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Article

Hadamard Compositions of Gelfond–Leont'ev Derivatives

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Abstract: For analytic functions $f_j(z) = \sum\limits_{n=0}^{\infty} a_{n,j}z^n$, $1 \le j \le p$, the notion of a Hadamard composition $(f_1 * \ldots * f_p)_m = \sum\limits_{n=0}^{\infty}\left(\sum\limits_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p} a_{n,1}^{k_1}\cdot\ldots\cdot a_{n,p}^{k_p}\right)z^n$ of genus m is introduced. The relationship between the growth of the Gelfond–Leont'ev derivative of the Hadamard composition of functions f_j and the growth Hadamard composition of Gelfond–Leont'ev derivatives of these functions is studied. We found conditions under which these derivatives and the composition have the same order and a lower order. For the maximal terms of the power expansion of these derivatives, I describe behavior of their ratios.

Keywords: analytic function; Gelfond–Leont'ev derivative; Hadamard composition

MSC: 30B10; 30B20; 30B40

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1. Introduction

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

and

$$f_j(z) = \sum_{n=0}^{\infty} a_{n,j} z^n, \quad 1 \le j \le p \tag{2}$$

be analytic functions. As in [1], I say that the function is similar to the Hadamard composition of the functions f_j if $a_n = w(a_{n,1}, \ldots, a_{n,p})$ for all n, where $w : \mathbb{C}^p \to \mathbb{C}$ is a continuous function. Clearly, if $p = 2$ and $w(a_{n,1}, a_{n,2}) = a_{n,1}a_{n,2}$, then $f = (f_1 * f_2)$ is [2] the Hadamard composition (product) of the functions f_1 and f_2. Obtained by J. Hadamard, the properties of this composition find the applications [3,4] in the theory of the analytic continuation of the functions represented by a power series.

Here, I consider the case when w is a homogeneous polynomial. Recall that a polynomial is named homogeneous if all monomials with nonzero coefficients have the identical degree. A polynomial $P(x_1, \ldots, x_p)$ is homogeneous to the degree m if, and only if, $P(tx_1, \ldots, tx_p) = t^m P(x_1, \ldots, x_p)$ for all t from the field above that a polynomial is defined. Function (1) is called a Hadamard composition of genus $m \ge 1$ of functions (2) if $a_n = P(a_{n,1}, \ldots, a_{n,p})$, where

$$P(x_1, \ldots, x_p) = \sum_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p} x_1^{k_1}\cdot\ldots\cdot x_p^{k_p}, \quad k_j \in \mathbb{Z}_+$$

is a homogeneous polynomial of degree $m \ge 1$ with constant fixed coefficients $c_{k_1\ldots k_p}$. I remark that the usual Hadamard composition is a special case of the Hadamard composition of the genus $m = 2$. The Hadamard composition of genus $m \geq 1$ of functions f_j I denote by $(f_1 * \ldots * f_p)_m$, i.e.,

$$(f_1 * \ldots * f_p)_m(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \left(\sum_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p} a_{n,1}^{k_1} \cdot \ldots \cdot a_{n,p}^{k_p} \right) z^n.$$

For a power series (1) with the convergence radius $R[f] \in [0, +\infty]$ and a power series $l(z) = \sum_{n=0}^{\infty} l_n z^n$ with the convergence radius $R[l] \in [0, +\infty]$ and coefficients $l_n > 0$ for all $n \geq 0$ the power series

$$D_l^{(k)} f(z) = \sum_{n=0}^{\infty} \frac{l_n}{l_{n+k}} a_{n+k} z^n \tag{3}$$

is called [5] Gelfond–Leont'ev derivative of the n-th order. If $l(z) = e^z$, then $D_l^{(k)} f(z) = f^{(k)}(z)$ is the usual derivative of the n-th order. The Gelfond–Leont'ev derivative is a very interesting object of investigations (see [6–8]). These derivatives found applications in the theory of univalent analytic functions. They allows researchers to describe the growth of these functions in other terms [7].

There are many papers on the Hadamard composition of analytic functions and the Dirichlet series [9–11]. For example, A. Gaisin and T. Belous [10] studied the maximal term of the Hadamard composition of the Dirichlet series with real exponents. Alower estimate for the sum of a Dirichlet series over a curve arbitrarily approaching the convergence line was obtained. Moreover, in [11] they established a criterion for the logarithm of the maximal term of a Dirichlet series whose absolute convergence domain is a half-plane to be equivalent to the logarithm of the maximal term of its Hadamard composition with another Dirichlet series of some class on the asymptotic set. S. Vakarchuk [9] investigated an interpolation problem for classes of analytic functions generated of the Hadamard compositions and obtained upper and lower bounds for various n-widths for these classes.

If $R[f] > 0$, then, for $0 \leq r \leq R[f]$, let $M(r, f) = \max\{|f(z)| : |z| = r\}$ and $\mu(r, f) = \max\{|a_n| r^n : n \geq 0\}$ be the maximal term of series (1). M. K. Sen [12,13] researched a connection between the growth of the maximal term of the derivative $(f_1 * f_2)^{(k)}$ of the usual Hadamard composition $f_1 * f_2$ of entire functions f and g and the growth of the maximal term of derivative $f_1^{(k)} * f_2^{(k)}$. In particular, he proved [13] that if the function has the order ϱ and the lower order λ then, for every $\varepsilon > 0$ and all $r \geq r_0(\varepsilon)$,

$$r^{(k+2)\lambda-1-\varepsilon} \leq \frac{\mu(r, f^{(k+1)} * g^{(k+1)})}{\mu(r, (f * g)^{(k)})} \leq r^{(k+2)\varrho-1+\varepsilon}.$$

The research of M.K. Sen was continued in [14], where, instead of ordinary derivatives, the Gelfond–Leont'ev derivatives are considered. In particular, in [14] (see also [15] p. 128), the following lemma is proved.

Lemma 1 ([14]). *In order for an arbitrary series (1) the equalities $R[f] = +\infty$ and $R[D_l^{(k)} f] = +\infty$ to be equivalent, it is necessary and sufficient that*

$$0 < q = \varliminf_{n\to\infty} \sqrt[n]{l_n/l_{n+1}} \leq \varlimsup_{n\to\infty} \sqrt[n]{l_n/l_{n+1}} = Q < +\infty, \tag{4}$$

and, for the equivalence of the equalities $R[f] = 1$ and $R[D_l^{(k)} f] = 1$, it is necessary and sufficient that

$$\lim_{n\to\infty} \sqrt[n]{l_n/l_{n+1}} = 1. \tag{5}$$

The generalization of the results from [14] to the case of Hadamard compositions of genus $m \geq 1$ has become an actual problem. It allows researchers to study the growth properties of these function classes and consider their applications in geometric function theory as it is achieved for usual the Gelfond–Leont'ev derivatives of univalent analytic functions in [6–8].

