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Hybrid Nil Radical of a Ring

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Abstract: The nature of universe problems is ambiguous due to the presence of asymmetric data in almost all disciplines, including engineering, mathematics, medical sciences, physics, computer science, operations research, artificial intelligence, and management sciences, and they involve various types of uncertainties when dealing with them on various occasions. To deal with the challenges of uncertainty and asymmetric information, different theories have been developed, including probability, fuzzy sets, rough sets, soft ideals, etc. The strategies of hybrid ideals, hybrid nil radicals, hybrid semiprime ideals, and hybrid products of rings are introduced in this paper and hybrid structures are used to examine the structural properties of rings.

Keywords: rings; hybrid structure; hybrid ideals; hybrid nil radical; hybrid product.

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Citation: Porselvi, K.; Muhiuddin, G.; Elavarasan, B.; Assiry, A. Hybrid Nil Radical of a Ring. *Symmetry* **2022**, *14*, 1367. https://doi.org/10.3390/sym14071367

Academic Editor: Evgeny Nikulchev

Received: 29 March 2022
Accepted: 18 April 2022
Published: 3 July 2022

1. Introduction

In [1], Zadeh developed and investigated fuzzy set theory and its features in relation to set theory, which gives a suitable structure for generalising fundamental algebraic ideas, including set theory, group theory, ring theory, etc. In group theory, Rosenfeld [2] explored the perception of a fuzzy set. Kuroki [3–5] has proposed fuzzy ideal structures and bi-ideals in the theory of semigroups. Liu [6] explored the fuzzy ideal of a ring and obtained significant results. Mukherjee et al. [7] defined and discussed the fuzzy prime ideal notion of a ring. Later, fuzzy prime ideals were further discussed by Swamy et al. [8]. Malik [9] studied the fuzzy maximal and radical of a ring. Kumbhojkar et al. [10] studied fuzzy nil radicals of a ring. A fuzzy nil radical in different algebraic structures was explored by several researchers [11–14]. The fuzzy subnear-ring theory concept was presented by Abou-Zaid [15]. Since then, many mathematicians have pursued their research in fuzzy algebraic structures in different directions and obtained significant results in semigroups, groups, rings, etc.

Molodtsov [16] introduced and examined the soft set theory idea to deal with uncertainty concepts. Later on, different researchers looked at these ideas. The hybrid structure idea, which is parallel to fuzzy set theory and soft set theory, was initiated by Jun et al. [17] and obtained various results, and then applied to the theory of BCI/BCK-algebras. Anis et al., in [18], presented hybrid structures in semigroups, including hybrid subsemigroups, and hybrid ideals, and analysed their significant properties.

Elavarasan et al. [19], investigated the concept of hybrid ideals and hybrid bi-ideals in semigroups and established equivalent conditions in terms of hybrid ideals and hybrid bi-ideals for a semigroup to be regular and intra-regular. They also used hybrid ideals and hybrid bi-ideals to characterise the left and right simple semigroups, as well as the completely regular semigroups. In [20], Porselvi et al. defined hybrid interior ideals and hybrid characteristic interior ideals of a semigroup and found some equivalent conditions for a hybrid structure to be a hybrid interior ideal of a semigroup. For a regular semigroup and an intra-regular semigroup, hybrid interior ideals and hybrid ideals are also shown to be the same.

In [21], Porselvi et al. introduced the concepts of hybrid interior ideals and hybrid simple in an ordered semigroup, discussed the characteristic hybrid structures using ideals and interior ideals, and defined the ordered semigroup in terms of various hybrid ideal structures. The equivalent condition for an ordered semigroup was also established to be simple. In [22], Porselvi et al. presented the ideas of hybrid AG-groupoid as well as equivalent claims for a semigroup to be regular, and explored several properties of hybrid ideal structures.

In [23], Elavarasan et al. hooked up the idea of hybrid ideals in a near-ring, established various properties, and discussed hybrid structure images using homomorphism. Muhiuddin et al. in [24] proposed and investigated a hybrid subsemimodule over semirings, investigated the representations of hybrid subsemimodules and hybrid ideals using hybrid products, and obtained some intriguing results on t-pure hybrid ideals in subsemimodules over semirings.

In a ternary semiring, Muhiuddin et al. [25] introduced hybrid ideals and k-hybrid ideals, proved various properties of k-hybrid ideals, and gave characterisations of hybrid intersections with respect to these k-hybrid ideals. There were also results based on the homomorphic hybrid preimage of a k-hybrid ideal. More concepts related to this study have been studied in [26–34].

The study of symmetry is one of the most important and beautiful themes uniting various areas of contemporary arithmetic. Rings are useful structures in pure mathematics for learning a geometrical object's symmetries. The most important functions in ring theory are those that preserve the ring operation, which are referred to as homomorphism. Showing a link from ring theory to real life is a simple way to see a possible connection between homomorphism and reality.

In this expository article, we discuss hybrid structures in ring theory, namely, hybrid ideals, hybrid semiprime ideals, hybrid nil radicals, hybrid products, and characteristic functions in rings, and present some results regarding these concepts. We also prove that the hybrid intersection of a hybrid radical will coincide with the hybrid radical of a hybrid intersection, the hybrid image of a hybrid radical will coincide with the hybrid radical of a hybrid image under certain conditions, and the hybrid preimage of a hybrid radical will coincide with the hybrid radical of a hybrid preimage. Further, we obtain an equivalent assertion for the hybrid semiprime ideal in the ring.

2. Definitions and Results

We will present the glossary of definitions that we will use in this sequel.

Definition 1. *A set $\mathscr{R}(\neq \phi)$ together with two binary operations '+' and '.' is said to be a ring if it fulfils the following assertions:*

(i) $\mathscr{R}$ *is an abelian group under '+',*
(ii) $\mathscr{R}$ *is associative under '.',*
(iii) $c.(u+k) = c.u + c.k$ *and* $(c+u).k = c.k + u.k$ *for all* $c, u, k \in \mathscr{R}$.

Throughout this paper, unless stated otherwise, $\mathscr{R}$ denotes a ring and $\mathscr{P}(\mathscr{X})$, the power set of a set $\mathscr{X}$.

Definition 2. *Let I be the unit interval and $\mathscr{U}$ be an initial universal set. Consider a mapping $\tilde{j}_\mu := (\tilde{j}, \mu) : \mathscr{R} \to \mathscr{P}(\mathscr{U}) \times I$, $x_1 \mapsto (\tilde{j}(x_1), \mu(x_1))$, where $\tilde{j} : \mathscr{R} \to \mathscr{P}(\mathscr{U})$ and $\mu : \mathscr{R} \to I$. Then, $\tilde{j}_\mu$ is described as a hybrid structure in $\mathscr{R}$ over $\mathscr{U}$.*

Let all the hybrid structures collected in $\mathscr{R}$ over $\mathscr{U}$ be described by $\mathscr{H}(\mathscr{R})$. An order $\ll$ in $\mathscr{H}(\mathscr{R})$ is outlined as follows: For every $\tilde{j}_\mu, \tilde{l}_\gamma \in \mathscr{H}(\mathscr{R})$, $\tilde{j}_\mu \ll \tilde{l}_\gamma$ if and only if $\tilde{j}(w) \subseteq \tilde{l}(w)$ and $\mu(w) \geq \gamma(w)$ for all $w \in \mathscr{R}$. For any $x_1, x_2 \in \mathscr{R}$, $\tilde{j}_\mu(x_1) = \tilde{l}_\gamma(x_2)$ if and only if $\tilde{j}_\mu(x_1) \ll \tilde{l}_\gamma(x_2)$ and $\tilde{l}_\gamma(x_2) \ll \tilde{j}_\mu(x_1)$. Additionally, $\tilde{j}_\mu \ll \tilde{l}_\gamma$ and $\tilde{l}_\gamma \ll \tilde{j}_\mu$ if and only if $\tilde{j}_\mu = \tilde{l}_\gamma$. It is noted that $(\mathscr{H}(\mathscr{R}), \ll)$ is a poset.

Definition 3. *Suppose $(\mathscr{G}, .)$ is a group. A hybrid structure $\tilde{i}_\Omega \in \mathscr{H}(\mathscr{R})$ is a hybrid subgroup in $\mathscr{G}$ if it fulfils the following assertions: $\forall\, a_1, a_2 \in \mathscr{G}$,*

(i) $\begin{pmatrix} \tilde{i}(a_1a_2) \supseteq \tilde{i}(a_1) \cap \tilde{i}(a_2) \\ \Omega(a_1a_2) \leq \Omega(a_1) \vee \Omega(a_2) \end{pmatrix}$,

(ii) $\begin{pmatrix} \tilde{i}(a_1^{-1}) = \tilde{i}(a_1) \\ \Omega(a_1^{-1}) = \Omega(a_1) \end{pmatrix}$.

Equivalently, $\tilde{i}(a_1a_2^{-1}) \supseteq \tilde{i}(a_1) \cap \tilde{i}(a_2)$ and $\Omega(a_1a_2^{-1}) \leq \Omega(a_1) \vee \Omega(a_2)$.

