)\}) - \mu.
 \end{aligned}$$

If $\max\{Q^b(\eta_1, \eta_2), Q^b(\eta_2, \eta_3), Q^b(\eta_3, \eta_4)\} = Q^b(\eta_3, \eta_4)$, then we obtain

$$F(Q^b(\eta_3, \eta_4)) \leq F(Q^b(\eta_3, \eta_4)) - \mu < F(Q^b(\eta_3, \eta_4)),$$

which is a contradiction. We obtain

$$\max\{Q^b(\eta_1, \eta_2), Q^b(\eta_2, \eta_3)\} > Q^b(\eta_3, \eta_4).$$

Therefore,

$$v > Q^b(\eta_2, \eta_3) > Q^b(\eta_3, \eta_4).$$

Inductively, continuing in this way, we prove that the sequence $\{Q^b(\eta_\beta, \eta_{\beta+1})\}$ is strictly decreasing in Λ . \square

Theorem 2. Let (Λ, Q) be a complete metric space, $\Gamma : \Lambda \rightarrow \Lambda$ a given map, and let $\alpha : \Lambda \times \Lambda \rightarrow [0, \infty)$ be a mapping. Suppose that the following affirmations hold:

(i) There exists $b \in [1, \infty)$ and $\mu > 0$ such that for all $\eta, m \in \Lambda$,

$$Q^b(\Gamma^2\eta, \Gamma^2m) > 0 \implies \mu + F(\alpha(\eta, m)Q^b(\Gamma^2\eta, \Gamma^2m)) \leq F(p^b(\eta, m)),$$

where

$$\begin{aligned}
 p^b(\eta, m) &= \max\{Q^b(\eta, m), Q^b(\eta, \Gamma\eta), Q^b(\Gamma\eta, \Gamma^2\eta), Q^b(m, \Gamma m), Q^b(\Gamma m, \Gamma^2m), \\
 &\quad \frac{Q^b(\eta, \Gamma m) + Q^b(m, \Gamma\eta)}{2}, \frac{Q^b(\Gamma\eta, \Gamma^2m) + Q^b(\Gamma m, \Gamma^2\eta)}{2}\}.
 \end{aligned}$$

- (ii) Γ is α -admissible;
- (iii) There exists $\eta_0 \in \Lambda$ such that $\alpha(\eta_0, \Gamma\eta_0) \geq 1$;
- (iv) Γ is continuous or orbitally continuous on Λ .

Then, Γ has a fixed point in Λ . Further, if Γ is α^* -admissible, then Γ has a unique fixed point $\mathfrak{o} \in \Lambda$. Moreover, for any $\eta_0 \in \Lambda$ if $\eta_{\beta+1} = \Gamma^{\beta+1}\eta_0 \neq \Gamma\eta_\beta$, for all $\beta \in \mathbb{N} \cup \{0\}$, then $\lim_{\beta \rightarrow \infty} \Gamma^\beta \eta_0 = \mathfrak{o}$.

Proof. Let $\eta_0 \in \Lambda$ be such that $\alpha(\Gamma\eta_0, \eta_0) \geq 1$ and construct a sequence $\{\eta_\beta\}$ by $\eta_{\beta+1} = \Gamma\eta_\beta$, for all $\beta \in \mathbb{N} \cup \{0\}$. If $\eta_{\beta_0} = \eta_{\beta_0+1}$, i.e., $\Gamma\eta_{\beta_0} = \eta_{\beta_0}$ for some $\beta_0 \in \mathbb{N} \cup \{0\}$, then η_{β_0} is a fixed point of Γ .

Now, we postulate that $\eta_\beta \neq \eta_{\beta+1} \forall \beta \geq 0$. Then, $\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1}) > 0$, for all $\beta \geq 0$. By (ii), we have $\alpha(\eta_0, \Gamma\eta_0) \geq 1 \Rightarrow \alpha(\eta_1, \eta_2) = \alpha(\Gamma\eta_0, \Gamma^2\eta_0) \geq 1$. Therefore, inductively, we show that $\alpha(\eta_\beta, \eta_{\beta+1}) = \alpha(\Gamma^\beta\eta_0, \Gamma^{\beta+1}\eta_0) \geq 1$, for all $\beta \geq 0$. Letting $v = \max\{\mathcal{Q}^b(\eta_0, \eta_1), \mathcal{Q}^b(\eta_1, \eta_2)\}$.

Now, from (3), taking $\eta = \eta_{\beta-2}$ and $m = \eta_{\beta-1}$ with $\beta \geq 2$, we have