2. Convergence of Hadamard Compositions of Genus m

Clearly, if function (1) is a Hadamard composition of genus m of functions (2) then

$$|a_n| \le \sum_{k_1+\cdots+k_p=m} |c_{k_1\ldots k_p}||a_{n,1}|^{k_1} \cdot \ldots \cdot |a_{n,p}|^{k_p}. \tag{6}$$

From $R[f]$, I denote the radius of the convergence of series (1) and suppose that $R[f_j] = R > 0$ for all $1 \le j \le p$. Then from the Cauchy–Hadamard formula, I have $\varlimsup\limits_{n\to\infty} \sqrt[n]{|a_{n,j}|} = 1/R[f_j] = 1/R$ and, thus, $|a_{n,j}| \le (1/R+\varepsilon)^n$ for every $\varepsilon > 0$ and all $n \ge n_0(\varepsilon)$. Therefore, (6) implies

$$|a_n| \le \sum_{k_1+\cdots+k_p=m} |c_{k_1\ldots k_p}| \left(\frac{1}{R}+\varepsilon\right)^{nk_1} \cdot \ldots \cdot \left(\frac{1}{R}+\varepsilon\right)^{nk_p} = \left(\frac{1}{R}+\varepsilon\right)^{nm} \sum_{k_1+\cdots+k_p=m} |c_{k_1\ldots k_p}|,$$

whence, $\sqrt[n]{|a_n|} \le (1+o(1))(1/R+\varepsilon)^m$ as $n \to \infty$, i.e., $1/R[f] \le (1/R+\varepsilon)^m$. In view of the arbitrariness of ε, I obtain the inequality $R[F] \ge R^m$.

Hence, it follows that, if $R[f_j] = +\infty$ for all j, then $R[(f_1 * \ldots * f_p)_m] = +\infty$, and, if $R[f_j] = 1$ for all j, then $R[(f_1 * \ldots * f_p)_m] \ge 1$.

In order for $R[(f_1 * \ldots * f_p)_m] = 1$, additional conditions on $a_{n,j}$ are required. For example, I say that the function f_1 is dominant if $|c_{m0\ldots0}||a_{n,1}|^m > 0$ for all $n \ge 0$ and $|a_{n,j}| = o(|a_{n,1}|)$ as $n \to \infty$ for all $2 \le j \le p$.

We put

$$\Sigma'_n = \sum_{k_1+\cdots+k_p=m,\, k_1\ne m} c_{k_1\ldots k_p} a_{n,1}^{k_1} \cdot \ldots \cdot a_{n,p}^{k_p} = \sum_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p} a_{n,1}^{k_1} \cdot \ldots \cdot a_{n,p}^{k_p} - c_{m0\ldots0} a_{n,1}^m.$$

Since, for each monomial of the polynomial Σ'_n, the sum of the exponents is equal to m, I have

$$\frac{|a_{n,1}|^{k_1} \cdot \ldots \cdot |a_{n,p}|^{k_p}}{|a_{n,1}|^m} = \frac{|a_{n,2}|^{k_2} \cdot \ldots \cdot |a_{n,p}|^{k_p}}{|a_{n,1}|^{m-k_1}} \to 0, \quad n \to \infty,$$

and, thus, $\Sigma'_n = o(|a_{n,1}|^m)$ as $n \to +\infty$.

Since

$$|c_{m0\ldots0}||a_{n,1}|^m - |\Sigma'_n| \le |a_n| \le |c_{m0\ldots0}||a_{n,1}|^m + |\Sigma'_n|,$$

we have $|a_n| = (1+o(1))|c_{m0\ldots0}||a_{n,1}|^m$ as $n \to \infty$, whence

$$1 = \frac{1}{R[f_j]} \le \frac{1}{R[f_1]} = \varlimsup_{n\to\infty} \sqrt[nm]{\frac{|a_n|}{|c_{m0\ldots0}|}} = \varlimsup_{n\to\infty} \left(\sqrt[n]{|a_n|}\right)^{1/m} = \left(\frac{1}{R[f]}\right)^{1/m},$$

i.e., $R[(f_1 * \ldots * f_p)_m] \le 1$ and, thus, $R[(f_1 * \ldots * f_p)_m] = 1$.

It is easy to check that

$$D_l^{(k)}(f_1 * \ldots * f_p)_m(z) = \sum_{n=0}^{\infty} \frac{l_n}{l_{n+k}} \left(\sum_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p} a_{n+k,1}^{k_1} \cdot \ldots \cdot a_{n+k,p}^{k_p} \right) z^n \tag{7}$$

is the Gelfond–Leont'ev derivative of the Hadamard composition of genus m, and

$$(D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m(z) = \sum_{n=0}^{\infty} \left(\frac{l_n}{l_{n+k}}\right)^m \left(\sum_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p} a_{n+k,1}^{k_1} \cdot \ldots \cdot a_{n+k,p}^{k_p} \right) z^n \tag{8}$$

is the Hadamard composition of genus m of the Gelfond–Leont'ev derivatives.

Lemma 2. *If condition* (4) *holds, then the equalities* $R[(D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m] = +\infty$ *and* $R[D_l^{(k)}(f_1 * \ldots * f_p)_m] = +\infty$ *are equivalent, and, if condition* (5) *holds, then the equalities* $R[(D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m] = 1$ *and* $R[D_l^{(k)}(f_1 * \ldots * f_p)_m] = 1$ *are equivalent.*

Proof. Indeed, using the Cauchy–Hadamard formula from (7) and (8) with

$$a_{n+k} = \sum_{k_1+\cdots+k_p=m} c_{k_1\ldots k_p} a_{n+k,1}^{k_1} \cdot \ldots \cdot a_{n+k,p}^{k_p}, \tag{9}$$

we have

$$\frac{1}{R[(D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m]} = \varlimsup_{n\to\infty} \sqrt[n]{|a_{n+k}| \frac{l_n}{l_{n+k}} \left(\frac{l_n}{l_{n+1}}\right)^{m-1}} \geq$$

$$\geq \varlimsup_{n\to\infty} \sqrt[n]{|a_{n+k}| \frac{l_n}{l_{n+k}}} \varliminf_{n\to\infty} \sqrt[n]{\left(\frac{l_n}{l_{n+1}}\right)^{m-1}} = \frac{1}{R[D_l^{(k)}(f_1 * \ldots * f_p)_m]} \varliminf_{n\to\infty} \left(\frac{l_n}{l_{n+k}}\right)^{(m-1)/n}$$

and, similarly,

$$\frac{1}{R[(D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m]} \leq \frac{1}{R[D_l^{(k)}(f_1 * \ldots * f_p)_m]} \varlimsup_{n\to\infty} \left(\frac{l_n}{l_{n+k}}\right)^{(m-1)/n}.$$

The last inequality yields the validity of Lemma 2. □

3. Hadamard Compositions of Gelfond–Leont'ev Derivatives of Entire Functions

We will remind that the most widely-used descriptions of entire transcendental function f are the lower order $\lambda[f] = \varliminf_{r\to+\infty} \frac{\ln\ln M(r,f)}{\ln r}$ and the order $\varrho[f] = \varlimsup_{r\to+\infty} \frac{\ln\ln M(r,f)}{\ln r}$. In view of the Cauchy inequality, I have

$$\mu(r,f) \leq M(r,f) \leq \sum_{n=0}^{\infty} |a_n|(2r)^n 2^{-n} \leq 2\mu(2r,f),$$

whence it follows that $\lambda[f] = \varliminf_{r\to+\infty} \frac{\ln\ln \mu(r,f)}{\ln r}$ and $\varrho[f] = \varlimsup_{r\to+\infty} \frac{\ln\ln \mu(r,f)}{\ln r}$.