Definition 4. *Let $\tilde{i}_\Omega \in \mathscr{H}(\mathscr{R})$. Then, $\forall\, k_1, w_2 \in \mathscr{R}$,*

(i) $\begin{pmatrix} \tilde{i}(k_1 - w_2) \supseteq \tilde{i}(k_1) \cap \tilde{i}(w_2) \\ \Omega(k_1 - w_2) \leq \Omega(k_1) \vee \Omega(w_2) \end{pmatrix}$,

(ii) $\begin{pmatrix} \tilde{i}(k_1w_2) \supseteq \tilde{i}(w_2) \\ \Omega(k_1w_2) \leq \Omega(w_2) \end{pmatrix}$,

(iii) $\begin{pmatrix} \tilde{i}(k_1w_2) \supseteq \tilde{i}(k_1) \\ \Omega(k_1w_2) \leq \Omega(k_1) \end{pmatrix}$.

$\tilde{i}_\Omega$ is a hybrid left ideal in $\mathscr{R}$ if it fulfils assertions (i) and (ii), $\tilde{i}_\Omega$ is a hybrid right ideal in $\mathscr{R}$ if it fulfils assertions (i) and (iii), and $\tilde{i}_\Omega$ is a hybrid ideal if it fulfils assertions (i), (ii), and (iii).

Example 1. *Consider the set $\mathbb{Z}_9 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Then, $(\mathbb{Z}_9, +_9, ._9)$ is a ring.*

Let $\tilde{i}_\Omega$ in $\mathscr{H}(\mathscr{R})$ over $\mathscr{U} = [0, 1]$ be described as $\tilde{i}(k) = \begin{cases} [0, 0.9] & if\ \ k = 0 \\ [0, 0.6] & if\ \ k = 3 \text{ or } 6 \\ [0, 0.2] & otherwise \end{cases}$ and the mapping $\Omega : \mathscr{R} \to I$ is constant. Then, $\tilde{i}_\Omega$ is a hybrid ideal in $\mathbb{Z}_9$.

Definition 5. *Let $\tilde{j}_\nu, \tilde{i}_\Omega \in \mathscr{H}(\mathscr{R})$. Then, the hybrid structure $\tilde{j}_\nu \odot \tilde{i}_\Omega := (\tilde{j}\tilde{\circ}\tilde{i}, \nu\tilde{\circ}\Omega)$ is the hybrid product of $\tilde{j}_\nu$ and $\tilde{i}_\Omega$, where*

$$(\tilde{j}\tilde{\circ}\tilde{i})(t) = \begin{cases} \bigcup\limits_{t=wk} \{\tilde{j}(w) \cap \tilde{i}(k)\} & if\ \exists w, k \in \mathscr{R} : t = wk \\ \phi & otherwise \end{cases}$$

and

$$(\nu\tilde{\circ}\Omega)(t) = \begin{cases} \bigwedge\limits_{t=wk} \{\nu(w) \vee \Omega(k)\} & if\ \exists w, k \in \mathscr{R} : t = wk \\ 1 & otherwise \end{cases}$$

$\forall\, t \in \mathscr{R}$.

Definition 6. *Let $\tilde{i}_\Omega, \tilde{j}_\gamma \in \mathscr{H}(\mathscr{R})$. Then, the hybrid intersection of $\tilde{i}_\Omega$ and $\tilde{j}_\gamma$ is defined as $\tilde{i}_\Omega \Cap \tilde{j}_\gamma : \mathscr{R} \to \mathscr{P}(\mathscr{U}) \times I$, $w \mapsto ((\tilde{i}\tilde{\cap}\tilde{j})(w), (\Omega \vee \gamma)(w))$, where $\tilde{i}\tilde{\cap}\tilde{j} : \mathscr{R} \to \mathscr{P}(\mathscr{U})$, $w \mapsto \tilde{i}(w) \cap \tilde{j}(w)$ and $\Omega \vee \gamma : \mathscr{R} \to I$, $w \mapsto \Omega(w) \vee \gamma(w)$.*

Definition 7 ([11]). *Let R be a commutative ring. An ideal $K \subseteq \mathscr{R}$ is termed as semiprime if, whenever $d^n \in K$ for some positive integer n and $d \in \mathscr{R}$, then $d \in K$.*

Definition 8 ([11]). *A ring $\mathscr{R}$ is referred to as regular if for each $d \in \mathscr{R}$; $\exists\, s \in \mathscr{R} \ni\ d = dsd$.*

Theorem 1 ([11]). *Suppose $\mathscr{R}$ is commutative. Then, every ideal of $\mathscr{R}$ is semiprime if and only if $\mathscr{R}$ is regular.*

Definition 9. *Consider a map $\Psi : \mathscr{S} \to \mathscr{T}$, where $\mathscr{S}$ and $\mathscr{T}$ are sets. For any hybrid structure $\tilde{i}_\Omega \in \mathscr{H}(\mathscr{T})$ over $\mathscr{U}$, consider $\Psi^{-1}(\tilde{i}_\Omega) := (\Psi^{-1}(\tilde{i}), \Psi^{-1}(\Omega)) \in \mathscr{H}(\mathscr{S})$ where $\Psi^{-1}(\tilde{i}(w)) =$*

$\tilde{i}(\Psi(w))$ *and* $\Psi^{-1}(\Omega)(w) = \Omega(\Psi(w)), \forall w \in \mathscr{S}$. *The hybrid preimage of* $\tilde{i}_\Omega$ *under the mapping* Ψ *is represented by* $\Psi^{-1}(\tilde{i}_\Omega)$.

For any $\tilde{i}_\Omega \in \mathscr{H}(\mathscr{S})$ *over* $\mathscr{U}$, *under the mapping* Ψ, *the hybrid image of* $\tilde{i}_\Omega$ *is considered as a hybrid structure* $\Psi(\tilde{i}_\Omega) := (\Psi(\tilde{i}), \Psi(\Omega))$ *in* $\mathscr{T}$, *where*

$$\Psi(\tilde{i}(c)) = \begin{cases} \bigcup_{t \in \Psi^{-1}(c)} \tilde{i}(t) & if\ \Psi^{-1}(c) \neq \phi, \\ \phi & otherwise, \end{cases}$$

$$\Psi(\Omega)(c) = \begin{cases} \bigwedge_{t \in \Psi^{-1}(c)} \Omega(t) & if\ \Psi^{-1}(c) \neq \phi, \\ 1 & otherwise \end{cases}$$

$\forall\, c \in \mathscr{T}$.

Definition 10. *For* $A \in \mathscr{P}(\mathscr{R})$, *define* $\chi_A(\tilde{i}_\Omega) \in \mathscr{H}(\mathscr{R})$ *as follows:*

$\chi_A(\tilde{i}_\Omega) = (\chi_A(\tilde{i}), \chi_A(\Omega))$, *where* $\chi_A(\tilde{i}) : \mathscr{R} \to \mathscr{P}(\mathscr{U}), a \mapsto \begin{cases} \mathscr{U} & if\ \ a \in A \\ \phi & otherwise \end{cases}$

and $\chi_A(\Omega) : \mathscr{R} \to I, a \mapsto \begin{cases} 0 & if\ \ a \in A \\ 1 & otherwise \end{cases}$.

This is called characteristic hybrid structure of A.

3. Hybrid Structures in Rings

We define the hybrid radical notion in a ring and explore its properties. We also show that for any two hybrid ideals, the hybrid radical of the hybrid intersection is equivalent to the hybrid intersection of the hybrid radical, and the hybrid radical of the hybrid sum is equivalent to the hybrid sum of the hybrid radical.

Definition 11. *Suppose* $\tilde{i}_\Omega \in \mathscr{H}(\mathscr{R})$ *is a hybrid ideal in* $\mathscr{R}$. *Then, the hybrid nil radical of* $\tilde{i}_\Omega$ *is the hybrid structure in* $\mathscr{R}$ *over* $\mathscr{U}$, *represented by* $\sqrt{\tilde{i}_\Omega} := (\sqrt{\tilde{i}}, \sqrt{\Omega})$ *where* $\sqrt{\tilde{i}}(x) = \bigcup_{n \geq 1} \tilde{i}(x^n)$ *and* $\sqrt{\Omega}(x) = \bigwedge_{n \geq 1} \Omega(x^n)$, *for* $x \in \mathscr{R}$.

Lemma 1. *Let* $\tilde{i}_\Omega, \tilde{j}_\gamma \in \mathscr{H}(\mathscr{R})$ *be hybrid ideals in* $\mathscr{R}$. *Then, the following assertions hold:*

(i) $\tilde{i}_\Omega \ll \sqrt{\tilde{i}_\Omega}$,

(ii) $\tilde{i}_\Omega \ll \tilde{j}_\gamma \Rightarrow \sqrt{\tilde{i}_\Omega} \ll \sqrt{\tilde{j}_\gamma}$,

(iii) $\sqrt{\sqrt{\tilde{i}_\Omega}} = \sqrt{\tilde{i}_\Omega}$.

Proof.