$$\begin{aligned} \mathfrak{p}^b(\eta_{\beta-2}, \eta_{\beta-1}) &= \max\{\mathfrak{d}^b(\eta_{\beta-2}, \eta_{\beta-1}), \mathfrak{d}^b(\eta_{\beta-2}, \Gamma\eta_{\beta-2}), \mathfrak{d}^b(\Gamma\eta_{\beta-2}, \Gamma^2\eta_{\beta-2}), \\ &\quad \mathfrak{d}^b(\eta_{\beta-1}, \Gamma\eta_{\beta-1}), \mathfrak{d}^b(\Gamma\eta_{\beta-1}, \Gamma^2\eta_{\beta-1}), \\ &\quad \frac{\mathfrak{d}^b(\eta_{\beta-2}, \Gamma\eta_{\beta-1}) + \mathfrak{d}^b(\eta_{\beta-1}, \Gamma\eta_{\beta-2})}{2}, \frac{\mathfrak{d}^b(\Gamma\eta_{\beta-2}, \Gamma^2\eta_{\beta-1}) + \mathfrak{d}^b(\Gamma\eta_{\beta-1}, \Gamma^2\eta_{\beta-2})}{2}\} \\ &= \max\{\mathfrak{d}^b(\eta_{\beta-2}, \eta_{\beta-1}), \mathfrak{d}^b(\eta_{\beta-2}, \eta_{\beta-1}), \mathfrak{d}^b(\eta_{\beta-1}, \eta_\beta), \\ &\quad \mathfrak{d}^b(\eta_{\beta-1}, \eta_\beta), \mathfrak{d}^b(\eta_\beta, \eta_{\beta+1}), \\ &\quad \frac{\mathfrak{d}^b(\eta_{\beta-2}, \eta_\beta) + \mathfrak{d}^b(\eta_{\beta-1}, \eta_{\beta-1})}{2}, \frac{\mathfrak{d}^b(\eta_{\beta-1}, \eta_{\beta+1}) + \mathfrak{d}^b(\eta_\beta, \eta_\beta)}{2}\} \\ &= \max\{\mathfrak{d}^b(\eta_{\beta-2}, \eta_{\beta-1}), \mathfrak{d}^b(\eta_{\beta-2}, \eta_{\beta-1}), \mathfrak{d}^b(\eta_{\beta-1}, \eta_\beta), \\ &\quad \mathfrak{d}^b(\eta_{\beta-1}, \eta_\beta), \mathfrak{d}^b(\eta_\beta, \eta_{\beta+1}), \frac{\mathfrak{d}^b(\eta_{\beta-2}, \eta_\beta)}{2}, \frac{\mathfrak{d}^b(\eta_{\beta-1}, \eta_{\beta+1})}{2}\} \\ &= \max\{\mathfrak{d}^b(\eta_{\beta-2}, \eta_{\beta-1}), \mathfrak{d}^b(\eta_{\beta-1}, \eta_\beta), \mathfrak{d}^b(\eta_\beta, \eta_{\beta+1})\}. \end{aligned}$$

By (F_1) , condition (ii), and Equation (2), we have

$$\begin{aligned} F(\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})) &= F(\mathcal{Q}^b(\Gamma^2\eta_{\beta-2}, \Gamma^2\eta_{\beta-1})) \\ &\leq F(\alpha(\eta_{\beta-2}, \eta_{\beta-1})\mathcal{Q}^b(\Gamma^2\eta_{\beta-2}, \Gamma^2\eta_{\beta-1})) \\ &\leq F(\mathfrak{p}^b(\eta_{\beta-2}, \eta_{\beta-1})) - \mu \\ &\leq F(\max\{\mathcal{Q}^b(\eta_{\beta-2}, \eta_{\beta-1}), \mathcal{Q}^b(\eta_{\beta-1}, \eta_\beta), \mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})\}) - \mu. \end{aligned}$$

If $\max\{\mathcal{Q}^b(\eta_{\beta-2}, \eta_{\beta-1}), \mathcal{Q}^b(\eta_{\beta-1}, \eta_\beta), \mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})\} = \mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})$, then we obtain

$$F(\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})) \leq F(\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})) - \mu < F(\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})).$$

This is a contradiction. Therefore,

$$F(\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})) \leq F(\max\{\mathcal{Q}^b(\eta_{\beta-2}, \eta_{\beta-1}), \mathcal{Q}^b(\eta_{\beta-1}, \eta_\beta)\}) - \mu.$$

Since $\{\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})\}$ is a strictly nonincreasing sequence, we obtain

$$F(\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})) \leq F(\mathcal{Q}^b(\eta_{\beta-2}, \eta_{\beta-1})) - \mu \leq \dots \leq F(v) - J\mu, \tag{7}$$

whenever $\beta = 2J$ or $\beta = 2J + 1$ for $J \geq 1$.

From (6), we obtain

$$\lim_{\beta \rightarrow \infty} F(\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})) = -\infty. \tag{8}$$

Therefore, by (F2) and by Equation (8), we have

$$\lim_{\beta \rightarrow \infty} \mathcal{Q}^b(\eta_\beta, \eta_{\beta+1}) = 0. \tag{9}$$

By (F3), there exists $0 < k < 1$ such that

$$\lim_{\beta \rightarrow \infty} [\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})]^k F(\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})) = 0. \tag{10}$$

Moreover, by Equation (7), we get

$$[\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})]^k [F(\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})) - F(v)] \leq -[\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})]^k J\mu \leq 0, \tag{11}$$

where $\beta = 2J$ or $\beta = 2J + 1$ for $J \geq 1$. Setting $\beta \rightarrow \infty$ in (11) along with (9) and (10), we have

$$\lim_{\beta \rightarrow \infty} J[\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})]^k = 0. \tag{12}$$

Now, two cases arise.

Case-(i): If β is even and $\beta \geq 2$, then by Equation (12), we have

$$\lim_{\beta \rightarrow \infty} \beta[\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})]^k = 0. \tag{13}$$

Case-(ii): If β is odd and $\beta \geq 3$, then by Equation (12), we have

$$\lim_{\beta \rightarrow \infty} (\beta - 1)[\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})]^k = 0. \tag{14}$$

Using (9), (14) gives

$$\lim_{\beta \rightarrow \infty} \beta[\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})]^k = 0. \tag{15}$$

We conclude for the above cases that, $\exists \beta_1 \in \mathbb{N}$ such that

$$\beta[\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1})]^k \leq 1 \quad \forall \beta \geq \beta_1.$$

Therefore, we obtain

$$\mathcal{Q}^b(\eta_\beta, \eta_{\beta+1}) \leq \frac{1}{\beta^k}, \quad \forall \beta \geq \beta_1.$$