Proposition 1. *If condition* (4) *holds, then for every* k

$$\lambda[(D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m] = \lambda[D_l^{(k)}(f_1 * \ldots * f_p)_m] = \lambda[(f_1 * \ldots * f_p)_m]$$

and

$$\varrho[(D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m] = \varrho[D_l^{(k)}(f_1 * \ldots * f_p)_m] = \varrho[(f_1 * \ldots * f_p)_m].$$

Proof. At first, let $k = 1$. From condition (4), the existence of the numbers $0 < q_1 \leq q_2 < +\infty$ follows such that $q_1^n \leq l_n/l_{n+1} \leq q_2^n$ for all $n \geq 0$. Therefore, using (9), I obtain

$$r\mu(r, D_l^{(1)}(f_1 * \ldots * f_p)_m) = \max\left\{\frac{l_n}{l_{n+1}}|a_{n+1}|r^{n+1} : n \geq 0\right\} \leq$$

$$\leq \frac{1}{q_2} \max\left\{|a_{n+1|}(q_2 r)^{n+1} : n \geq 0\right\} \leq \frac{\mu(q_2 r, (f_1 * \ldots * f_p)_m)}{q_2}$$

and, by analogy,

$$r\mu(r, D_l^{(1)}(f_1 * \ldots * f_p)_m) \geq \frac{\mu(q_1 r,\ (f_1 * \ldots * f_p)_m)}{q_1}$$

for all large-enough r.

$\ln r = o(\ln \mu(r, f))$ as $r \to +\infty$ for each entire transcendental function, hence, I obtain

$$(1 + o(1)) \ln \mu(q_1 r,\ (f_1 * \ldots * f_p)_m) \leq \ln \mu(r, D_l^{(1)}(f_1 * \ldots * f_p)_m) \leq \\ \leq (1 + o(1)) \ln \mu(q_2 r,\ (f_1 * \ldots * f_p)_m)$$

as $r \to +\infty$. Hence, it follows that $\lambda[D_l^{(1)}(f_1 * \ldots * f_p)_m] = \lambda[(f_1 * \ldots * f_p)_m]$ and $\varrho[D_l^{(1)}(f_1 * \ldots * f_p)_m] = \varrho[(f_1 * \ldots * f_p)_m]$.

Since $D_l^{(k+1)} f(z) = D_l^{(1)} D_l^{(k)} f(z)$, the equalities $\lambda[D_l^{(k)}(f_1 * \ldots * f_p)_m] = \lambda[(f_1 * \ldots * f_p)_m]$ and $\varrho[D_l^{(k)}(f_1 * \ldots * f_p)_m] = \varrho[(f_1 * \ldots * f_p)_m]$ are proved.

On the other hand,

$$\mu(r,\ (D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m) = \max\left\{ \left(\frac{l_n}{l_{n+k}}\right)^{m-1} \frac{l_n}{l_{n+k}} |a_{n+k}| r^n :\ n \geq 0 \right\} \geq$$

$$\geq \max\left\{ q_1^{n(m-1)} \frac{l_n}{l_{n+k}} |a_{n+k}| r^n :\ n \geq 0 \right\} = \max\left\{ \frac{l_n}{l_{n+k}} |a_{n+k}| (q_1^{(m-1)} r)^n :\ n \geq 0 \right\} =$$

$$= \mu(q_1^{(m-1)} r,\ D_l^{(k)}(f_1 * \ldots * f_p)_m)$$

and, by analogy,

$$\mu(r,\ (D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m) \leq \mu(q_1^{(m-1)} r,\ D_l^{(k)}(f_1 * \ldots * f_p)_m),$$

whence, as above, I obtain $\lambda[(D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m] = \lambda[D_l^{(k)}(f_1 * \ldots * f_p)_m]$ and $\varrho[(D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m] = \varrho[D_l^{(k)}(f_1 * \ldots * f_p)_m]$. $\square$

Now let me establish a connection between the growth of a function $f = (f_1 * \ldots * f_p)_m$ and the growth of functions f_j. Since $|a_n| \leq \sum_{k_1 + \cdots + k_p = m} |c_{k_1 \ldots k_p}| |a_{n,1}|^{k_1} \cdot \ldots \cdot |a_{n,p}|^{k_p}$, I have

$$|a_n| r^{mn} \leq \sum_{k_1 + \cdots + k_p = m} |c_{k_1 \ldots k_p}| (|a_{n,1}| r^n)^{k_1} \cdot \ldots \cdot (|a_{n,p}| r^n)^{k_p},$$

i.e.,

$$\mu(r^m, f) \leq \sum_{k_1 + \cdots + k_p = m} |c_{k_1 \ldots k_p}| \mu(r, f_1)^{k_1} \cdot \ldots \cdot \mu(r, f_p)^{k_p},$$

whence, for all r large enough, I get

$$\ln \mu(r^m, f) \leq \sum_{k_1 + \cdots + k_p = m} \ln(|c_{k_1 \ldots k_p}| \mu(r, f_1)^{k_1} \cdot \ldots \cdot \mu(r, f_p)^{k_p}) + K_1 =$$

$$= \sum_{k_1 + \cdots + k_p = m} (\ln |c_{k_1 \ldots k_p}| + k_1 \ln \mu(r, f_1) + \ldots + k_p \ln \mu(r, f_p)) + K_1 \leq$$

$$\leq \sum_{k_1 + \cdots + k_p = m} (k_1 \ln \mu(r, f_1) + \ldots + k_p \ln \mu(r, f_p)) + K_2, \quad (K_j \equiv \text{const} > 0). \tag{10}$$

Let $\max\{\varrho[f_j] : 1 \le j \le p\} = \varrho < +\infty$. Then, $\ln \mu(r, f_j) \le r^{\varrho+\varepsilon}$ for every $\varepsilon > 0$ all $r \ge r_0(\varepsilon)$ and all j. Therefore, in view of (10)

$$\ln \mu(r^m, f) \le r^{\varrho+\varepsilon} \sum_{k_1+\dots+k_p=m} (k_1 + \ldots + k_p) + K_2,$$

whence

$$\varrho[f] = \overline{\lim_{r\to+\infty}} \frac{\ln \ln \mu(r^m, f)}{\ln r^m} = \overline{\lim_{r\to+\infty}} \frac{\ln \ln \mu(r^m, f)}{m \ln r} \le \frac{\varrho+\varepsilon}{m},$$

and, in view of the arbitrariness of ε, I obtain $m\varrho[f] \le \varrho$.