(i) Let $s \in \mathscr{R}$. Then, $\sqrt{\tilde{i}}(s) = \bigcup_{k \geq 1} \tilde{i}(s^k) \supseteq \tilde{i}(s^k) \supseteq \tilde{i}(s)$, and $\sqrt{\Omega}(s) = \bigwedge_{k \geq 1} \Omega(s^k) \leq \Omega(s^k) \leq \Omega(s)$. So $\tilde{i}_\Omega \ll \sqrt{\tilde{i}_\Omega}$.

(ii) Let $s \in \mathscr{R}$. Then, $\sqrt{\tilde{i}}(s) = \bigcup_{k \geq 1} \tilde{i}(s^k) \subseteq \bigcup_{k \geq 1} \tilde{j}(s^k) = \sqrt{\tilde{j}}(s)$ and $\sqrt{\Omega}(s) = \bigwedge_{k \geq 1} \Omega(s^k) \geq \bigwedge_{k \geq 1} \gamma(s^k) = \sqrt{\gamma}(s)$. So $\sqrt{\tilde{i}_\Omega} \ll \sqrt{\tilde{j}_\gamma}$.

(iii) Let $s \in \mathscr{R}$. Then,

$$\sqrt{\sqrt{\tilde{i}}}(s) = \bigcup_{r \geq 1} \sqrt{\tilde{i}}(s^r) = \bigcup_{r \geq 1} \bigcup_{m \geq 1} \tilde{i}(s^r)^m = \bigcup_{t \geq 1} \tilde{i}(s^t) = \sqrt{\tilde{i}}(s) \text{ and}$$
$$\sqrt{\sqrt{\Omega}}(s) = \bigwedge_{r \geq 1} \sqrt{\Omega}(s^r) = \bigwedge_{r \geq 1} \bigwedge_{m \geq 1} \Omega(s^r)^m = \bigwedge_{t \geq 1} \Omega(s^t) = \sqrt{\Omega}(s).$$

So, $\sqrt{\sqrt{\tilde{i}_\Omega}} = \sqrt{\tilde{i}_\Omega}$. □

Theorem 2. *Let $\mathscr{R}$ be a commutative ring and $\tilde{i}_\Omega \in \mathscr{H}(\mathscr{R})$ a hybrid ideal in $\mathscr{R}$. Then, $\sqrt{\tilde{i}_\Omega}$ is a hybrid ideal in $\mathscr{R}$.*

Proof. Let $v, u \in \mathscr{R}$. Then, for positive integers r, s, we have
$\bigcap\{\sqrt{\tilde{i}}(v), \sqrt{\tilde{i}}(u)\} = \bigcap\{\bigcup_{r\geq 1} \tilde{i}(v^r), \bigcup_{s\geq 1} \tilde{i}(u^s)\} = \bigcup_{r\geq 1}\{\bigcup_{s\geq 1}\{\tilde{i}(v^r) \cap \tilde{i}(u^s)\}\}$, and
$\bigvee\{\sqrt{\Omega}(v), \sqrt{\Omega}(u)\} = \bigvee\{\bigwedge_{r\geq 1} \Omega(v^r), \bigwedge_{s\geq 1} \Omega(u^s)\} = \bigwedge_{r\geq 1}\{\bigwedge_{s\geq 1}\{\Omega(v^r) \vee \Omega(u^s)\}\}$.

Since $\mathscr{R}$ is commutative, all the terms in $(v+u)^{r+s}$ contain either v^r or u^s as a factor. Therefore, $(v+u)^{r+s} = xv^r + yu^s$ for some $x, y \in \mathscr{R}$. Now,

$$\begin{aligned}
\bigcap\{\tilde{i}(v^r), \tilde{i}(u^s)\} &\subseteq \bigcap\{\{\tilde{i}(v^r) \cup \tilde{i}(x)\}, \{\tilde{i}(u^s) \cup \tilde{i}(y)\}\} \\
&\subseteq \bigcap\{\tilde{i}(xv^r), \tilde{i}(yu^s)\} \\
&\subseteq \tilde{i}(xv^r + yu^s) \\
&= \tilde{i}(v+u)^{r+s} \\
&\subseteq \bigcup_{k\geq 1} \tilde{i}(v+u)^k \\
&= \sqrt{\tilde{i}}(v+u),
\end{aligned}$$

$$\begin{aligned}
\bigvee\{\Omega(v^r), \Omega(u^s)\} &\geq \bigvee\{\{\Omega(v^r) \wedge \Omega(x)\}, \{\Omega(u^s) \wedge \Omega(y)\}\} \\
&\geq \bigvee\{\Omega(xv^r), \Omega(yu^s)\} \\
&\geq \Omega(xv^r + yu^s) \\
&= \Omega(v+u)^{r+s} \\
&\geq \bigwedge_{k\geq 1} \Omega(v+u)^k \\
&= \sqrt{\Omega}(v+u).
\end{aligned}$$

Therefore, we have $\bigcap\{\sqrt{\tilde{i}}(v), \sqrt{\tilde{i}}(u)\} \subseteq \sqrt{\tilde{i}}(v+u)$ and $\bigvee\{\sqrt{\Omega}(v), \sqrt{\Omega}(u)\} \geq \sqrt{\Omega}(v+u)$. Now, for a positive integer r,

$$\bigcup\{\sqrt{\tilde{i}}(v), \sqrt{\tilde{i}}(u)\} = \bigcup\{\bigcup_{r\geq 1} \tilde{i}(v^r), \bigcup_{r\geq 1} \tilde{i}(u^r)\} = \bigcup_{r\geq 1}\{\tilde{i}(v^r) \cup \tilde{i}(u^r)\},$$
$$\bigwedge\{\sqrt{\Omega}(v), \sqrt{\Omega}(u)\} = \bigwedge\{\bigwedge_{r\geq 1} \Omega(v^r), \bigwedge_{r\geq 1} \Omega(u^r)\} = \bigwedge_{r\geq 1}\{\Omega(v^r) \wedge \Omega(u^r)\}.$$

Then,

$$\begin{aligned}
\bigcup\{\tilde{i}(v^r), \tilde{i}(u^r)\} &\subseteq \tilde{i}(v^r u^r) \\
&= \tilde{i}((vu)^r) \\
&\subseteq \bigcup_{k\geq 1} \tilde{i}((vu)^k) \\
&= \sqrt{\tilde{i}}(vu),
\end{aligned}$$

$$\begin{aligned}
\bigwedge\{\Omega(v^r), \Omega(u^r)\} &\geq \Omega(v^r u^r) \\
&= \Omega((vu)^r) \\
&\geq \bigwedge_{k\geq 1} \Omega((vu)^k) \\
&= \sqrt{\Omega}(vu).
\end{aligned}$$

Thus, $\bigcup\{\sqrt{\tilde{i}}(v), \sqrt{\tilde{i}}(u)\} \subseteq \sqrt{\tilde{i}}(vu)$ and $\bigwedge\{\sqrt{\Omega}(v), \sqrt{\Omega}(u)\} \geq \sqrt{\Omega}(vu)$ and hence $\sqrt{\tilde{i}_\Omega}$ is a hybrid ideal in $\mathscr{R}$. $\square$

Definition 12. *Let $\tilde{i}_\Omega, \tilde{j}_\gamma \in \mathscr{H}(\mathscr{R})$ be hybrid ideals in $\mathscr{R}$. Then, the hybrid intrinsic product $\tilde{i}_\Omega \tilde{*} \tilde{j}_\gamma$ is the hybrid structure in $\mathscr{R}$ stated as below: For $x \in \mathscr{R}$, define*

$$(\tilde{i}\tilde{*}\tilde{j})(x) = \begin{cases} \bigcup\limits_{x=\sum_{finite} a_i w_i} \cap\{\tilde{i}(a_1), \tilde{i}(a_2), ..., \tilde{i}(a_m), \tilde{j}(w_1), \tilde{j}(w_2), ..., \tilde{j}(w_m)\} \\ \quad if\ x = a_1w_1 + a_2w_2 + ... + a_mw_m\ for\ some\ a_i, w_i \in \mathscr{R}, \\ \quad where\ each\ a_iw_i \neq 0,\ m \in N \\ \phi, \qquad otherwise \end{cases}$$

$$(\Omega\tilde{*}\gamma)(x) = \begin{cases} \bigwedge\limits_{x=\sum_{finite} a_i w_i} \vee\{\Omega(a_1), \Omega(a_2), ..., \Omega(a_m), \gamma(w_1), \gamma(w_2), ..., \gamma(w_m)\} \\ \quad if\ x = a_1w_1 + a_2w_2 + ... + a_mw_m\ for\ some\ a_i, w_i \in \mathscr{R}, \\ \quad where\ each\ a_iw_i \neq 0,\ m \in N \\ 1, \qquad otherwise \end{cases}$$

It is evident that if $\mathscr{R}$ is commutative, then $\tilde{i}_\Omega \tilde{*} \tilde{j}_\gamma$ is commutative.