Now, we prove the sequence $\{\eta_\beta\}$ is a Cauchy sequence. For all $b > q \geq \beta_1$, we have

$$\mathcal{Q}^b(\eta_b, \eta_q) \leq \mathcal{Q}^b(\eta_b, \eta_{b-1}) + \mathcal{Q}^b(\eta_{b-1}, \eta_{b-2}) + \dots + \mathcal{Q}^b(\eta_{q+1}, \eta_q) < \sum_{\ell=q}^{\infty} \mathcal{Q}^b(\eta_\ell, \eta_{\ell+1}) \leq \sum_{\ell=q}^{\infty} \frac{1}{\ell^k}.$$

Taking $q \rightarrow \infty$, we get $\lim_{b, q \rightarrow \infty} \mathcal{Q}^b(\eta_b, \eta_q) = 0$, since $\sum_{\ell=q}^{\infty} \frac{1}{\ell^k}$ is convergent. This proves that the sequence $\{\eta_\beta\}$ is a Cauchy sequence in Λ . By the completeness property, there exists $\sigma \in \Lambda$ such that $\lim_{\beta \rightarrow \infty} \eta_\beta = \sigma$. Now, we show that σ is a fixed point of Γ . Since Γ is continuous,

$$\mathcal{Q}^b(\sigma, \Gamma\sigma) = \lim_{\beta \rightarrow \infty} \mathcal{Q}^b(\eta_\beta, \Gamma\eta_\beta) = \lim_{\beta \rightarrow \infty} \mathcal{Q}^b(\eta_\beta, \eta_{\beta+1}) = 0.$$

This implies that σ is a fixed point of Γ .

Again, we postulate that Γ is orbitally continuous on Λ , then

$$\eta_{\beta+1} = \Gamma\eta_\beta = \Gamma(\Gamma^\beta\eta_0) \rightarrow \Gamma\mathfrak{o} \text{ as } \beta \rightarrow \infty.$$

By completeness, we obtain $\Gamma\mathfrak{o} = \mathfrak{o}$. Therefore, $Fix(\Gamma) \neq \emptyset$.

Further, postulating that Γ is α^* -admissible, $\forall \mathfrak{o}, \mathfrak{o}^* \in Fix(\Gamma)$, we have $\alpha(\mathfrak{o}, \mathfrak{o}^*) \geq 1$. By Equations (2) and (3), we have

$$\begin{aligned} F(\mathcal{Q}^b(\mathfrak{o}, \mathfrak{o}^*)) &= F(\mathcal{Q}^b(\Gamma^2\mathfrak{o}, \Gamma^2\mathfrak{o}^*)) \\ &= F(\alpha(\mathfrak{o}, \mathfrak{o}^*)\mathcal{Q}^b(\Gamma^2\mathfrak{o}, \Gamma^2\mathfrak{o}^*)) \\ &\leq F(\mathfrak{p}^b(\mathfrak{o}, \mathfrak{o}^*)) - \mu \\ &= F(\max\{\mathcal{Q}^b(\mathfrak{o}, \mathfrak{o}^*), \mathcal{Q}^b(\mathfrak{o}, \Gamma\mathfrak{o}), \mathcal{Q}^b(\Gamma\mathfrak{o}, \Gamma^2\mathfrak{o}), \mathcal{Q}^b(\mathfrak{o}^*, \Gamma\mathfrak{o}^*), \mathcal{Q}^b(\Gamma\mathfrak{o}^*, \Gamma^2\mathfrak{o}^*), \\ &\quad \frac{\mathcal{Q}^b(\mathfrak{o}, \Gamma\mathfrak{o}^*) + \mathcal{Q}^b(\mathfrak{o}^*, \Gamma\mathfrak{o})}{2}, \frac{\mathcal{Q}^b(\Gamma\mathfrak{o}, \Gamma^2\mathfrak{o}^*) + \mathcal{Q}^b(\Gamma\mathfrak{o}^*, \Gamma^2\mathfrak{o})}{2}\}) - \mu \\ &= F(\mathcal{Q}^b(\mathfrak{o}, \mathfrak{o}^*)) - \mu. \end{aligned}$$

Since $\mu > 0$ and using (F_1) , we obtain

$$F(\mathcal{Q}^b(\mathfrak{o}, \mathfrak{o}^*)) < F(\mathcal{Q}^b(\mathfrak{o}, \mathfrak{o}^*)).$$

This is a contradiction. Therefore, Γ has a unique fixed point in Λ . \square

Example 10. Let $\Lambda = [0, 1]$ and $\mathcal{Q} : \Lambda \times \Lambda \rightarrow \mathbb{R}_+$ be given by

$$\mathcal{Q}^b(\eta, \mathfrak{m}) = |\eta - \mathfrak{m}|,$$

for all $\eta, \mathfrak{m} \in \Lambda$. Then, (Λ, \mathcal{Q}) is a complete metric space. Define a mapping $\Gamma : \Lambda \rightarrow \Lambda$ by

$$\Gamma\eta = \frac{\eta^2}{2} + \frac{1}{8},$$

for all $\eta \in \Lambda$ with $\alpha(\eta, \mathfrak{m}) = 1$, for all $\eta, \mathfrak{m} \in \Lambda$. Then, Γ is α -admissible. Let $F \in \mathfrak{S}$ be $F(\mathfrak{x}) = \ln \mathfrak{x}, \mathfrak{x} > 0$. Since we have

$$|\Gamma\eta, \Gamma\mathfrak{m}| = \frac{1}{2}|\eta^2 - \mathfrak{m}^2| \leq |\eta - \mathfrak{m}|,$$