Suppose now that the function f_1 is dominant. Then $|a_n| = (1 + o(1))|c_{m0\ldots0}||a_{n,1}|^m$ as $n \to \infty$ and, thus, $|a_n|r^{mn} = (1 + o(1))|c_{m0\ldots0}|(|a_{n,1}|r^n)^m$ as $n \to \infty$, whence it follows that

$$A_1\mu(r, f_1)^m \le \mu(r^m, f) \le A_2\mu(r, f_1)^m, \quad (A_1, A_2 = \text{const} > 0). \tag{11}$$

Using (11), as above, I obtain $m\varrho[f] = \varrho[f_1]$ and $m\lambda[f] = \lambda[f_1]$. Thus, the following statement is proved.

Proposition 2. *Let $f_1, \ldots, f_p$ be entire transcendental functions, and condition (4) holds. If $\varrho[f_1] = \ldots = \varrho[f_p] = \varrho$, then $m\varrho[(f_1 * \ldots * f_p)_m] \le \varrho$, and, if among functions $f_1, \ldots, f_p$ there is a dominant function f_1, then $m\varrho[(f_1 * \ldots * f_p)_m] = \varrho$. Moreover, if among functions $f_1, \ldots, f_p$ there is a dominant function f_1 then $m\lambda[(f_1 * \ldots * f_p)_m] = \lambda[f_1]$.*

Let us now examine the growth of the ratio $\dfrac{\mu(r, (D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m)}{\mu(r, D_l^{(k)}(f_1 * \ldots * f_p)_m)}$, $j \ge k$. Let $\nu(r, f) = \max\{n : |a_n|r^n = \mu(r, f)\}$ be the central index of series (1). Then, $\mu(r, f) = |a_{\nu(r,f)}|r^{\nu(r,f)}$ and, therefore,

$$\mu(r, D_l^{(k)}(f_1 * \ldots * f_p)_m) = \frac{l_{\nu(r,D_l^{(k)} f)}}{l_{\nu(r,D_l^{(k)} f)+k}} |a_{\nu(r,D_l^{(k)} f)+k}| r^{\nu(r,D_l^{(k)} f)} =$$

$$= \left(\frac{l_{\nu(r,D_l^{(k)} f)+k}}{l_{\nu(r,D_l^{(k)} f+k-j)}}\right)^{m-1} \frac{l_{\nu(r,D_l^{(k)} f)}}{l_{\nu(r,D_l^{(k)} f)+k-j}} \times$$

$$\times \left(\frac{l_{\nu(r,D_l^{(k)} f)+k-j}}{l_{\nu(r,D_l^{(k)} f)+k-j+j}}\right)^{m} |a_{\nu(r,D_l^{(k)})+k-j+j}| r^{\nu(r,D_l^{(k)} f)+k-j} r^{j-k} \le$$

$$\le \left(\frac{l_{\nu(r,D_l^{(k)} f)+k}}{l_{\nu(r,D_l^{(k)} f)+k-j}}\right)^{m-1} \frac{l_{\nu(r,D_l^{(k)} f)}}{l_{\nu(r,D_l^{(k)} f)+k-j}} r^{j-k} \mu(r, (D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m)$$

and

$$\mu(r, (D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m) =$$

$$= \left(\frac{l_{\nu(r,(D_l^{(j)} f_1*\ldots*D_l^{(j)} f_p)_m}}{l_{\nu(r,(D_l^{(j)} f_1*\ldots*D_l^{(j)} f_p)_m+j}}\right)^{m} |a_{\nu(r,(D_l^{(j)} f_1*\ldots*D_l^{(j)} f_p)_m)+j}| r^{\nu(r,(D_l^{(j)} f_1*\ldots*D_l^{(j)} f_p)_m} =$$

$$= \left(\frac{l_{\nu(r,(D_l^{(j)} f_1*\ldots*D_l^{(j)} f_p)_m}}{l_{\nu(r,(D_l^{(j)} f_1*\ldots*D_l^{(j)} f_p)_m+j}}\right)^{m} |a_{\nu(r,(D_l^{(j)} f_1*\ldots*D_l^{(j)} f_p)_m)+j-k+k}| r^{\nu(r,(D_l^{(j)} f_1*\ldots*D_l^{(j)} f_p)_m+j-k} r^{k-j}$$

$$= \left(\frac{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m}}{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m + j}} \right)^{m-1} \frac{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m}}{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m + j-k}} \times$$

$$\times \frac{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m + j-k}}{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m + j}} |a_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m) + j-k+k}| r^{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m + j-k} r^{k-j} \leq$$

$$\leq \left(\frac{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m}}{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m + j}} \right)^{m-1} \frac{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m}}{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m + j-k}} r^{k-j} \mu(r, D_l^{(k)} (f_1 * \ldots * f_p)_m).$$

Hence, it follows that

$$\left(\frac{l_{\nu(r,D_l^{(k)} f)+k-j}}{l_{\nu(r,D_l^{(k)} f)+k}} \right)^{m-1} \frac{l_{\nu(r,D_l^{(k)} f)+k-j}}{l_{\nu(r,D_l^{(k)} f)}} \leq \frac{r^{j-k} \mu(r, (D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m)}{\mu(r, D_l^{(k)} (f_1 * \ldots * f_p)_m)} \leq$$

$$\leq \left(\frac{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m}}{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m + j}} \right)^{m-1} \frac{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m}}{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m + j-k}}. \tag{12}$$

Using (12), I prove such a theorem.

Theorem 1. *Let f_j be entire transcendental functions, $1 \leq j \leq p$, and (4) hold with $q > 1$. Then, for each $j \geq k$*

$$\underline{\lim}_{r \to +\infty} \frac{1}{\ln r} \ln \ln \frac{r^{j-k} \mu(r, (D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m)}{\mu(r, D_l^{(k)} (f_1 * \ldots * f_p)_m)} = \lambda[(f_1 * \ldots * f_p)_m] \tag{13}$$

and

$$\overline{\lim}_{r \to +\infty} \frac{1}{\ln r} \ln \ln \frac{r^{j-k} \mu(r, (D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m)}{\mu(r, D_l^{(k)} (f_1 * \ldots * f_p)_m)} = \varrho[(f_1 * \ldots * f_p)_m]. \tag{14}$$

