

Article

The Wave-Front Equation of Gravitational Signals in Classical General Relativity

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Abstract: In this paper the dynamical equation for propagating wave-fronts of gravitational signals in classical general relativity (GR) is determined. The work relies on the manifestly-covariant Hamilton and Hamilton–Jacobi theories underlying the Einstein field equations recently discovered (Cremaschini and Tassarotto, 2015–2019). The Hamilton–Jacobi equation obtained in this way yields a wave-front description of gravitational field dynamics. It is shown that on a suitable subset of configuration space the latter equation reduces to a Klein–Gordon type equation associated with a 4-scalar field which identifies the wave-front surface of a gravitational signal. Its physical role and mathematical interpretation are discussed. Radiation-field wave-front solutions are pointed out, proving that according to this description, gravitational wave-fronts propagate in a given background space-time as waves characterized by the invariant speed-of-light c . The outcome is independent of the actual shape of the same wave-fronts and includes the case of gravitational waves which are characterized by an eikonal representation and propagate in a generic curved space-time along a null geodesics. The same waves are shown: (a) to correspond to the geometric-optics limit of the same curved space-time solutions; (b) to propagate in a flat space-time as plane waves with constant amplitude; (c) to display also the corresponding form of the wave-front in curved space-time. The result is consistent with the theory of the linearized Einstein field equations and the existence of gravitational waves achieved in such an asymptotic regime. Consistency with the non-linear Trautman boundary-value theory is also displayed.

Keywords: General Relativity Hamilton equations; General Relativity Hamilton–Jacobi theory; wave-front theory; speed of propagation

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1. Introduction

A number of basic issues remain still unsolved today regarding the theory of wave solutions for the classical gravitational field dynamics to be achieved in the framework of the standard formulation of general relativity (SF-GR); namely, the theory of the Einstein field equations [1]. These concern, in particular, the prescription of the wave-front dynamics for gravitational signals and the related problem of the determination of their speed of propagation. Indeed, both still represent theoretical challenges after more than one hundred years since the mathematical foundations of the theory originally were laid down (in 1915) by Einstein in terms of the famous Einstein–Hilbert variational principle [2]. This occurs because, as explained below, the mathematical setting of the theory concerning both its continuous Hamilton equations and the proper form of the corresponding Hamilton–Jacobi formulation has remained undetermined until recently [3–5].

Concerning, in particular, the wave theory itself, and the determination of the related wave-front dynamics, both should be regarded:

- As consistency tests for the Einstein field equations specifically in reference with the determination of its appropriate Hamiltonian variational representation. This is true especially in view of the experimental evidence of gravitational waves reached after intensive and long-time efforts by experimental astrophysics [6]. The matter concerns in particular highly non-linear phenomena of GR that could be explored by studies of gravitational waves produced in compact binaries mergers [7–10].
- As test-beds for theory of SF-GR itself, due to the possibility of still unexplored and yet to be found experimental predictions and crucial observational implications [11,12].
- As theoretical frameworks for the interpretation of observational features characterizing gravitational waves, with particular reference to the constraints that can be placed on their speed of propagation [13], in turn permitting the implementation of additional confirmations to SF-GR or restriction of the validity of alternative gravity theories [14–18].

Indeed, contrary to the case of the Maxwell equations for the classical electromagnetic field, the Einstein equations for the space-time metric tensor are non-linear second-order PDEs, which, in their proper tensor form, are not formally cast in terms of so-called wave equations. Nevertheless, the concept of wave equation can acquire different meanings in the context of either classical mechanics or field theory, which actually correspond to disparate mathematical problems [19]. Most frequently this refers to a second-order linear PDE supplied with appropriate initial/boundary conditions. In its invariant formulation this representation describes the dynamics of a 4-scalar or tensorial quantity governed by the D'Alembertian differential operator (\square). Such a category includes the Klein–Gordon equation, either for massless or massive scalar fields, and also the Maxwell equations for the EM 4-potential. However, in the same context, a different kind of wave equation is provided by a first-order PDE; namely, the Hamilton–Jacobi equation for a classical Hamiltonian system. It is well known that, thanks to the symplectic nature of the canonical representation, its initial-value solution is completely equivalent to that of an underlying set of first-order ODEs identified with the corresponding Hamilton equations.