for all $\eta \in \Lambda$, we have

$$\begin{aligned} \alpha(\eta, \mathfrak{m})(|\Gamma^2\eta, \Gamma^2\mathfrak{m}|^2) &= \frac{1}{256}|(2\eta^2 - \eta^4) - (2\mathfrak{m}^2 - \mathfrak{m}^4)|^2 \\ &\leq \frac{1}{64}(|\eta^4 - \mathfrak{m}^4 + \eta^2 - \mathfrak{m}^2|^2) \\ &\leq \frac{1}{64}(|\eta^4 - \mathfrak{m}^4|^2 + |\eta^2 - \mathfrak{m}^2|^2) \\ &= \frac{1}{64}|\eta^2 - \mathfrak{m}^2|^2 + \frac{1}{64}|\eta^4 - \mathfrak{m}^4|^2 \\ &\leq \frac{1}{2}(|\Gamma\eta - \Gamma\mathfrak{m}|^2 + \frac{1}{4}|\eta - \mathfrak{m}|^2) \\ &\leq \frac{5}{8} \max\{|\eta - \mathfrak{m}|^2, |\Gamma\eta - \Gamma\mathfrak{m}|^2\} \\ &\leq \frac{5}{8}\mathfrak{p}^2(\eta, \mathfrak{m}) \\ &= e^{-\mu}\mathfrak{p}^2(\eta, \mathfrak{m}). \end{aligned}$$

where $-\mu = \ln(\frac{5}{8})$. Applying the logarithm on both sides, we get

$$\mu + F(\alpha(\eta, m)Q^b(\Gamma^2\eta, \Gamma^2m)) \leq F(p^1(\eta, m)).$$

This shows that Γ is an α - F -convex contraction mapping. We define a sequence $\{\eta_\beta\}$ by

$$\eta_\beta = \frac{\beta}{\beta + 1} - \frac{1}{\sqrt{2}},$$

then $\eta_\beta \rightarrow 1 - \frac{1}{\sqrt{2}}$, as $\beta \rightarrow \infty$.

Therefore,

$$\eta_{\beta+1} = \Gamma\eta_\beta = [\frac{1}{4}(\frac{\beta}{\beta + 1} - \frac{1}{\sqrt{2}})^2 + \frac{1}{8}] \rightarrow 1 - \frac{1}{\sqrt{2}},$$

as $\beta \rightarrow \infty$. Thus, all conditions of Theorem 2 are satisfied and $\eta = 1 - \frac{1}{\sqrt{2}}$ is the unique fixed point of Γ in Λ .

Corollary 1. Let (Λ, Q) be a complete metric space and $\alpha : \Lambda \times \Lambda \rightarrow [0, \infty)$ a mapping. Postulating that $\Gamma : \Lambda \rightarrow \Lambda$ is a self-mapping, the following affirmations hold:

(i) $\forall \eta, m \in \Lambda,$

$$\alpha(\eta, m)Q^b(\Gamma^2\eta, \Gamma^2m) \leq \mathbb{k} \max\{Q^b(\eta, m), Q^b(\eta, \Gamma\eta), Q^b(\Gamma\eta, \Gamma^2\eta), Q^b(m, \Gamma m), Q^b(\Gamma m, \Gamma^2m), \frac{Q^b(\eta, \Gamma m) + Q^b(m, \Gamma\eta)}{2}, \frac{Q^b(\Gamma\eta, \Gamma^2m) + Q^b(\Gamma m, \Gamma^2\eta)}{2}\} \tag{16}$$

where $\mathbb{k} \in (0, 1)$;

(ii) Γ is α -admissible;

(iii) There exists $\eta_0 \in \Lambda$ such that $\alpha(\eta_0, \Gamma\eta_0) \geq 1$;

(iv) Γ is continuous or orbitally continuous on Λ .

Then, Γ has a fixed point in Λ . Further, if Γ is an α^* -admissible mapping, then Γ has a unique fixed point $\mathfrak{o} \in \Lambda$. Moreover, for any $\eta_0 \in \Lambda$ if $\eta_{\beta+1} = \Gamma^{\beta+1}\eta_0 \neq \Gamma^\beta\eta_0$, for all $\beta \in \mathbb{N} \cup \{0\}$, then $\lim_{\beta \rightarrow \infty} \Gamma^\beta\eta_0 = \mathfrak{o}$.

Proof. Setting $F(x) = \ln(x), x > 0$. Obviously, $F \in \mathfrak{S}$. Applying the logarithm on both sides of (16), we get

$$-\ln \mathbb{k} + \ln \alpha(\eta, m)Q^b(\Gamma^2\eta, \Gamma^2m) \leq \ln(\max\{Q^b(\eta, m), Q^b(\eta, \Gamma\eta), Q^b(\Gamma\eta, \Gamma^2\eta), Q^b(m, \Gamma m), Q^b(\Gamma m, \Gamma^2m), \frac{Q^b(\eta, \Gamma m) + Q^b(m, \Gamma\eta)}{2}, \frac{Q^b(\Gamma\eta, \Gamma^2m) + Q^b(\Gamma m, \Gamma^2\eta)}{2}\}),$$

which implies that

$$\mu + F(\alpha(\eta, m)Q^b(\Gamma^2\eta, \Gamma^2m)) \leq F(p^1(\eta, m)),$$

for all $\eta, m \in \Lambda$ with $\eta \neq m$ where $\mu = -\ln \mathbb{k}$. It follows that Γ is an α - F -convex contraction with $b = 1$. Thus, all the affirmations of Theorem 2 are held and hence, Γ has a unique fixed point in Λ . \square