Let $\tilde{i}_\Omega, \tilde{j}_\gamma \in \mathscr{H}(\mathscr{R})$ be hybrid ideals in $\mathscr{R}$. Then, the hybrid intrinsic product $\tilde{i}_\Omega \tilde{*} \tilde{j}_\gamma$ is also a hybrid ideal, and whenever $\tilde{i}_\Omega$ is a hybrid ideal, $\forall\ n \geq 1$, $(\tilde{i}_\Omega)^n = \tilde{i}_\Omega \tilde{*} \tilde{i}_\Omega \tilde{*} .... \tilde{*} \tilde{i}_\Omega$(n times) is also a hybrid ideal.

Theorem 3. *Let $\tilde{i}_\Omega, \tilde{j}_\gamma \in \mathscr{H}(\mathscr{R})$ be hybrid ideals in $\mathscr{R}$. Then, $\sqrt{\tilde{i}_\Omega \tilde{*} \tilde{j}_\gamma} = \sqrt{\tilde{i}_\Omega \Cap \tilde{j}_\gamma} = \sqrt{\tilde{i}_\Omega} \Cap \sqrt{\tilde{j}_\gamma}$.*

Proof. Let $s \in \mathscr{R}$ with $s = a_1w_1 + a_2w_2 + ... + a_mw_m$, and $a_iw_i \neq 0$ for each $i \in \{1, 2, ..., m\}$.

Then, $\tilde{i}(a_i) \cap \tilde{j}(w_i) \subseteq \tilde{i}(a_i) \subseteq \tilde{i}(a_iw_i)$ and $\Omega(a_i) \vee \gamma(w_i) \geq \Omega(a_i) \geq \Omega(a_iw_i)$. Now,

$$\begin{aligned} \bigcap\{\tilde{i}(a_1), \tilde{i}(a_2), ..., \tilde{i}(a_m), \tilde{j}(w_1), \tilde{j}(w_2), ..., \tilde{j}(w_m)\} &\subseteq \bigcap\{\tilde{i}(a_1w_1), \tilde{i}(a_2w_2), ..., \tilde{i}(a_mw_m)\} \\ &\subseteq \tilde{i}(a_1w_1 + a_2w_2 + ... + a_mw_m) \\ &= \tilde{i}(s), \end{aligned}$$

$$\begin{aligned} \bigvee\{\Omega(a_1), \Omega(a_2), ..., \Omega(a_m), \gamma(w_1), \gamma(w_2), ..., \gamma(w_m)\} &\geq \bigvee\{\Omega(a_1w_1), \Omega(a_2w_2), ..., \Omega(a_mw_m)\} \\ &\geq \Omega(a_1w_1 + a_2w_2 + ... + a_mw_m) \\ &= \Omega(s). \end{aligned}$$

So, $(\tilde{i}\tilde{*}\tilde{j})(s) \subseteq \tilde{i}(s)$ and $(\Omega\tilde{*}\gamma)(s) \geq \Omega(s)$. Hence $\tilde{i}_\Omega \tilde{*} \tilde{j}_\gamma \ll \tilde{i}_\Omega$. Similarly, we can show that $\tilde{i}_\Omega \tilde{*} \tilde{j}_\gamma \ll \tilde{j}_\gamma$. So, $(\tilde{i}\tilde{*}\tilde{j})(s) \subseteq \tilde{i}(s) \cap \tilde{j}(s) = (\tilde{i}\tilde{\cap}\tilde{j})(s)$ and $(\Omega\tilde{*}\gamma)(s) \geq \Omega(s) \vee \gamma(s) = (\Omega \vee \gamma)(s)$ and hence $\tilde{i}_\Omega \tilde{*} \tilde{j}_\gamma \ll \tilde{i}_\Omega \Cap \tilde{j}_\gamma$.

Therefore, by Lemma 1(ii), $\sqrt{\tilde{i}_\Omega \tilde{*} \tilde{j}_\gamma} \ll \sqrt{\tilde{i}_\Omega \Cap \tilde{j}_\gamma}$.

Let $s \in \mathscr{R}$. Then,

$$\begin{aligned} \sqrt{\tilde{i}\tilde{*}\tilde{j}}(s) &= \bigcup_{k\geq 1} (\tilde{i}\tilde{*}\tilde{j})(s^k) \\ &\supseteq (\tilde{i}\tilde{*}\tilde{j})(s^{2n}) \\ &\supseteq \tilde{i}(s^n) \cap \tilde{j}(s^n) \\ &= (\tilde{i}\tilde{\cap}\tilde{j})(s^n)\ \ \forall\ n \geq 1, \end{aligned}$$

$$\begin{aligned}\sqrt{\Omega\tilde{*}\gamma}(s) &= \bigwedge_{k\geq 1}(\Omega\tilde{*}\gamma)(s^k)\\ &\leq (\Omega\tilde{*}\gamma)(s^{2n})\\ &\leq \Omega(s^n)\vee\gamma(s^n)\\ &= (\Omega\vee\gamma)(s^n)\ \ \forall\, n\geq 1.\end{aligned}$$

Therefore, $\tilde{i}_\Omega \Cap \tilde{j}_\gamma \ll \tilde{i}_\Omega\tilde{*}\tilde{j}_\gamma$ which implies $\sqrt{\tilde{i}_\Omega \Cap \tilde{j}_\gamma} \ll \sqrt{\tilde{i}_\Omega\tilde{*}\tilde{j}_\gamma}$, by Lemma 1(ii). Hence, $\sqrt{\tilde{i}_\Omega \Cap \tilde{j}_\gamma} = \sqrt{\tilde{i}_\Omega\tilde{*}\tilde{j}_\gamma}$.

Now, by Lemma 1(ii), we have
$\sqrt{\tilde{i}_\Omega \Cap \tilde{j}_\gamma} \ll \sqrt{\tilde{i}_\Omega}$ and $\sqrt{\tilde{i}_\Omega \Cap \tilde{j}_\gamma} \ll \sqrt{\tilde{j}_\gamma}$ which imply $\sqrt{\tilde{i}_\Omega \Cap \tilde{j}_\gamma} \ll \sqrt{\tilde{i}_\Omega} \Cap \sqrt{\tilde{j}_\gamma}$.
Conversely, let $v \in \mathscr{R}$. Then, for any two positive integers t, r, we have

$$\begin{aligned}(\sqrt{\tilde{i}}\tilde{\bigcap}\sqrt{\tilde{j}})(v) &= \bigcap\{\bigcup_{t\geq 1}\tilde{i}(v^t), \bigcup_{r\geq 1}\tilde{j}(v^r)\}\\ &= \bigcup_{t\geq 1}(\bigcup_{r\geq 1}\tilde{i}(v^t)\cap\tilde{j}(v^r)),\end{aligned}$$

$$\begin{aligned}(\sqrt{\Omega}\bigvee\sqrt{\gamma})(v) &= \bigvee\{\bigwedge_{t\geq 1}\Omega(v^t), \bigwedge_{r\geq 1}\gamma(v^r)\}\\ &= \bigwedge_{t\geq 1}(\bigwedge_{r\geq 1}\Omega(v^t)\vee\gamma(v^r)).\end{aligned}$$

Now,

$$\begin{aligned}\tilde{i}(v^t)\cap\tilde{j}(v^r) &\subseteq \tilde{i}(v^{tr})\cap\tilde{j}(v^{tr})\\ &= (\tilde{i}\tilde{\cap}\tilde{j})(v^{tr})\\ &\subseteq \bigcup_{k\geq 1}(\tilde{i}\tilde{\cap}\tilde{j})(v^k)\\ &= \sqrt{(\tilde{i}\cap\tilde{j})}(v)\end{aligned}$$

which implies $(\sqrt{\tilde{i}}\cap\sqrt{\tilde{j}})(v) \subseteq \sqrt{(\tilde{i}\cap\tilde{j})}(v)$.

Additionally,

$$\begin{aligned}\Omega(v^t)\vee\gamma(v^r) \geq \Omega(v^{tr})\vee\gamma(v^{tr}) &= (\Omega\vee\gamma)(v^{tr})\\ &\geq \bigwedge_{k\geq 1}(\Omega\vee\gamma)(v^k)\\ &= \sqrt{(\Omega\vee\gamma)}(v)\end{aligned}$$

which implies $(\sqrt{\Omega}\vee\sqrt{\gamma})(v) \geq \sqrt{(\Omega\vee\gamma)}(v)$.