The case of GR is, however, peculiar, since the exact Einstein field equations seem to escape this classification. Nevertheless, it was soon realized (originally by Einstein himself) that the corresponding linearized gravitational field (LGF) equations for small-amplitude perturbations actually formally recover a wave-type equation analogous to the one occurring in the case of Maxwell equations for the electromagnetic 4-potential [20]. According to this kind of approach, the wave-like behavior belongs to a perturbation—denoted as gravitational wave—of a given background space-time metric tensor with respect to which the notions of covariance and manifest covariance (i.e., respectively, the transformation and tensor properties) of the theory are prescribed. Notice that the consequent perturbative representation of the metric tensor achieved in this way is peculiar. In fact, it actually differs from the exact one provided by the solution of the complete nonlinear exact set of Einstein equations, in terms of which exact reference frame (i.e., coordinate system) should be properly prescribed. As such, the gravitational wave thus determined is actually regarded as a separate field over the background one. This treatment permits, in principle, a straightforward representation of the LGF equations indicated above [21]. For example, in its simplest treatment, upon introducing a metric tensor decomposition of the type $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$ with respect to the flat Minkowski background space-time metric tensor $\eta_{\mu\nu}$ which raises/lowers tensor indices, and upon letting $\varepsilon \ll 1$, it can be shown that for a suitable choice of gauge condition the perturbation field $h_{\mu\nu}$ satisfies in vacuum the LGF equation

$$\square h_{\mu\nu} = 0, \quad (1)$$

where in such a context $\square \equiv \eta_{\alpha\beta} \partial^\alpha \partial^\beta$ identifies the D'Alembertian operator in flat space-time. It is then straightforward to show that analytical solutions of this equation for the linearized field $h_{\mu\nu}$

are expressed in terms of superposition of plane waves traveling at the speed of light c . Nowadays, within the framework of linearized GR, additional realizations of the same gravitational wave equation have been carried out, depending on the physical context to which they apply and the approximations retained; e.g., including non-vacuum contributions, curvature effects due to the background space-time, and self-interaction due to the energy-momentum density carried by the same waves [20].

Despite these outcomes, because of the mathematical structure of the Einstein equations, the issue of the possible wave-like character of exact GR field equations, the existence of corresponding wave-like solutions for the full metric tensor (i.e., without introducing any perturbative decomposition), and the identification of their speed of propagation, involve a deeper understanding, which goes beyond the framework of linearized theory, and at the same time, still preserves in a suitable sense the physical interpretation in terms of gravitational waves. The problems of determining the existence of GR exact non-linear wave solutions for the gravitational field, together with that of reaching an exact mathematical proof of the corresponding wave-front speed of propagation for arbitrary gravitational signals as well, represent fundamental theoretical concerns of GR [22]. These issues, to be regarded as intimately connected, refer actually to two distinct problems:

Problem (1) The first one concerns the classical equations of space-time obtained by ignoring possible quantum effects. In the context of SF-GR these equations are identified with the exact nonlinear system of the Einstein field equations. For this purpose either the presence of vacuum, or more generally of arbitrary local classical sources, needs to be taken into account. In addition, in both cases the allowance is given for the presence of a non-vanishing classical cosmological constant. The latter is regarded as a 4-scalar universal constant to be ultimately prescribed on phenomenological/theoretical grounds of some sort (see also related discussion reported in reference [23]). Thus, the issue is whether, upon due account of all such effects, the same equations may admit exact solutions in which either: Problem 1a—can be interpreted exactly, i.e., in the whole space-time, as wave-like ones (exact wave-like solutions); or Problem 1b—reproduces asymptotically, in a suitable sense, the gravitational waves predicted by linearized theory, i.e., when the conditions of validity of the same perturbative scheme apply (asymptotic wave-like solutions).