Corollary 2. Let (Λ, Q) be a complete metric space and $\alpha : \Lambda \times \Lambda \rightarrow [0, \infty)$ a mapping. Postulating that $\Gamma : \Lambda \rightarrow \Lambda$ is a self-mapping, the following affirmations hold:

(i) $\forall \eta, m \in \Lambda,$

$$\begin{aligned} \alpha(\eta, m) \mathcal{Q}^b(\Gamma^2\eta, \Gamma^2m) &\leq \alpha_1 \mathcal{Q}^b(\eta, m) + \alpha_2 \mathcal{Q}^b(\eta, \Gamma\eta) + \alpha_3 \mathcal{Q}^b(\Gamma\eta, \Gamma^2\eta) \\ &\quad + \alpha_4 \mathcal{Q}^b(m, \Gamma m) + \alpha_5 \mathcal{Q}^b(\Gamma m, \Gamma^2m) + \alpha_6 \left(\frac{\mathcal{Q}^b(\eta, \Gamma m) + \mathcal{Q}^b(m, \Gamma\eta)}{2} \right) \\ &\quad + \alpha_7 \left(\frac{\mathcal{Q}^b(\Gamma\eta, \Gamma^2m) + \mathcal{Q}^b(\Gamma m, \Gamma^2\eta)}{2} \right), \end{aligned}$$

where $0 \leq \alpha_{\mathfrak{k}} < 1, \mathfrak{k} = 1, 2, \dots, 7$ such that $\sum_{\mathfrak{k}=1}^7 \alpha_{\mathfrak{k}} < 1$;

- (ii) Γ is α -admissible;
- (iii) There exists $\eta_0 \in \Lambda$ such that $\alpha(\eta_0, \Gamma\eta_0) \geq 1$;
- (iv) Γ is continuous or orbitally continuous on Λ .

Then, Γ has a fixed point in Λ . Further, if Γ is an α^* -admissible mapping, then Γ has a unique fixed point $\mathfrak{o} \in \Lambda$. Moreover, for any $\eta_0 \in \Lambda$ if $\eta_{\beta+1} = \Gamma^{\beta+1}\eta_0 \neq \Gamma^\beta\eta_0$, for all $\beta \in \mathbb{N} \cup \{0\}$, then $\lim_{\beta \rightarrow \infty} \Gamma^\beta\eta_0 = \mathfrak{o}$.

Proof. Setting $F(x) = \ln(x), x > 0$. Obviously, $F \in \mathfrak{S}$. For all $\eta, m \in \Lambda$ with $\eta \neq m$, we obtain

$$\begin{aligned} \alpha(\eta, m) \mathcal{Q}^b(\Gamma^2\eta, \Gamma^2m) &= \mathcal{Q}^b(\Gamma^2\eta, \Gamma^2m) \\ &\leq \alpha_1 \mathcal{Q}^b(\eta, m) + \alpha_2 \mathcal{Q}^b(\eta, \Gamma\eta) + \alpha_3 \mathcal{Q}^b(\Gamma\eta, \Gamma^2\eta) \\ &\quad + \alpha_4 \mathcal{Q}^b(m, \Gamma m) + \alpha_5 \mathcal{Q}^b(\Gamma m, \Gamma^2m), \\ &\quad + \alpha_6 \left(\frac{\mathcal{Q}^b(\eta, \Gamma m) + \mathcal{Q}^b(m, \Gamma\eta)}{2} \right) + \alpha_7 \left(\frac{\mathcal{Q}^b(\Gamma\eta, \Gamma^2m) + \mathcal{Q}^b(\Gamma m, \Gamma^2\eta)}{2} \right) \\ &\leq \mathbb{k} \max \left\{ \mathcal{Q}^b(\eta, m), \mathcal{Q}^b(\eta, \Gamma\eta), \mathcal{Q}^b(\Gamma\eta, \Gamma^2\eta), \mathcal{Q}^b(m, \Gamma m), \mathcal{Q}^b(\Gamma m, \Gamma^2m), \right. \\ &\quad \left. \frac{\mathcal{Q}^b(\eta, \Gamma m) + \mathcal{Q}^b(m, \Gamma\eta)}{2}, \frac{\mathcal{Q}^b(\Gamma\eta, \Gamma^2m) + \mathcal{Q}^b(\Gamma m, \Gamma^2\eta)}{2} \right\}, \end{aligned}$$

where $\mathbb{k} = \sum_{\mathfrak{k}=1}^7 \alpha_{\mathfrak{k}} < 1$. By Corollary 1, Γ has a unique fixed point in Λ . \square

Corollary 3. Consider a continuous self-mapping Γ on a complete metric space (Λ, \mathcal{Q}) . If there exists $\mathbb{k} \in (0, 1)$ satisfying the following inequality

$$\begin{aligned} \mathcal{Q}^b(\Gamma^2\eta, \Gamma^2m) &\leq \mathbb{k} \max \left\{ \mathcal{Q}^b(\eta, m), \mathcal{Q}^b(\eta, \Gamma\eta), \mathcal{Q}^b(\Gamma\eta, \Gamma^2\eta), \mathcal{Q}^b(m, \Gamma m), \mathcal{Q}^b(\Gamma m, \Gamma^2m), \right. \\ &\quad \left. \frac{\mathcal{Q}^b(\eta, \Gamma m) + \mathcal{Q}^b(m, \Gamma\eta)}{2}, \frac{\mathcal{Q}^b(\Gamma\eta, \Gamma^2m) + \mathcal{Q}^b(\Gamma m, \Gamma^2\eta)}{2} \right\} \end{aligned}$$

for all $\eta, m \in \Lambda$, then Γ has a unique fixed point in Λ .