So, $\sqrt{\tilde{i}_\Omega} \Cap \sqrt{\tilde{j}_\gamma} \ll \sqrt{\tilde{i}_\Omega \Cap \tilde{j}_\gamma}$. Hence, $\sqrt{\tilde{i}_\Omega} \Cap \sqrt{\tilde{j}_\gamma} = \sqrt{\tilde{i}_\Omega \Cap \tilde{j}_\gamma}$. □

Corollary 1. *Let $\tilde{i}_\Omega \in \mathscr{H}(\mathscr{R})$ be a hybrid ideal. Then, $\sqrt{\tilde{i}^n_\Omega} = \sqrt{\tilde{i}_\Omega}$ for all $n \geq 1$, where $\tilde{i}^n_\Omega = \tilde{i}_\Omega\tilde{*}\tilde{i}_\Omega\tilde{*}...\tilde{*}\tilde{i}_\Omega$ (n - times).*

Proof. Taking $\tilde{j}_\gamma = \tilde{i}_\Omega$ in Theorem 3, we obtain $\sqrt{\tilde{i}_\Omega\tilde{*}\tilde{i}_\Omega} = \sqrt{\tilde{i}_\Omega}$. So, by induction principle, $\sqrt{\tilde{i}^n_\Omega} = \sqrt{\tilde{i}_\Omega}$ for all $n \geq 1$. □

Corollary 2. *Let $\tilde{i}_\Omega, \tilde{j}_\gamma \in \mathscr{H}(\mathscr{R})$ be hybrid ideals in $\mathscr{R}$. If $\tilde{i}^k_\Omega \ll \tilde{j}_\gamma$ for some $k \geq 1$, then $\sqrt{\tilde{i}_\Omega} \ll \sqrt{\tilde{j}_\gamma}$.*

Proof. By applying Corollary 1, $\sqrt{\tilde{i}^k_\Omega} = \sqrt{\tilde{i}_\Omega}$ for all $k \geq 1$. Then, by Lemma 1(ii), $\sqrt{\tilde{j}_\gamma} \gg \sqrt{\tilde{i}^k_\Omega} = \sqrt{\tilde{i}_\Omega}$. □

Definition 13. *For any $\tilde{i}_\Omega, \tilde{j}_\gamma \in \mathscr{H}(\mathscr{R})$, the hybrid sum of $\tilde{i}_\Omega$ and $\tilde{j}_\gamma$, represented by $\tilde{i}_\Omega \oplus \tilde{j}_\gamma$, is defined as $\tilde{i}_\Omega \oplus \tilde{j}_\gamma = (\tilde{i} + \tilde{j}, \Omega + \gamma) \in \mathscr{H}(\mathscr{R})$, where*

$$(\tilde{i} + \tilde{j})(a') = \begin{cases} \bigcup\limits_{a'=s'+s''} \{\tilde{i}(s') \cap \tilde{j}(s'')\} & if\ \exists\ s', s'' \in \mathscr{R} \ni a'=s'+s'' \\ \phi & otherwise \end{cases}$$

and

$$(\Omega + \gamma)(a') = \begin{cases} \bigwedge\limits_{a'=s'+s''} \{\Omega(s') \vee \gamma(s'')\} & if\ \exists\, s', s'' \in \mathscr{R} \ni a' = s' + s'' \\ 1 & otherwise \end{cases}$$

for all $a' \in \mathscr{R}$.

Lemma 2. *Let $\tilde{i}_\Omega, \tilde{j}_\gamma \in \mathscr{H}(\mathscr{R})$ be hybrid ideals in $\mathscr{R}$. Then, $\tilde{i}_\Omega \oplus \tilde{j}_\gamma$ is a hybrid ideal in $\mathscr{R}$.*

Proof. As $\tilde{i}_\Omega, \tilde{j}_\gamma \in \mathscr{H}(\mathscr{R})$ are hybrid subgroups of commutative group $(\mathscr{R}, +)$, $\tilde{i}_\Omega \oplus \tilde{j}_\gamma$ is also a hybrid subgroup of commutative group $(\mathscr{R}, +)$. Let $v, u \in \mathscr{R}$. If $v = a + b$ for $a, b \in \mathscr{R}$, then

$$\begin{aligned}
(\tilde{i} + \tilde{j})(vu) &= \bigcup_{vu=(a+b)u} \{\tilde{i}(au) \cap \tilde{j}(bu)\} \\
&\supseteq \bigcup_{vu=(a+b)u} \{\tilde{i}(a) \cap \tilde{j}(b)\} \\
&= \bigcup_{v=a+b} \{\tilde{i}(a) \cap \tilde{j}(b)\} \\
&= (\tilde{i} + \tilde{j})(v), \\
(\Omega + \gamma)(vu) &= \bigwedge_{vu=(a+b)u} \{\Omega(au) \vee \gamma(bu)\} \\
&\leq \bigwedge_{vu=(a+b)u} \{\Omega(a) \vee \gamma(b)\} \\
&= \bigwedge_{v=a+b} \{\Omega(a) \vee \gamma(b)\} \\
&= (\Omega + \gamma)(v).
\end{aligned}$$

In addition,

$$\begin{aligned}
(\tilde{i} + \tilde{j})(vu) &= \bigcup_{vu=v(r+s)} \{\tilde{i}(vr) \cap \tilde{j}(vs)\} \\
&\supseteq \bigcup_{vu=v(r+s)} \{\tilde{i}(r) \cap \tilde{j}(s)\} \\
&= \bigcup_{u=r+s} \{\tilde{i}(r) \cap \tilde{j}(s)\} \\
&= (\tilde{i} + \tilde{j})(u), \\
(\Omega + \gamma)(vu) &= \bigwedge_{vu=v(r+s)} \{\Omega(vr) \vee \gamma(vs)\} \\
&\leq \bigwedge_{vu=v(r+s)} \{\Omega(r) \vee \gamma(s)\} \\
&= \bigwedge_{u=r+s} \{\Omega(r) \vee \gamma(s)\} \\
&= (\Omega + \gamma)(u).
\end{aligned}$$

Thus, $\tilde{i}_\Omega \oplus \tilde{j}_\gamma$ is a hybrid ideal in $\mathscr{R}$. $\square$

Theorem 4. *Suppose $\tilde{j}_\Omega, \tilde{i}_\gamma, \tilde{f}_\nu, \tilde{p}_\mu \in \mathscr{H}(\mathscr{R})$. If $\tilde{j}_\Omega \ll \tilde{f}_\nu$ and $\tilde{i}_\gamma \ll \tilde{p}_\mu$, then $\tilde{j}_\Omega \oplus \tilde{i}_\gamma \ll \tilde{f}_\nu \oplus \tilde{p}_\mu$.*

Proof. Suppose $w \in \mathscr{R}$. If $w \neq s + r$ for $r, s \in \mathscr{R}$, then clearly $\tilde{j}_\Omega \oplus \tilde{i}_\gamma \ll \tilde{f}_\nu \oplus \tilde{p}_\mu$.

Assume $w = s + r$ for some $s, r \in \mathscr{R}$. Then,

$$\begin{aligned}(\tilde{j} + \tilde{i})(w) &= \bigcup_{w=s+r} \{\tilde{j}(s) \cap \tilde{i}(r)\} \\ &\subseteq \bigcup_{w=s+r} \{\tilde{f}(s) \cap \tilde{p}(r)\} \\ &= (\tilde{f} + \tilde{p})(w).\end{aligned}$$

Additionally,

$$\begin{aligned}(\Omega + \gamma)(w) &= \bigwedge_{w=s+r} \Omega(s) \vee \gamma(r) \\ &\geq \bigwedge_{w=s+r} \nu(s) \vee \mu(r) \\ &= (\nu + \mu)(w).\end{aligned}$$

Therefore, $\tilde{j}_\Omega \oplus \tilde{i}_\gamma \ll \tilde{f}_\nu \oplus \tilde{p}_\mu$. $\square$

Theorem 5. *Let $\mathscr{R}$ be a commutative ring and $\tilde{i}_\Omega, \tilde{j}_\gamma \in \mathscr{H}(\mathscr{R})$ be hybrid ideals in $\mathscr{R}$. Then, $\sqrt{\tilde{i}_\Omega \oplus \tilde{j}_\gamma} = \sqrt{\sqrt{\tilde{i}_\Omega} \oplus \sqrt{\tilde{j}_\gamma}} \gg \sqrt{\tilde{i}_\Omega} \oplus \sqrt{\tilde{j}_\gamma}$.*

Proof. By Lemma 1(i), we have $\tilde{i}_\Omega \ll \sqrt{\tilde{i}_\Omega}$ and $\tilde{j}_\gamma \ll \sqrt{\tilde{j}_\gamma}$. Again, by Lemma 1(i) and Theorem 4, we have $\sqrt{\tilde{i}_\Omega \oplus \tilde{j}_\gamma} \ll \sqrt{\sqrt{\tilde{i}_\Omega} \oplus \sqrt{\tilde{j}_\gamma}}$.