Proof. From condition (4), with $q > 1$ the existence of numbers $1 < q_1 \leq q_2 < +\infty$, it follows such that $q_1^{nk} \leq l_n / l_{n+k} \leq q_2^{nk}$ for all $n \geq n_0$. Therefore,

$$\left(\frac{l_{\nu(r,D_l^{(k)} f)+k-j}}{l_{\nu(r,D_l^{(k)} f)+k}} \right)^{m-1} \frac{l_{\nu(r,D_l^{(k)} f)+k-j}}{l_{\nu(r,D_l^{(k)} f)}} \geq q_1^{(m-1)(\nu(r,D_l^{(k)} f)+k-j)j} q_1^{(\nu(r,D_l^{(k)} f)+k-j)(j-k)} =$$

$$= q_1^{(mj-k)(\nu(r,D_l^{(k)} f)+k-j)},$$

$$\left(\frac{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m)}}{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m) + j}} \right)^{m-1} \frac{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m)}}{l_{\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m) + j-k}} \leq$$

$$q_2^{(m-1)\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m)} q_2^{(j-k)\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m)} = q_2^{(jm-k)\nu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m)},$$

and, thus, (12) implies

$$(mj-k)(\nu(r, D_l^{(k)} (f_1 * \ldots * f_p)_m) + k - j) \ln q_1 \leq \ln \frac{r^{j-k} \mu(r, (D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m)}{\mu(r, D_l^{(k)} (f_1 * \ldots * f_p)_m)} \leq$$

$$\leq (mj-k) \nu(r, (D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m) \ln q_2. \tag{15}$$

It is well known [16, p. 13] that

$$\ln\,\mu(r,f) = \ln\,\mu(1,f) + \int_1^r \frac{\nu(t,f)}{t}dt. \tag{16}$$

Hence, I get $\ln\,\mu(r,f) \le \ln\,\mu(1,f) + \nu(r,f)\ln r$, $\ln\,\mu(r,f) \ge \ln\,\mu(1,f) + \nu(r/2,f)\ln 2$ and, thus, $\lambda[f] = \varliminf\limits_{r\to+\infty} \frac{\ln\,\nu(r,f)}{\ln r}$, $\varrho[f] = \varlimsup\limits_{r\to+\infty} \frac{\ln\,\nu(r,f)}{\ln r}$. Therefore, by Proposition 1

$$\varliminf_{r\to+\infty} \frac{\ln\,\nu(r, D_l^{(k)}(f_1 * \ldots * f_p)_m)}{\ln r} = \lambda[D_l^{(k)}(f_1 * \ldots * f_p)_m] = \lambda[(f_1 * \ldots * f_p)_m],$$

$$\varlimsup_{r\to+\infty} \frac{\ln\,\nu(r, D_l^{(k)}(f_1 * \ldots * f_p)_m)}{\ln r} = \varrho[D_l^{(k)}(f_1 * \ldots * f_p)_m] = \varrho[(f_1 * \ldots * f_p)_m],$$

and these equalities hold good if, instead of $D_l^{(k)}(f_1 * \ldots * f_p)_m$, it is possible to put $(D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m$. From this and (15), I obtain (13) and (14). Theorem 1 is proved. □

For $j = k$, Theorem 1 implies the following statement.

Corollary 1. *Let f_j be entire functions, $1 \le j \le p$, and (4) holds with $q > 1$. Then, for each $k \ge 1$*

$$\varliminf_{r\to+\infty} \frac{1}{\ln r} \ln\ln \frac{\mu(r,(D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m)}{\mu(r, D_l^{(k)}(f_1 * \ldots * f_p)_m)} = \lambda[(f_1 * \ldots * f_p)_m]$$

and

$$\varlimsup_{r\to+\infty} \frac{1}{\ln r} \ln\ln \frac{\mu(r,(D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m)}{\mu(r, D_l^{(k)}(f_1 * \ldots * f_p)_m)} = \varrho[(f_1 * \ldots * f_p)_m].$$

From Corollary 1 and Proposition 2, I obtain the following corollary.

Corollary 2. *Suppose that, among entire functions $f_1, \ldots, f_p$, there is a dominant function f_1. If condition (4) holds with $q > 1$ then, for each $k \ge 1$,*

$$\varliminf_{r\to+\infty} \frac{1}{\ln r} \ln\ln \frac{\mu(r,(D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m)}{\mu(r, D_l^{(k)}(f_1 * \ldots * f_p)_m)} = \frac{\lambda[f_1]}{m}$$

and

$$\varlimsup_{r\to+\infty} \frac{1}{\ln r} \ln\ln \frac{\mu(r,(D_l^{(k)} f_1 * \ldots * D_l^{(k)} f_p)_m)}{\mu(r, D_l^{(k)}(f_1 * \ldots * f_p)_m)} = \frac{\varrho[f_1]}{m}.$$

Let us now consider the case when, instead of condition (5), the stronger condition

$$0 < \varlimsup_{n\to\infty} \frac{l_n}{(n+1)l_{n+1}} \le \varlimsup_{n\to\infty} \frac{l_n}{(n+1)l_{n+1}} < +\infty \tag{17}$$

is fulfilled.

Theorem 2. *Let f_j be entire functions, $1 \le j \le p$, and (17) hold, then, for each $j \ge k$,*

$$\varliminf_{r\to+\infty} \frac{1}{\ln r} \ln \frac{\mu(r,(D_l^{(j)} f_1 * \ldots * D_l^{(j)} f_p)_m)}{\mu(r, D_l^{(k)}(f_1 * \ldots * f_p)_m)} = (mj-k)\lambda[(f_1 * \ldots * f_p)_m] + k - j \tag{18}$$

and

$$\overline{\lim_{r\to+\infty}}\frac{1}{\ln r}\ln\frac{\mu(r,(D_l^{(j)}f_1*\ldots*D_l^{(j)}f_p)_m)}{\mu(r,D_l^{(k)}(f_1*\ldots*f_p)_m)}=(mj-k)\varrho[(f_1*\ldots*f_p)_m]+k-j. \tag{19}$$

Proof. From condition (17), the existence of numbers $0<h_1\le h_2<+\infty$ follows such that $(h_1n)^k\le l_n/l_{n+k}\le(h_2n)^k$. Therefore,

$$\left(\frac{l_{\nu(r,D_l^{(k)}f)+k-j}}{l_{\nu(r,D_l^{(k)}f)+k}}\right)^{m-1}\frac{l_{\nu(r,D_l^{(k)}f)+k-j}}{l_{\nu(r,D_l^{(k)}f)}}\ge$$

$$\ge h_1^{(m-1)j}(\nu(r,D_l^{(k)}f)+k-j)^{(m-1)j}h_1^{j-k}(\nu(r,D_l^{(k)}f)+k-j)^{j-k}=$$

$$=h_1^{mj-k}(\nu(r,D_l^{(k)}f)+k-j)^{mj-k},$$

$$\left(\frac{l_{\nu(r,(D_l^{(j)}f_1*\ldots*D_l^{(j)}f_p)_m)}}{l_{\nu(r,(D_l^{(j)}f_1*\ldots*D_l^{(j)}f_p)_m)+j}}\right)^{m-1}\frac{l_{\nu(r,(D_l^{(j)}f_1*\ldots*D_l^{(j)}f_p)_m)}}{l_{\nu(r,(D_l^{(j)}f_1*\ldots*D_l^{(j)}f_p)_m)+j-k}}\le$$