3. Application

In this application part, we provide a nonlinear integral equation application of our main results.

Consider a real-valued continuous function $\Lambda = \zeta[a, b]$ defined on $[a, b]$ with metric $d(\varphi, \psi) = |\varphi - \psi| = \max_{s \in [a, b]} |\varphi(s) - \psi(s)| \forall \varphi, \psi \in \zeta[a, b]$. Then, (Λ, d) is a complete metric space.

Consider

$$\eta(s) = v(s) + \frac{1}{b-a} \int_a^b K(s, t, \eta(t)) dt, \tag{17}$$

where $s, t \in [a, b]$, $v(s)$ is a given function in Λ and $K : [a, b] \times [a, b] \times \Lambda \rightarrow \mathbb{R}, v : [a, b] \rightarrow \mathbb{R}$ are given continuous functions.

Theorem 3. Let (Λ, d) be a metric space with metric $d(\varphi, \psi) = |\varphi - \psi| = \max_{s \in [a, b]} |\varphi(s) - \psi(s)| \forall \varphi, \psi \in \Lambda$ and define a continuous operator $\Gamma : \Lambda \rightarrow \Lambda$ on Λ by

$$\Gamma\eta(s) = v(s) + \frac{1}{b-a} \int_a^b K(s, t, \eta(t)) dt. \tag{18}$$

If there exists $\mathbb{k} \in (0, 1)$ such that $\forall \eta, m \in \Lambda$ with $\eta \neq m$ and $t, s \in [a, b]$ satisfying the following inequality

$$\begin{aligned} |K(s, t, \Gamma\eta(t)) - K(s, t, \Gamma m(t))| \leq \mathbb{k} \max \left\{ & |\eta(t) - m(t)|, |\Gamma\eta - \Gamma m|, |\eta - \Gamma\eta|, |\Gamma\eta - \Gamma^2\eta|, \right. \\ & |m - \Gamma m|, |\Gamma m - \Gamma^2 m|, \left(\frac{|\eta - \Gamma m| + |m - \Gamma\eta|}{2} \right), \\ & \left. \left(\frac{|\Gamma\eta - \Gamma^2 m| + |\Gamma m - \Gamma^2\eta|}{2} \right) \right\}, \end{aligned} \tag{19}$$

then, by (18), the integral operator has a unique solution $o \in \Lambda$ and for each $\eta_0 \in \Lambda, \Gamma\eta_\beta \neq \eta_\beta \forall \beta \in \mathbb{N} \cup \{0\}$, we have $\lim_{\beta \rightarrow \infty} \Gamma\eta_\beta = o$.

Proof. Define a mapping $\alpha : \Lambda \times \Lambda \rightarrow \mathbb{R}_+$ by $\alpha(\eta, m) = 1 \forall \eta, m \in \Lambda$. Therefore, Γ is α -admissible. Let $F \in \mathfrak{S}$ such that $F(x) = In(x), x > 0$. Let $\eta_0 \in \Lambda$ and a sequence $\{\eta_\beta\}$ in Λ defined by $\eta_{\beta+1} = \Gamma\eta_\beta = \Gamma^{\beta+1}\eta_0 \forall \beta \geq 0$. By Equation (18), we have

$$\eta_{\beta+1} = \Gamma\eta_\beta(s) = v(s) + \frac{1}{b-a} \int_a^b K(s, t, \eta_\beta(t)) dt. \tag{20}$$

We prove that Γ is an α - F -convex contraction on $\zeta[a, b]$. By Equations (18) and (19), we obtain

$$\begin{aligned} |\Gamma^2\eta(s) - \Gamma^2m(s)| &= \frac{1}{|b-a|} \left| \int_a^b K(s, t, \Gamma\eta(t)) dt - \int_a^b K(s, t, \Gamma m(t)) dt \right| \\ &\leq \frac{1}{|b-a|} \int_a^b |K(s, t, \Gamma\eta(t)) - K(s, t, \Gamma m(t))| dt \\ &\leq \frac{\mathbb{k}}{|b-a|} \int_a^b \max \left\{ |\eta(t) - m(t)|, |\Gamma\eta - \Gamma m|, |\eta - \Gamma\eta|, |\Gamma\eta - \Gamma^2\eta|, \right. \\ &\quad |m - \Gamma m|, |\Gamma m - \Gamma^2 m|, \left(\frac{|\eta - \Gamma m| + |m - \Gamma\eta|}{2} \right), \\ &\quad \left. \left(\frac{|\Gamma\eta - \Gamma^2 m| + |\Gamma m - \Gamma^2\eta|}{2} \right) \right\} dt. \end{aligned}$$