On the other hand, let $x \in \mathscr{R}$. Then,

$$\begin{aligned}(\sqrt{\tilde{i}} + \sqrt{\tilde{j}})(x) &= \bigcup_{x=v+u} \{\sqrt{\tilde{i}}(v) \cap \sqrt{\tilde{j}}(u)\} \\ &= \bigcup_{x=v+u} \{\bigcup_{m\geq 1} \tilde{i}(v^m) \cap \bigcup_{n\geq 1} \tilde{j}(u^n)\} \\ &= \bigcup_{x=v+u} (\bigcup_{m\geq 1} (\bigcup_{n\geq 1} \{\tilde{i}(v^m) \cap \tilde{j}(u^n)\})),\end{aligned}$$

and

$$\begin{aligned}(\sqrt{\Omega} + \sqrt{\gamma})(x) &= \bigwedge_{x=v+u} \{\sqrt{\Omega}(v) \vee \sqrt{\gamma}(u)\} \\ &= \bigwedge_{x=v+u} \{\bigwedge_{m\geq 1} \Omega(v^m) \vee \bigwedge_{n\geq 1} \gamma(u^n)\} \\ &= \bigwedge_{x=v+u} (\bigwedge_{m\geq 1} (\bigwedge_{n\geq 1} \{\Omega(v^m) \vee \gamma(u^n)\})).\end{aligned}$$

Now, let $k = u + v$; $u, v \in \mathscr{R}$; n, m any positive integers. As $\mathscr{R}$ is commutative, for some $t, r \in \mathscr{R}$, we have $k^{m+n} = tu^m + rv^n$. Now,

$$\begin{aligned}\tilde{i}(u^m) \cap \tilde{j}(v^n) &\subseteq \tilde{i}(tu^m) \cap \tilde{j}(rv^n) \\ &\subseteq (\tilde{i} + \tilde{j})(tu^m + rv^n) \\ &= (\tilde{i} + \tilde{j})(k^{m+n}) \\ &\subseteq \bigcup_{a\geq 1} (\tilde{i} + \tilde{j})(k^a) \\ &= \sqrt{\tilde{i} + \tilde{j}}(k),\end{aligned}$$

$$\begin{aligned}\Omega(u^m) \vee \gamma(v^n) &\geq \Omega(tu^m) \vee \gamma(rv^n) \\ &\geq (\Omega+\gamma)(tu^m+rv^n) \\ &= (\Omega+\gamma)(k^{m+n}) \\ &\geq \bigwedge_{a\geq 1}(\Omega+\gamma)(k^a) \\ &= \sqrt{\Omega+\gamma}(k).\end{aligned}$$

Thus, $(\sqrt{\tilde{i}}+\sqrt{\tilde{j}})(k) \subseteq \sqrt{\tilde{i}+\tilde{j}}(k)$ and $(\sqrt{\Omega}+\sqrt{\gamma})(k) \geq \sqrt{\Omega+\gamma}(k)$.

Hence, $\sqrt{\tilde{i}_\Omega} \oplus \sqrt{\tilde{j}_\gamma} \ll \sqrt{\tilde{i}_\Omega \oplus \tilde{j}_\gamma}$. By Lemma 1, $\sqrt{\sqrt{\tilde{i}_\Omega} \oplus \sqrt{\tilde{j}_\gamma}} \ll \sqrt{\tilde{i}_\Omega \oplus \tilde{j}_\gamma}$. Therefore, $\sqrt{\sqrt{\tilde{i}_\Omega} \oplus \sqrt{\tilde{j}_\gamma}} = \sqrt{\tilde{i}_\Omega \oplus \tilde{j}_\gamma}$. □

Definition 14. *Consider the function $\Psi : \mathscr{S} \to \mathscr{T}$. A hybrid structure $\tilde{i}_\Omega$ in $\mathscr{S}$ is stated as Ψ-invariant whenever $\Psi(u) = \Psi(v) \Rightarrow \tilde{i}(u) = \tilde{i}(v)$ and $\Omega(u) = \Omega(v)$ for any $u, v \in \mathscr{S}$.*

Theorem 6. *Consider the ring homomorphism $\Psi : \mathscr{R} \to \mathscr{S}$. If $\tilde{i}_\Omega$ and $\tilde{j}_\gamma$ are hybrid ideals in $\mathscr{R}$ and $\mathscr{S}$, respectively, then the following assertions hold:*

(i) $\sqrt{\Psi^{-1}(\tilde{j}_\gamma)} = \Psi^{-1}(\sqrt{\tilde{j}_\gamma})$.

(ii) $\sqrt{\Psi(\tilde{i}_\Omega)} = \Psi(\sqrt{\tilde{i}_\Omega})$, *provided Ψ is onto and $\tilde{i}_\Omega$ is Ψ-invariant.*

Proof.

(i) Let $v \in \mathscr{R}$. Then,

$$\begin{aligned}\sqrt{\Psi^{-1}(\tilde{j})}(v) &= \bigcup_{r\geq 1}(\Psi^{-1}(\tilde{j}))(v^r) \\ &= \bigcup_{r\geq 1}(\tilde{j}(\Psi(v^r))) \\ &= \bigcup_{r\geq 1}(\tilde{j}(\Psi(v))^r) \\ &= \sqrt{\tilde{j}}(\Psi(v)) \\ &= (\Psi^{-1}(\sqrt{\tilde{j}}))(v), \\ \sqrt{\Psi^{-1}(\gamma)}(v) &= \bigwedge_{r\geq 1}(\Psi^{-1}(\gamma))(v^r) \\ &= \bigwedge_{r\geq 1}(\gamma(\Psi(v^r))) \\ &= \bigwedge_{r\geq 1}(\gamma(\Psi(v))^r) \\ &= \sqrt{\gamma}(\Psi(v)) \\ &= (\Psi^{-1}(\sqrt{\gamma}))(v).\end{aligned}$$

So, $\sqrt{\Psi^{-1}(\tilde{j}_\gamma)} = \Psi^{-1}(\sqrt{\tilde{j}_\gamma})$.

(ii) Since Ψ is onto, for $v \in \mathscr{S}$, $\Psi(u) = v$ for some $u \in \mathscr{R}$. In addition, $\Psi^{-1}(\Psi(\tilde{i}_\Omega)) = \tilde{i}_\Omega$, as $\tilde{i}_\Omega$ is Ψ-invariant. Now, for some integer m, we have

$$\begin{aligned}
(\sqrt{\Psi(\tilde{i})})(v) &\subseteq (\Psi(\tilde{i}))(v^m) \\
&= (\Psi(\tilde{i}))(\Psi(u^m)) \\
&= \Psi^{-1}(\Psi(\tilde{i}))(u^m) \\
&= \tilde{i}(u^m) \\
&\subseteq \sqrt{\tilde{i}}(u) \\
&\subseteq \bigcup_{t\in\Psi^{-1}(v)} \sqrt{\tilde{i}}(t) \\
&= (\Psi(\sqrt{\tilde{i}}))(v),
\end{aligned}$$

$$\begin{aligned}
(\sqrt{\Psi(\Omega)})(v) &> (\Psi(\Omega))(v^m) \\
&= (\Psi(\Omega))(\Psi(u^m)) \\
&= \Psi^{-1}(\Psi(\Omega))(u^m) \\
&= \Omega(u^m) \\
&\geq \sqrt{\Omega}(u) \\
&\geq \bigwedge_{t\in\Psi^{-1}(v)} \sqrt{\Omega}(t) \\
&= (\Psi(\sqrt{\Omega}))(v).
\end{aligned}$$

So, $\sqrt{\Psi(\tilde{i}_\Omega)} \ll \Psi(\sqrt{\tilde{i}_\Omega})$.

On the other hand, $\exists\, u_0 \in \mathscr{R} \ni \Psi(u_0) = v$, and $(\Psi(\sqrt{\tilde{i}}))(v) \subset \sqrt{\tilde{i}}(u_0) = \bigcup_{n\geq 1} \tilde{i}(u_0^n)$ and $(\Psi(\sqrt{\Omega}))(v) > \sqrt{\Omega}(u_0) = \bigwedge_{n\geq 1} \Omega(u_0^n)$. Then, $\exists$ a positive integer $k \ni$

$$\begin{aligned}
(\Psi(\sqrt{\tilde{i}}))(v) &\subseteq \tilde{i}(u_0^k) \\
&\subseteq \bigcup_{w\in\Psi^{-1}(v^k)} \tilde{i}(w) \\
&= (\Psi(\tilde{i}))(v^k) \\
&\subseteq \bigcup_{n\geq 1} (\Psi(\tilde{i}))(v^n) \\
&= \sqrt{\Psi(\tilde{i})}(v)
\end{aligned}$$

and

$$\begin{aligned}
(\Psi(\sqrt{\Omega}))(v) &> \Omega(u_0^k) \\
&\geq \bigwedge_{w\in\Psi^{-1}(v^k)} \Omega(w) \\
&= (\Psi(\Omega))(v^k) \\
&\geq \bigwedge_{n\geq 1} (\Psi(\Omega))(v^n) \\
&= \sqrt{\Psi(\Omega)}(v).
\end{aligned}$$

Thus, $\Psi(\sqrt{\tilde{i}_\Omega}) \ll \sqrt{\Psi(\tilde{i}_\Omega)}$ and hence $\sqrt{\Psi(\tilde{i}_\Omega)} = \Psi(\sqrt{\tilde{i}_\Omega})$. □

Theorem 7. *Consider $\Psi : \mathscr{R} \to \mathscr{S}$ as the ring homomorphism. If $\tilde{i}_\Omega$ is a hybrid left (resp, right, ideal) ideal in $\mathscr{S}$, then $\Psi^{-1}(\tilde{i}_\Omega)$ is a hybrid left (resp, right, ideal) in $\mathscr{R}$.*