$$(h_2\nu(r,(D_l^{(j)}f_1*\ldots*D_l^{(j)}f_p)_m))^{j(m-1)}(h_2\nu(r,(D_l^{(j)}f_1*\ldots*D_l^{(j)}f_p)_m))^{j-k}=$$

$$=(h_2\nu(r,(D_l^{(j)}f_1*\ldots*D_l^{(j)}f_p)_m))^{mj-k},$$

and, thus, (12) implies

$$(mj-k)\ln(\nu(r,D_l^{(k)}(f_1*\ldots*f_p)_m))+k-j)+(mj-k)\ln h_1\le$$

$$\le(j-k)\ln r+\ln\frac{r^{j-k}\mu(r,(D_l^{(j)}f_1*\ldots*D_l^{(j)}f_p)_m)}{\mu(r,D_l^{(k)}(f_1*\ldots*f_p)_m)}\le$$

$$\le(mj-k)\ln(\nu(r,(D_l^{(j)}f_1*\ldots*D_l^{(j)}f_p)_m+(mj-k)\ln h_2.$$

Hence, as above, I obtain

$$j-k+\underline{\lim_{r\to+\infty}}\frac{1}{\ln r}\ln\frac{\mu(r,(D_l^{(j)}f_1*\ldots*D_l^{(j)}f_p)_m)}{\mu(r,D_l^{(k)}(f_1*\ldots*f_p)_m)}=(mj-k)\lambda[(f_1*\ldots*f_p)_m]$$

and

$$j-k+\overline{\lim_{r\to+\infty}}\frac{1}{\ln r}\ln\frac{\mu(r,(D_l^{(j)}f_1*\ldots*D_l^{(j)}f_p)_m)}{\mu(r,D_l^{(k)}(f_1*\ldots*f_p)_m)}=(mj-k)\varrho[(f_1*\ldots*f_p)_m],$$

i.e., (18) and (19) hold. Theorem 2 is proved. □

For $j=k+1$, Theorem 2 implies the following statement.

Corollary 3. *Let f_j be entire functions, $1\le j\le p$, and (17) hold, then, for each $k\ge1$,*

$$\underline{\lim_{r\to+\infty}}\frac{1}{\ln r}\ln\frac{\mu(r,(D_l^{(k+1)}f_1*\ldots*D_l^{(k+1)}f_p)_m)}{\mu(r,D_l^{(k)}(f_1*\ldots*f_p)_m)}=((m-1)k+m)\lambda[(f_1*\ldots*f_p)_m]-1$$

and

$$\overline{\lim_{r\to+\infty}}\frac{1}{\ln r}\ln\frac{\mu(r,(D_l^{(k+1)}f_1*\ldots*D_l^{(k+1)}f_p)_m)}{\mu(r,D_l^{(k)}(f_1*\ldots*f_p)_m)}=((m-1)k+m)\varrho[(f_1*\ldots*f_p)_m]-1.$$

Choosing $l_n=1/n!$ and $m=2$ from Corollary 3, I get the above result of M.K. Sen [12], i.e., Lemma 1.

4. Hadamard Compositions of Gelfond–Leont'ev Derivatives for Functions Analytic in a Disk

For the functions analytic in the disk $U=\{z:|z|<1\}$, the lower order $\lambda_U[f]$ and the order $\varrho_U[f]$ are defined as

$$\lambda_U[f]=\underline{\lim_{r\uparrow1}}\frac{\ln^+\ln M(r,f)}{-\ln(1-r)},\quad \varrho_U[f]=\overline{\lim_{r\uparrow1}}\frac{\ln^+\ln M(r,f)}{-\ln(1-r)}.$$

Since

$$\mu(r,f)\le M(r,f)\le\sum_{n=0}^{\infty}|a_n|\left(\frac{1+r}{2}\right)^n\left(\frac{2r}{1+r}\right)^n\le\frac{1+r}{1-r}\mu\left(\frac{1+r}{2},f\right),$$

and in view of (16) for $r\ge1/e$

$$\nu\left(\frac{1+r}{2},f\right)\ge\int\limits_{1/e}^{(1+r)/2}\frac{\nu(t,f)}{t}dt=\ln\mu\left(\frac{1+r}{2},f\right)-\ln\mu\left(\frac{1}{2},f\right)\ge$$

$$\ge\int\limits_{r}^{(1+r)/2}\frac{\nu(t,f)}{t}dt\ge\nu(r,f)\ln\frac{1+r}{2r}=(1+o(1)\frac{1-r}{2}\nu(r,f),\quad r\uparrow1,$$

I obtain

$$\underline{\lim_{r\uparrow1}}\frac{\ln^+\ln\mu(r,f)}{-\ln(1-r)}=\lambda_U[f],\quad \overline{\lim_{r\uparrow1}}\frac{\ln^+\ln\mu(r,f)}{-\ln(1-r)}=\varrho_U[f]$$

and

$$\lambda_U[f]\le\underline{\lim_{r\uparrow1}}\frac{\ln^+\nu(r,f)}{-\ln(1-r)}\le\lambda_U[f]+1,\quad \varrho_U[f]\le\overline{\lim_{r\uparrow1}}\frac{\ln^+\nu(r,f)}{-\ln(1-r)}\le\varrho_U[f]+1. \tag{20}$$

Proposition 3. *If condition (17) holds, then, for every k,*

$$\lambda_U[(D_l^{(k)}f_1*\ldots*D_l^{(k)}f_p)_m]=\lambda_U[D_l^{(k)}(f_1*\ldots*f_p)_m]=\lambda_U[(f_1*\ldots*f_p)_m]$$

and

$$\varrho_U[(D_l^{(k)}f_1*\ldots*D_l^{(k)}f_p)_m]=\varrho_U[D_l^{(k)}(f_1*\ldots*f_p)_m]=\varrho_U[(f_1*\ldots*f_p)_m].$$

Proof. At first, I remark that, for each function f and analytic in U, the equalities $\lambda_U[f']=\lambda_U[f]$ and $\varrho_U[f']=\varrho_U[f]$ are true.

Indeed, from Cauchy formula $f'(z)=\dfrac{1}{2\pi i}\displaystyle\int\limits_{|\tau-z|=(1-|z|)/2}\frac{f(\tau)d\tau}{(\tau-z)^2}$ I have $M(r,f')\le 2M((1+r)/2,f)/(1-r)$, and since $f(z)-f(0)=\displaystyle\int\limits_0^z f'(\tau)d\tau$, the inequality $M(r,f)\le M(r,f')+|f(0)|$ holds, from which the necessary equalities follow.