Taking the maximum on both sides, for all $s \in [a, b]$, we have

$$\begin{aligned}
 d(\Gamma^2\eta, \Gamma^2\mathfrak{m}) &= \max_{s \in [a,b]} |\Gamma^2\eta(s) - \Gamma^2\mathfrak{m}(s)| \\
 &\leq \frac{\mathbb{k}}{|b-a|} \max_{s \in [a,b]} \int_a^b \max \left\{ |\eta(t) - \mathfrak{m}(t)|, |\Gamma\eta(t) - \Gamma\mathfrak{m}(t)|, |\eta(t) - \Gamma\eta(t)|, \right. \\
 &\quad |\Gamma\eta(t) - \Gamma^2\eta(t)|, |\mathfrak{m}(t) - \Gamma\mathfrak{m}(t)|, |\Gamma\mathfrak{m}(t) - \Gamma^2\mathfrak{m}(t)|, \\
 &\quad \left. \left(\frac{|\eta(t) - \Gamma\mathfrak{m}(t)| + |\mathfrak{m}(t) - \Gamma\eta(t)|}{2} \right), \right. \\
 &\quad \left. \left(\frac{|\Gamma\eta(t) - \Gamma^2\mathfrak{m}(t)| + |\Gamma\mathfrak{m}(t) - \Gamma^2\eta(t)|}{2} \right) \right\} dt. \\
 &\leq \frac{\mathbb{k}}{|b-a|} \max \left[\max_{\vartheta \in [a,b]} \left\{ |\eta(\vartheta) - \mathfrak{m}(\vartheta)|, |\Gamma\eta(\vartheta) - \Gamma\mathfrak{m}(\vartheta)|, |\eta(\vartheta) - \Gamma\eta(\vartheta)|, \right. \right. \\
 &\quad |\Gamma\eta(\vartheta) - \Gamma^2\eta(\vartheta)|, |\mathfrak{m}(\vartheta) - \Gamma\mathfrak{m}(\vartheta)|, |\Gamma\mathfrak{m}(\vartheta) - \Gamma^2\mathfrak{m}(\vartheta)|, \\
 &\quad \left. \left(\frac{|\eta(\vartheta) - \Gamma\mathfrak{m}(\vartheta)| + |\mathfrak{m}(\vartheta) - \Gamma\eta(\vartheta)|}{2} \right), \right. \\
 &\quad \left. \left(\frac{|\Gamma\eta(\vartheta) - \Gamma^2\mathfrak{m}(\vartheta)| + |\Gamma\mathfrak{m}(\vartheta) - \Gamma^2\eta(\vartheta)|}{2} \right) \right] \int_a^b dt \\
 &= \mathbb{k} \max \left\{ d(\eta, \mathfrak{m}), d(\Gamma\eta, \Gamma\mathfrak{m}), d(\eta, \Gamma\eta), d(\Gamma\eta, \Gamma^2\eta), d(\mathfrak{m}, \Gamma\mathfrak{m}), d(\Gamma\mathfrak{m}, \Gamma^2\mathfrak{m}), \right. \\
 &\quad \left. \left(\frac{d(\eta, \Gamma\mathfrak{m}) + d(\mathfrak{m}, \Gamma\eta)}{2} \right), \left(\frac{d(\Gamma\eta, \Gamma^2\mathfrak{m}) + d(\Gamma\mathfrak{m}, \Gamma^2\eta)}{2} \right) \right\} \\
 &= \mathbb{k}p^1(\eta, \mathfrak{m}).
 \end{aligned}$$

Therefore, $d(\Gamma^2\eta, \Gamma^2\mathfrak{m}) \leq \mathbb{k}p^1(\eta, \mathfrak{m})$. Hence, we have

$$\alpha(\eta, \mathfrak{m})d(\Gamma^2\eta, \Gamma^2\mathfrak{m}) \leq \mathbb{k}p^1(\eta, \mathfrak{m}).$$

Now, applying the logarithm on both sides, we get

$$-\ln \mathbb{k} + \ln[\alpha(\eta, \mathfrak{m})d(\Gamma^2\eta, \Gamma^2\mathfrak{m})] \leq \ln p^1(\eta, \mathfrak{m}).$$

Therefore, we have

$$\mathbb{k} + F(\alpha(\eta, \mathfrak{m})d(\Gamma^2\eta, \Gamma^2\mathfrak{m})) \leq F(p^1(\eta, \mathfrak{m})),$$

where $-\ln \mathbb{k} = \mu$. It follows that Γ is an α - F -convex contraction with $b = 1 \forall \eta, \mathfrak{m} \in \Lambda$ with $\eta \neq \mathfrak{m}$. Since Γ is α -admissible and $\Lambda = \zeta[a, b]$ is a complete metric space, the iteration scheme converges to some point $\mathfrak{o} \in \Lambda$, i.e., $\lim_{\beta \rightarrow \infty} \eta_\beta \rightarrow \mathfrak{o}$. From the continuity, we show that \mathfrak{o} is a fixed point of Γ . It follows that $\mathcal{T}\mathfrak{o} = \mathfrak{o}$. Clearly, $\text{Fix}(\mathcal{T}) \neq \emptyset$. Now, $\forall \eta, \mathfrak{m} \in \text{Fix}(\Gamma), \alpha(\mathfrak{o}, \mathfrak{o}^*) = 1$. This shows that Γ is α^* -admissible. Thus, all the affirmations of Theorem 2 are held and hence, Γ has a unique fixed point solution $\mathfrak{o} \in \Lambda$. \square

The example below demonstrates the existence of a singular integral operator solution meeting each of the conditions in Theorem 3.

Example 11. Let $\Lambda = \zeta[0, 1]$ be a set of all continuous function on $[0, 1]$, $\mathfrak{v}(s) = \frac{7}{15}s^2$ and $K(s, t, \eta(t)) = \frac{1}{4}s^2(1 + \frac{t}{2})(\eta(t) + 1)$. Then, (18) becomes

$$\Gamma\eta(s) = \frac{7}{15}s^2 + \int_0^1 \frac{1}{4}s^2(1 + \frac{t}{2})(\eta(t) + 1)dt. \tag{21}$$