Proof. Let $c, v \in \mathscr{R}$ and $\tilde{i}_\Omega$ be a hybrid left ideal in $\mathscr{S}$. Then,

$$\begin{aligned}\Psi^{-1}(\tilde{i}(c-v)) &= \tilde{i}(\Psi(c-v)) \\ &= \tilde{i}(\Psi(c)-\Psi(v)) \\ &\supseteq \tilde{i}(\Psi(c)) \cap \tilde{i}(\Psi(v)) \\ &= \Psi^{-1}(\tilde{i}(c)) \cap \Psi^{-1}(\tilde{i}(v)),\end{aligned}$$

$$\begin{aligned}\Psi^{-1}(\Omega(c-v)) &= \Omega(\Psi(c-v)) \\ &= \Omega(\Psi(c)-\Psi(v)) \\ &\leq \Omega(\Psi(c)) \vee \Omega(\Psi(v)) \\ &= \Psi^{-1}(\Omega(c)) \vee \Psi^{-1}(\Omega(v)).\end{aligned}$$

Additionally,

$$\begin{aligned}\Psi^{-1}(\tilde{i}(cv)) &= \tilde{i}(\Psi(cv)) \\ &= \tilde{i}(\Psi(c)\Psi(v)) \\ &\supseteq \tilde{i}(\Psi(v)) \\ &= \Psi^{-1}(\tilde{i}(v)),\end{aligned}$$

$$\begin{aligned}\Psi^{-1}(\Omega(cv)) &= \Omega(\Psi(cv)) \\ &= \Omega(\Psi(c)\Psi(v)) \\ &\leq \Omega(\Psi(v)) \\ &= \Psi^{-1}(\Omega(v)).\end{aligned}$$

So, $\Psi^{-1}(\tilde{i}_\Omega) \in \mathscr{H}(\mathscr{R})$ is hybrid left ideal. □

Theorem 8. *Consider $\Psi : \mathscr{R} \to \mathscr{S}$ as onto ring homomorphism. If $\tilde{j}_\gamma \in \mathscr{H}(\mathscr{R})$ is a Ψ-invariant hybrid left (resp, right) ideal in $\mathscr{R}$, then $\Psi(\tilde{j}_\gamma)$ is a hybrid left (resp, right) ideal in $\mathscr{S}$.*

Proof. Let $w_1, w_2 \in \mathscr{S}$. Then, $\{k : k \in \Psi^{-1}(w_1 - w_2)\} \supseteq \{u_1 - r_2 : u_1 \in \Psi^{-1}(w_1), r_2 \in \Psi^{-1}(w_2)\}$. Now,

$$\begin{aligned}\Psi(\tilde{j})(w_1 - w_2) &= \bigcup_{k \in \Psi^{-1}(w_1 - w_2)} \tilde{j}(k) \\ &\supseteq \bigcup_{u_1 \in \Psi^{-1}(w_1), r_2 \in \Psi^{-1}(w_2)} \tilde{j}(u_1 - r_2) \\ &\supseteq \bigcup_{u_1 \in \Psi^{-1}(w_1), r_2 \in \Psi^{-1}(w_2)} \{\tilde{j}(u_1) \cap \tilde{j}(r_2)\} \\ &= \{\bigcup_{u_1 \in \Psi^{-1}(w_1)} \tilde{j}(u_1)\} \cap \{\bigcup_{r_2 \in \Psi^{-1}(w_2)} \tilde{j}(r_2)\} \\ &= \Psi(\tilde{j})(w_1) \cap \Psi(\tilde{j})(w_2),\end{aligned}$$

$$\begin{aligned}\Psi(\gamma)(w_1 - w_2) &= \bigwedge_{k \in \Psi^{-1}(w_1 - w_2)} \gamma(k) \\ &\leq \bigwedge_{u_1 \in \Psi^{-1}(w_1), r_2 \in \Psi^{-1}(w_2)} \gamma(u_1 - r_2) \\ &\leq \bigwedge_{u_1 \in \Psi^{-1}(w_1), r_2 \in \Psi^{-1}(w_2)} \{\gamma(u_1) \vee \gamma(r_2)\} \\ &= \{\bigwedge_{u_1 \in \Psi^{-1}(w_1)} \gamma(u_1)\} \vee \{\bigwedge_{r_2 \in \Psi^{-1}(w_2)} \gamma(r_2)\} \\ &= \Psi(\gamma)(w_1) \vee \Psi(\gamma)(w_2),\end{aligned}$$

$$
\begin{aligned}
\Psi(\tilde{j})(w_1w_2) &= \bigcup_{k\in\Psi^{-1}(w_1w_2)} \tilde{j}(k) \\
&\supseteq \bigcup_{u_1\in\Psi^{-1}(w_1), r_2\in\Psi^{-1}(w_2)} \tilde{j}(u_1r_2) \\
&\supseteq \bigcup_{u_1\in\Psi^{-1}(w_1), r_2\in\Psi^{-1}(w_2)} \{\tilde{j}(u_1)\cap\tilde{j}(r_2)\} \\
&= \{\bigcup_{u_1\in\Psi^{-1}(w_1)} \tilde{j}(u_1)\}\cap\{\bigcup_{r_2\in\Psi^{-1}(w_2)} \tilde{j}(r_2)\} \\
&= \Psi(\tilde{j})(w_1)\cap\Psi(\tilde{j})(w_2), \\
\Psi(\gamma)(w_1w_2) &= \bigwedge_{k\in\Psi^{-1}(w_1w_2)} \gamma(k) \\
&\leq \bigwedge_{u_1\in\Psi^{-1}(w_1), r_2\in\Psi^{-1}(w_2)} \gamma(u_1r_2) \\
&\leq \bigwedge_{u_1\in\Psi^{-1}(w_1), r_2\in\Psi^{-1}(w_2)} \{\gamma(u_1)\vee\gamma(r_2)\} \\
&= \{\bigwedge_{u_1\in\Psi^{-1}(w_1)} \gamma(u_1)\}\vee\{\bigwedge_{r_2\in\Psi^{-1}(w_2)} \gamma(r_2)\} \\
&= \Psi(\gamma)(w_1)\vee\Psi(\gamma)(w_2).
\end{aligned}
$$

So, $\Psi(\tilde{j}_\gamma)$ is hybrid ideal. □

4. Hybrid Semiprime Ideals in Commutative Rings

We discuss some properties of hybrid semiprime ideals of a ring and we characterise rings in terms of hybrid semiprime ideals. We also obtain an equivalent assertion for the hybrid semiprime ideal. Throughout this section, unless otherwise stated, $\mathscr{R}$ denotes a commutative ring.

Definition 15. *A hybrid ideal $\tilde{i}_\Omega \in \mathscr{H}(\mathscr{R})$ in $\mathscr{R}$ is hybrid semiprime whenever $\sqrt{\tilde{i}_\Omega} = \tilde{i}_\Omega$.*

Example 2. *Let $\mathscr{K}$ be a semiprime ideal of $\mathscr{R}$. Let $\tilde{i}_\Omega$ in $\mathscr{H}(\mathscr{R})$ over $\mathscr{U} = [0,1]$ be given by $\tilde{i}(k) = \begin{cases} [0,0.5] & if\ k\in\mathscr{K} \\ [0,0.1] & if\ k\notin\mathscr{K} \end{cases}$ and a mapping $\Omega : \mathscr{R} \to I$ be constant. Then, $\tilde{i}_\Omega$ is a hybrid semiprime ideal in $\mathscr{R}$.*

Lemma 3. *Let $\tilde{i}_\Omega, \tilde{j}_\gamma \in \mathscr{H}(\mathscr{R})$ be hybrid semiprime ideals in $\mathscr{R}$. Then, $\tilde{i}_\Omega \Cap \tilde{j}_\gamma \in \mathscr{H}(\mathscr{R})$ is a hybrid semiprime ideal.*

Proof. By Theorem 3, $\sqrt{\tilde{i}_\Omega \Cap \tilde{j}_\gamma} = \sqrt{\tilde{i}_\Omega} \Cap \sqrt{\tilde{j}_\gamma} = \tilde{i}_\Omega \Cap \tilde{j}_\gamma$. Therefore, $\tilde{i}_\Omega \Cap \tilde{j}_\gamma$ is a hybrid semiprime ideal. □

Theorem 9. *For any $Q \in \mathscr{P}(\mathscr{R})$ and $\tilde{i}_\Omega \in \mathscr{H}(\mathscr{R})$, we have that Q is a semiprime ideal if and only if $\chi_Q(\tilde{i}_\Omega)$ is a hybrid semiprime ideal.*

Proof. Assume Q is a semiprime ideal. Let $u \in \mathscr{R}$. If $u^2 \in Q$, then $u \in Q$. Thus, $\chi_Q(\tilde{i})(u) = \mathscr{U} = \chi_Q(\tilde{i})(u^2)$ and $\chi_Q(\Omega)(u) = 0 = \chi_Q(\Omega)(u^2)$ which imply $\sqrt{\chi_Q(\tilde{i}_\Omega)} = \chi_Q(\tilde{i}_\Omega)$. If $u^2 \notin Q$, then $\chi_Q(\tilde{i})(u^2) = \phi \subseteq \chi_Q(\tilde{i})(u)$ and $\chi_Q(\Omega)(u^2) = 1 \geq \chi_Q(\Omega)(u)$ which imply $\sqrt{\chi_Q(\tilde{i}_\Omega)} = \chi_Q(\tilde{i}_\Omega)$. So, $\chi_Q(\tilde{i}_\Omega)$ is a hybrid semiprime ideal.