Now, in view of (17) $h_1(n+1) \le l_n/l_{n+1} \le h_2(n+1)$, I have, for $k=1$,

$$\mu(r, D_l^{(1)}((f_1 * \ldots * f_p)_m)) = \max\left\{\frac{l_n}{l_{n+1}}|a_{n+1}|r^n : n \ge 0\right\} \le$$

$$\le h_2 \max\{(n+1)|a_{n+1}|r^n : n \ge 0\} = h_2\mu(r, ((f_1 * \ldots * f_p)_m)')$$

and by analogy $\mu(r, D_l^{(1)}(f_1 * \ldots * f_p)_m) \ge h_1\mu(r, ((f_1 * \ldots * f_p)_m)')$. Hence, it follows that $\lambda_U[D_l^{(1)}(f_1 * \ldots * f_p)_m] = \lambda_U[(f_1 * \ldots * f_p)_m]$ and $\varrho_U[D_l^{(1)}(f_1 * \ldots * f_p)_m] = \varrho_U[(f_1 * \ldots * f_p)_m]$ and, thus, $\lambda_U[D_l^{(k)}(f_1 * \ldots * f_p)_m] = \lambda_U[(f_1 * \ldots * f_p)_m]$ and $\varrho_U[D_l^{(k)}(f_1 * \ldots * f_p)_m] = \varrho_U[(f_1 * \ldots * f_p)_m]$.

Since $l_n/l_{n+1} \ge h_1(n+1) \ge h_1$, I have $(l_n/l_{n+k})^{m-1} \ge h_1^{k(m-1)}$, i. e.

$$\mu(r,\ (D_l^{(k)}f_1 * \ldots * D_l^{(k)}f_p)_m) = \max\left\{\left(\frac{l_n}{l_{n+k}}\right)^{m-1}\frac{l_n}{l_{n+k}}|a_{n+k}|r^n : n \ge 0\right\} \ge$$

$$\ge h_1^{k(m-1)}\mu(r, D_l^{(1)}((f_1 * \ldots * f_p)_m))$$

and, thus, $\lambda_U[D_l^{(k)}(f_1 * \ldots * f_p)_m] \le \lambda_U[(D_l^{(k)}f_1 * \ldots * D_l^{(k)}f_p)_m]$ and $\varrho_U[D_l^{(k)}(f_1 * \ldots * f_p)_m] \le \varrho_U[(D_l^{(k)}f_1 * \ldots * D_l^{(k)}f_p)_m]$.

On the other hand,

$$\left(\frac{l_n}{l_{n+k}}\right)^{m-1} \le \left(h_2^k(n+1)\ldots(n+k)\right)^{m-1} \le$$

$$\le h_2^{k(m-1)}(n+1)\ldots(n+k)(n+k+1)\ldots(n+2k)\ldots(n+(m-2)k+1)\ldots(n+(m-1)k).$$

Therefore,

$$\mu(r,\ (D_l^{(k)}f_1 * \ldots * D_l^{(k)}f_p)_m) \le h_2^{k(m-1)}\max\left\{(n+1)\ldots(n+(m-1)k)\frac{l_n}{l_{n+k}}|a_{n+k}|r^n : n \ge 0\right\} =$$

$$= h_2^{k(m-1)}\max\left\{\left(\frac{l_n}{l_{n+k}}|a_{n+k}|r^{n+(m-1)k}\right)^{((m-1)k)} : n \ge 0\right\} =$$

$$= h_2^{k(m-1)}\max\left\{\left(r^{k(m-1)}\frac{l_n}{l_{n+k}}|a_{n+k}|r^n\right)^{(k(m-1))} : n \ge 0\right\} = h_2^{k(m-1)}\mu(r, F^{(k(m-1))}),$$

where $F(z) = z^{k(m-1)}D_l^{(k)}(f_1 * \ldots * f_p)_m(z)$. Hence it follows that $\lambda_U[(D_l^{(k)}f_1 * \ldots * D_l^{(k)}f_p)_m] \le \lambda_U[F^{(k(m-1))}] = \lambda_U[F] = \lambda_U[D_l^{(k)}(f_1 * \ldots * f_p)_m]$ and $\varrho_U[(D_l^{(k)}f_1 * \ldots * D_l^{(k)}f_p)_m] \le \varrho_U[D_l^{(k)}(f_1 * \ldots * f_p)_m]$. Proposition 3 is proved. □

The following statement is an analog of Proposition 2.

Proposition 4. *Let $f_1, \ldots, f_p$ be functions analytic in U and condition (17) holds. If $\varrho_U[f_1] = \ldots = \varrho_U[f_p] = \varrho$, then $\varrho_U[(f_1 * \ldots * f_p)_m] \le m\varrho$, and if. among functions $f_1, \ldots, f_p$. there is a dominant function f_1, then $\varrho_U[(f_1 * \ldots * f_p)_m] = m\varrho$. Moreover, if, among functions $f_1, \ldots, f_p$, there is a dominant function f_1, then $\lambda_U[(f_1 * \ldots * f_p)_m] = m\lambda_U[f_1]$.*

Proof. Indeed, since $\varrho_U[f_1] = \ldots = \varrho_U[f_p] = \varrho$, for every $\varepsilon > 0$ and all $r \in [r_0(\varepsilon, 1)$ I have $\ln\,\mu(r, f_j) \le (1/(1-r))^{\varrho+\varepsilon}$, and (10) implies $\ln\,\mu(r^m, f) \le C(1/(1-r))^{\varrho+\varepsilon}$ for $r \in [r_0(\varepsilon, 1)$, where $C = \text{const}$. Therefore,

$$\varrho_U[f] = \overline{\lim_{r\uparrow 1}}\frac{\ln^+\ln\,\mu(r^m, f)}{-\ln(1-r^m)} \le (\varrho+\varepsilon)\overline{\lim_{r\uparrow 1}}\frac{-\ln(1-r)}{-\ln(1-r^m)} \le (\varrho+\varepsilon)\overline{\lim_{r\uparrow 1}}\frac{1-r^m}{1-r} = (\varrho+\varepsilon)m,$$

and, in view of the arbitrariness of ε, I obtain $\varrho_U[(f_1 * \ldots * f_p)_m] \le m\varrho$.

If the function f_1 is dominant, then, from (11), I obtain $\varrho_U[f] = m\varrho_U[f_1]$ and $\lambda_U[f] = m\lambda_U[f_1]$. $\square$

Using (12) and (20), I can investigate the growth of the ratio $\dfrac{\mu(r,(D_l^{(j)}f_1 * \ldots * D_l^{(j)}f_p)_m)}{\mu(r,D_l^{(k)}(f_1 * \ldots * f_p)_m)}$ in the case of analytic functions in U. I will not dwell on this, but rather study the growth of the ratio $\dfrac{\mu(r,D_l^{(j)}(f_1 * \ldots * f_p)_m)}{\mu(r,D_l^{(k)}(f_1 * \ldots * f_p)_m)}$ for $j > k$ and the ratio $\dfrac{\mu(r,(D_l^{(j)}f_1 * \ldots * D_l^{(j)}f_p)_m)}{\mu(r,(D_l^{(k)}f_1 * \ldots * D_l^{(k)}f_p)_m)}$ for $j > k$.