Now,

1. $\max | \frac{1}{4} s^2 (1 + \frac{t}{2}) | \leq \frac{1}{2}$ for all $(s, t) \in [0, 1] \times [0, 1]$;
2. For all $\eta, m \in \Lambda$ with $\eta \neq m$ and $(s, t) \in [0, 1] \times [0, 1]$ and using (18), we obtain

$$|\Gamma \eta - \Gamma m| \leq |\eta - m|.$$

By the above inequality, we obtain Γ is not an F-contraction. Now, we obtain

$$\begin{aligned} \left| \Gamma^2 \eta(s) - \Gamma^2 m(s) \right| &= \left| \int_0^1 K(s, t, v(s) + \int_0^1 K(s, t, \eta(s)) dt) dt - \int_0^1 K(s, t, v(s) \right. \\ &\quad \left. + \int_0^1 K(s, t, m(s)) dt) dt \right| \\ &\leq \int_0^1 \int_0^1 \left| \left[\frac{s^2(1 + \frac{t}{2})}{4} \right] (\eta(t) - m(t)) \right| dt dt \\ &\leq \max_{s \in [0,1]} \int_0^1 \int_0^1 \left| \left[\frac{s^2(1 + \frac{t}{2})}{4} \right] (\eta(t) - m(t)) \right| dt dt \\ &\leq \frac{1}{2} |\eta - m| \\ &= e^{-\mu} p^1(\eta, m). \end{aligned}$$

Therefore, $\left| \Gamma^2 \eta(s) - \Gamma^2 m(s) \right| \leq e^{-\mu} p^1(\eta, m)$, where $\ln \frac{1}{2} = -\mu$. Set $\alpha : \Lambda \times \Lambda \rightarrow [0, \infty)$ by $\alpha(\eta, m) = 1$, for all $\eta, m \in \Lambda$ and $F \in \mathfrak{S}$ such that $F(x) = \ln(x), x > 0$. Therefore, we obtain

$$\alpha(\eta, m) |\Gamma^2 \eta - \Gamma^2 m| \leq p^1(\eta, m).$$

Applying the logarithm on both sides, we get

$$\mu + \ln \alpha(\eta, m) d(\Gamma^2 \eta, \Gamma^2 m) \leq \ln p^1(\eta, m),$$

that is,

$$\mu + F(\alpha(\eta, m) d(\Gamma^2 \eta, \Gamma^2 m)) \leq F(p^1(\eta, m)).$$

We conclude that Γ is an α -F-convex contraction with $\flat = 1 \forall \eta, m \in \Lambda$. Thus, all the affirmations of Theorem 2 are held and therefore, Equation (18) has a unique solution. It follows that $\eta(s) = s^2$ is the exact solution of Equation (18). Using Equations (20), (21) becomes

$$\eta_{\beta+1}(s) = \Gamma \eta_{\beta}(s) = \frac{7}{15} s^2 + \int_0^1 \frac{1}{4} s^2 (1 + \frac{t}{2}) (\eta_{\beta}(t) + 1) dt. \tag{22}$$

Letting $\eta_0(s) = 0$ be an initial solution. Letting $\beta = 0, 1, 2, \dots$, respectively, in (22), we get

$$\begin{aligned} \eta_1(s) &= 0.7791666667s^2, & \eta_2(s) &= 0.8684461806s^2, & \eta_3(s) &= 0.8786761249s^2, \\ \eta_4(s) &= 0.879848306s^2, & \eta_5(s) &= 0.8799826184s^2, & \eta_6(s) &= 0.8799980084s^2, \\ \eta_7(s) &= 0.879997718s^2, & \eta_8(s) &= 0.879999739s^2, & \eta_9(s) &= 0.879999959s^2, \\ \eta_{10}(s) &= 0.879999984s^2, & \eta_{11}(s) &= 0.879999998s^2, & \eta_{12}(s) &= 0.88s^2, \\ \eta_{13}(s) &= 0.88s^2. \end{aligned}$$

Figure 1 discuss about the convergence criterion by using the $\eta(s)$ numerical values.

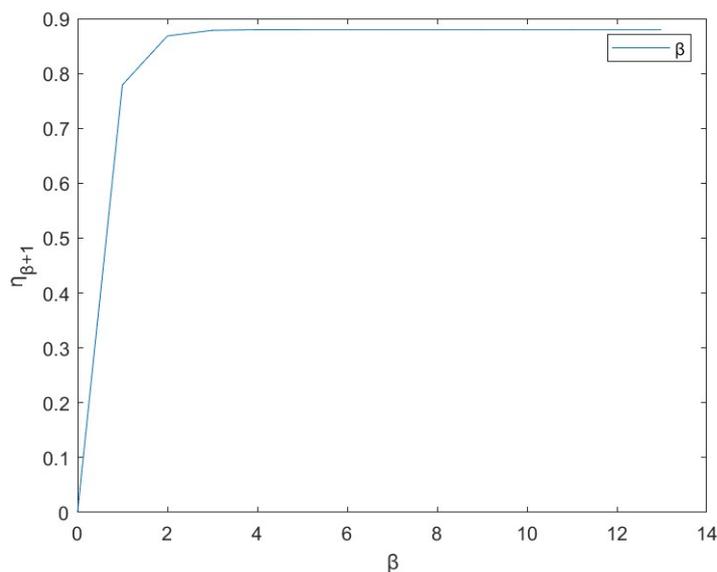


Figure 1. Convergence criterion.

Therefore, $\eta(s) = 0.88s^2$ is the unique solution.

4. Conclusions

This study introduced convexity conditions to α - F -contraction mappings with possible seven values. This research proved that the fixed point for α - F two-sided convex contraction mappings in a complete metric space was unique. The solution of a nonlinear integral equation was obtained via α - F -convex contraction mappings. This research work has many potential applications as the fixed point for these newly introduced convex contraction mappings can be established in different abstract spaces. Faraji and Radenovic provided some fixed-point results for convex contraction mappings on F -metric spaces. This will provide a structural method for finding a value of a fixed point. It is an interesting open problem to study the fixed-point results α - F -convex contraction mappings on complete F -metric spaces.

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