Conversely, suppose $\chi_Q(\tilde{i}_\Omega)$ is a hybrid semiprime ideal and $u^2 \in Q$. Then, $\chi_Q(\tilde{i})(u) \supseteq \chi_Q(\tilde{i})(u^2) = \mathscr{U}$ and $\chi_Q(\Omega)(u) \leq \chi_Q(\Omega)(u^2) = 0$ which imply $u \in Q$. Hence Q is a semiprime ideal. □

Theorem 10. *Let $\tilde{i}_\Omega \in \mathscr{H}(\mathscr{R})$ be a hybrid ideal in $\mathscr{R}$. Then, the following assertions are equivalent:*

(i) $\tilde{i}_\Omega$ *is a hybrid semiprime ideal,*
(ii) $\tilde{i}(u^n) = \tilde{i}(u)$ *and* $\Omega(u^n) = \Omega(u)\ \forall\, u \in \mathscr{R}, \forall\, n \geq 1,$
(iii) $\tilde{i}(u^m) = \tilde{i}(u)$ *and* $\Omega(u^m) = \Omega(u)\ \forall\, u \in \mathscr{R}$ *with some* $m \geq 2$.

Proof. $(i) \Rightarrow (ii)$ Let n_1 be any positive integer and $u \in \mathscr{R}$. Then,

$$\tilde{i}(u) \subseteq \tilde{i}(u^{n_1}) \subseteq \bigcup_{n_1 \geq 1} \tilde{i}(u^{n_1}) = \sqrt{\tilde{i}}(u) = \tilde{i}(u)$$
$$\Rightarrow \tilde{i}(u^{n_1}) = \tilde{i}(u)$$

and

$$\Omega(u) \geq \Omega(u^{n_1}) \geq \bigwedge_{n_1 \geq 1} \Omega(u^{n_1}) = \sqrt{\Omega}(u) = \Omega(u)$$
$$\Rightarrow \Omega(u^{n_1}) = \Omega(u).$$

$(ii) \Rightarrow (iii)$ Evident.

$(iii) \Rightarrow (i)$ Suppose $w \in \mathscr{R}$ with $m < s$. Then, for some positive integer k, $m^k > s$. Now, for all $s > m$

$$\tilde{i}(w) = \tilde{i}(w^m) = \tilde{i}(w^{m^2}) = \ldots = \tilde{i}(w^{m^k}) \supseteq \tilde{i}(w^s) = \tilde{i}(w),$$
$$\Omega(w) = \Omega(w^m) = \Omega(w^{m^2}) = \ldots = \Omega(w^{m^k}) \leq \Omega(w^s) = \Omega(w)$$
$$\Rightarrow \tilde{i}(w) = \tilde{i}(w^s), \Omega(w) = \Omega(w^s).$$

Thus, $\sqrt{\tilde{i}}(w) = \bigcup_{s \geq 1} \tilde{i}(w^s) = \tilde{i}(w)$ and $\sqrt{\Omega}(w) = \bigwedge_{s \geq 1} \Omega(w^s) = \Omega(w)$, and hence $\sqrt{\tilde{i}_\Omega} = \tilde{i}_\Omega$. □

Theorem 11. *If $\tilde{i}_\Omega \in \mathscr{H}(\mathscr{R})$ is a hybrid ideal in $\mathscr{R}$, then $\sqrt{\tilde{i}_\Omega} = \Cap\{\tilde{j}_\gamma : \tilde{i}_\Omega \ll \tilde{j}_\gamma,\ \tilde{j}_\gamma$ is a hybrid semiprime ideal in $\mathscr{R}\}$.*

Proof. Let $\{\tilde{j}_{\gamma_i}\}_i \in \mathscr{H}(\mathscr{R})$ be hybrid semiprime ideals in $\mathscr{R}$ containing $\tilde{i}_\Omega$. Let $\sigma = \Cap_i \tilde{j}_{\gamma_i}$. By Lemma 1(i), $\sqrt{\tilde{i}_\Omega} \ll \sqrt{\tilde{j}_{\gamma_i}} = \tilde{j}_{\gamma_i}\ \forall\, i$, and hence $\sqrt{\tilde{i}_\Omega} \ll \Cap_i \tilde{j}_{\gamma_i} = \sigma$. So, $\sqrt{\tilde{i}_\Omega}$ is hybrid semiprime, by Theorem 2. In other words, $\sqrt{\tilde{i}_\Omega}$ is one of the $\tilde{j}_{\gamma_i}$. So, $\sigma \ll \sqrt{\tilde{i}_\Omega}$ and hence $\sigma = \sqrt{\tilde{i}_\Omega}$. □

Theorem 12. *Let $\tilde{i}_\Omega \in \mathscr{H}(\mathscr{R})$. Then, the equivalent assertions are as follows:*

(i) *Every hybrid ideal $\tilde{i}_\Omega$ in $\mathscr{R}$ is a hybrid semiprime ideal,*
(ii) *$\mathscr{R}$ is regular.*

Proof. $(i) \Rightarrow (ii)$ Suppose X is an ideal of $\mathscr{R}$. Then, $\chi_X(\tilde{i}_\Omega)$ is a hybrid semiprime ideal in $\mathscr{R}$. By Theorem 9, X is semiprime. Thus, $\mathscr{R}$ is regular by Theorem 1.

$(ii) \Rightarrow (i)$ Assume $\mathscr{R}$ is regular with $x \in \mathscr{R}$. Then, $\exists\, r \in \mathscr{R} \ni x = xrx = x^2 r$ which gives $x = x^n r^{n-1}\ \forall\, n \geq 2$. Let $\tilde{i}_\Omega \in \mathscr{H}(\mathscr{R})$ be a hybrid ideal. Then, $\tilde{i}(x) = \tilde{i}(x^n r^{n-1}) \supseteq \tilde{i}(x^n) \supseteq \tilde{i}(x)$ and $\Omega(x) = \Omega(x^n r^{n-1}) \leq \Omega(x^n) \leq \Omega(x)$ which imply $\tilde{i}(x^n) = \tilde{i}(x)$ and $\Omega(x^n) = \Omega(x)\ \forall\, x \in \mathscr{R}, n \geq 1$. Therefore, by Theorem 10, $\tilde{i}_\Omega$ is hybrid semiprime. □

Theorem 13. *Consider the ring homomorphism $\Psi : \mathscr{R} \to \mathscr{S}$. If $\tilde{i}_\Omega \in \mathscr{H}(\mathscr{R})$ and $\tilde{j}_\gamma \in H(\mathscr{S})$ are hybrid semiprime in $\mathscr{R}$ and $\mathscr{S}$, respectively, then we have*

(i) $\Psi^{-1}(\tilde{j}_\gamma)$ *is hybrid semiprime in* $\mathscr{R}$*,*
(ii) $\Psi(\tilde{i}_\Omega)$ *is hybrid semiprime in* $\mathscr{S}$*, provided* Ψ *is onto and* $\tilde{i}_\Omega$ *is* Ψ*-invariant.*

Proof.

(i) By Theorem 6, we can have $\sqrt{\Psi^{-1}(\tilde{j}_\gamma)} = \Psi^{-1}(\sqrt{\tilde{j}_\gamma}) = \Psi^{-1}(\tilde{j}_\gamma)$, and so $\Psi^{-1}(\tilde{j}_\gamma)$ is hybrid semiprime ideal in $\mathscr{R}$.
(ii) By Theorem 6, we can have $\sqrt{\Psi(\tilde{i}_\Omega)} = \Psi(\sqrt{\tilde{i}_\Omega}) = \Psi(\tilde{i}_\Omega)$, and so $\Psi(\tilde{i}_\Omega)$ is hybrid semiprime ideal in $\mathscr{S}$.
□

5. Conclusions

We developed an understanding of how hybrid nil radicals, hybrid ideals, hybrid products, and hybrid semiprime ideals operate in rings and discussed several of their properties. We have shown that for any hybrid ideal, the hybrid radical of the hybrid intersection is equivalent to the hybrid intersection of the hybrid radical. We also obtained an equivalent assertion for the hybrid radical of the hybrid intrinsic product, as well as a hybrid semiprime ideal. A future research project explores hybrid primes and hybrid quasi-primes in algebraic structures and establishes their structural properties.

Author Contributions: Conceptualization, methodology, K.P., G.M., B.E. and A.A.; writing—original draft preparation, G.M. and B.E.; writing—review and editing, K.P., G.M., B.E. and A.A. All authors contributed equally to the final manuscript and discussed the findings. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by Deanship of Scientific Research at Umm Al-Qura University. Grant number: 22UQU4270201DSR02

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: This article does not contain any studies with human participants or animals performed by any of the authors.

Data Availability Statement: Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Acknowledgments: The authors would like to express their gratitude to the anonymous referees for their insightful comments and suggestions, which significantly improved the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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