As above, I have

$$\mu(r,D_l^{(k)}(f_1 * \ldots * f_p)_m) = \frac{l_{\nu(r,D_l^{(k)}f)}}{l_{\nu(r,D_l^{(k)}f)+k}}|a_{\nu(r,D_l^{(k)}f)+k}|r^{\nu(r,D_l^{(k)}f)} =$$

$$= \frac{l_{\nu(r,D_l^{(k)}f)}}{l_{\nu(r,D_l^{(k)}f)+k-j}}\frac{l_{\nu(r,D_l^{(k)}f)+k-j}}{l_{\nu(r,D_l^{(k)}f)+k-j+j}}|a_{\nu(r,D_l^{(k)}f)+k-j+j}|r^{\nu(r,D_l^{(k)}f)+k-j}r^{j-k} \le$$

$$\le \frac{l_{\nu(r,D_l^{(k)}f)}}{l_{\nu(r,D_l^{(k)}f)+k-j}}r^{j-k}\mu(r,D_l^{(j)}(f_1 * \ldots * f_p)_m)$$

and, similarly,

$$\mu(r,D_l^{(j)}(f_1 * \ldots * f_p)_m) \le \frac{l_{\nu(r,D_l^{(j)}f)}}{l_{\nu(r,D_l^{(j)}f)+j-k}}r^{k-j}\mu(r,D_l^{(k)}(f_1 * \ldots * f_p)_m).$$

Therefore,

$$\frac{l_{\nu(r,D_l^{(k)}f)+k-j}}{l_{\nu(r,D_l^{(k)}f)}} \le \frac{r^{j-k}\mu(r,D_l^{(j)}(f_1 * \ldots * f_p)_m)}{\mu(r,D_l^{(k)}(f_1 * \ldots * f_p)_m)} \le \frac{l_{\nu(r,D_l^{(j)}f)}}{l_{\nu(r,D_l^{(j)}f)+j-k}}. \tag{21}$$

Using (21), I prove following theorem.

Theorem 3. *Let f_j be analytic functions in U, $1 \le j \le p$, and let (17) hold, then, for each $j \ge k$,*

$$(j-k)\lambda_U[(f_1 * \ldots * f_p)_m] \le \varliminf_{r\uparrow 1}\frac{1}{-\ln(1-r)}\ln\frac{\mu(r,D_l^{(j)}(f_1 * \ldots * f_p)_m)}{\mu(r,D_l^{(k)}(f_1 * \ldots * f_p)_m)} \le (j-k)(\lambda_U[(f_1 * \ldots * f_p)_m]+1)$$

and

$$(j-k)\varrho_U[(f_1 * \ldots * f_p)_m] \le \varlimsup_{r\uparrow 1}\frac{1}{-\ln(1-r)}\ln\frac{\mu(r,D_l^{(j)}(f_1 * \ldots * f_p)_m)}{\mu(r,D_l^{(k)}(f_1 * \ldots * f_p)_m)} \le (j-k)(\varrho_U[(f_1 * \ldots * f_p)_m]+1).$$

Proof. Since $(h_1 n)^k \le l_n/l_{n+k} \le (h_2 n)^k$, I have

$$\frac{l_{\nu(r,D_l^{(k)}f)+k-j}}{l_{\nu(r,D_l^{(k)}f)}} = \frac{l_{\nu(r,D_l^{(k)}f)+k-j}}{l_{\nu(r,D_l^{(k)}f)+k-j+j-k}} \ge (h_1\nu(r,D_l^{(k)}f)+k-j)^{j-k}$$

and

$$\frac{l_{\nu(r,D_l^{(j)}f)}}{l_{\nu(r,D_l^{(j)}f)+j-k}} \le (h_2\nu(r,D_l^{(j)}f))^{j-k}.$$

Therefore, (21) implies

$$(1+o(1))(j-k)\ln \nu(r,D_l^{(k)}f) \le \ln \frac{\mu(r,D_l^{(j)}(f_1*\ldots*f_p)_m)}{\mu(r,D_l^{(k)}(f_1*\ldots*f_p)_m)} \le (1+o(1))(j-k)\ln \nu(r,D_l^{(j)}f)$$

as $r\uparrow 1$, i.e., in view of (20) and Proposition 3, I obtain

$$(j-k)\lambda_U[f] \le \varliminf_{r\uparrow 1} \frac{1}{-\ln(1-r)} \ln \frac{\mu(r,D_l^{(j)}(f_1*\ldots*f_p)_m)}{\mu(r,D_l^{(k)}(f_1*\ldots*f_p)_m)} \le (j-k)(\lambda_U[f]+1)$$

and

$$(j-k)\varrho_U[f] \le \varlimsup_{r\uparrow 1} \frac{1}{-\ln(1-r)} \ln \frac{\mu(r,D_l^{(j)}(f_1*\ldots*f_p)_m)}{\mu(r,D_l^{(k)}(f_1*\ldots*f_p)_m)} \le (j-k)(\varrho_U[f]+1).$$

Q.E.D. $\square$

Since series (8) differs from series (7), only that instead l_k/l_{k+1}, it contains $(l_k/l_{k+1})^m$, I will easily prove the inequalities

$$\left(\frac{l_{\nu(r,D_l^{(k)}f)+k-j}}{l_{\nu(r,D_l^{(k)}f)}})\right)^m \le \frac{r^{j-k}\mu(r,D_l^{(j)}(f_1*\ldots*f_p)_m)}{\mu(r,D_l^{(k)}(f_1*\ldots*f_p)_m)} \le \left(\frac{l_{\nu(r,D_l^{(j)}f)}}{l_{\nu(r,D_l^{(j)}f)+j-k}}\right)^m,$$

whence, as in the proof of Theorem 3, I will come to the next theorem.

Theorem 4. *Let f_j be analytic functions in U, $1\le j\le p$, and let (17) hold, then, for each $j\ge k$,*

$$m(j-k)\lambda_U[(f_1*\ldots*f_p)_m] \le \varliminf_{r\uparrow 1} \frac{1}{-\ln(1-r)} \ln \frac{\mu(r,(D_l^{(j)}f_1*\ldots*D_l^{(j)}f_p)_m)}{\mu(r,(D_l^{(k)}f_1*\ldots*D_l^{(k)}f_p)_m)} \le$$
$$\le m(j-k)(\lambda_U[(f_1*\ldots*f_p)_m]+1)$$

and

$$m(j-k)\varrho_U[(f_1*\ldots*f_p)_m] \le \varlimsup_{r\uparrow 1} \frac{1}{-\ln(1-r)} \ln \frac{\mu(r,(D_l^{(j)}f_1*\ldots*D_l^{(j)}f_p)_m)}{\mu(r,(D_l^{(k)}f_1*\ldots*D_l^{(k)}f_p)_m)} \le$$
$$\le m(j-k)(\varrho_U[(f_1*\ldots*f_p)_m]+1).$$

5. Discussion

In conclusion, I note that, in addition to the analytic continuation of functions, the usual Hadamard composition was used in other aspects of complex analysis (in particular, in the geometric function theory). One can naturally hope that the Hadamard composition of the genus m will also find applications in similar areas of mathematics.

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