Regarding the second question, early progress was made in the late fifties thanks to the initial efforts of people such as Bondi, Pirani, and Robinson [24–26]. A notable mention should be made to the contributions by A. Trautman (1958 [27,28]) and by I. Robinson and A. Trautman (1960–1962 [29,30]) (for a review see also reference [22]) in which it was shown that, in analogy with the wave equation in Maxwell's theory, it is possible to formulate appropriate boundary conditions at infinity for the metric tensor solution of the GR equations such that it coincides with a radiative field; namely, describing outgoing gravitational waves carrying gravitational radiation. This effectively amounts to require that at infinity the solution of the Einstein equations recovers the linearized GR theory in a flat background space-time. However, the actual proof of existence of this type of solution for the metric tensor should be reached in whole generality; i.e., without any asymptotic decomposition and imposing only the aforementioned radiative boundary conditions, or by constructing explicitly exact wave-like solutions. In this connection, it is worth pointing out recent investigations that have analyzed the role of the cosmological constant on the detection of gravitational waves and their propagation in the de Sitter space-time [31,32], as well as the related issue of radiation fields generated by accelerated charges in similar cosmological scenarios [33].

Problem (2) The second problem, which remains still unsolved to date, is about the determination of an invariant; i.e., 4-scalar, dynamical equation describing the propagation and evolution of gravitational front-surfaces in the full nonlinear GR theory and whether this can be realized as a wave equation according to the classification given above, so that the same front-surfaces can be effectively interpreted as wave-fronts of the gravitational field. Thus, independently of the particular solution of the Einstein field equations and without invoking a priori, any perturbative scheme, the goal should be that of prescribing in a suitable way the corresponding wave-front equation holding for the same solution and describing the dynamics of the gravitational wave-fronts. This should involve,

in particular, the identification of the corresponding speed of propagation, showing whether or not it coincides with the speed of light c . Of course, for consistency, a theory of this type should be in agreement both with the wave-front dynamics of gravitational waves arising in the linearized GR Einstein equations and with the radiative boundary conditions at infinity proposed by Trautman. In addition, a theory of gravitational wave-fronts should necessarily satisfy the principle of manifest covariance and should be able to treat the dynamics of wave-fronts as belonging to the full metric tensor solution of the Einstein equations, for any kind of gravitational signal, not distinguishing between background and perturbative wave tensors.

The subject of this paper deals primarily with Problem 2; namely, the determination of the front dynamics of gravitational signals, the specification of their tensorial properties and their identification as wave-fronts associated with generally non-linear solutions of the Einstein equations for the metric tensor. In this context we adopt the definition of wave-fronts given by A.N. Temchin [34]: “Mathematically, wave fronts are characteristic manifolds of the equations that describe the field. On them there are discontinuities of the field functions or their derivatives, and a characteristic surface can therefore separate a region of space in which there is as yet no field from one in which a field is already present; the motion of such a surface corresponds very precisely to the notion of propagation of disturbances”. However, as we intend to show below, another possible occurrence (of wave-front) is the one in which, more generally, on the same surface, the background field tensor remains smooth together with its partial derivatives. Thus, in principle, in both cases, the motion of such surfaces may actually be assumed to identify the notion of propagation of gravitational disturbances.

This paper intends to contribute to a better understanding of the classical gravitational wave-front dynamics in the framework of the standard formulation of GR (SF-GR); namely, the Einstein field equations. The goal is to point out some crucial mathematical aspects of the problem, which are actually related to the innermost nature of the Einstein equations; i.e., their Hamiltonian representation and corresponding wave propagation phenomena. For this purpose the theoretical framework adopted here is based on the synchronous variational theory of covariant classical gravity (CCG) recently developed in series of papers (see reference [35] and also references [4,5]), which permits us to formulate a continuum Hamiltonian representation of the Einstein field equations.

In contrast to earlier Hamiltonian representations to be found in prevailing literature [21,36], and which ultimately date back to Dirac [37–41], a notable feature of CCG is that it is based on a manifestly-covariant approach. In this sense, the concept of covariance requires the identification of a “background” space-time with respect to which it can be expressed. The latter is represented by a differential 4-dimensional manifold $\{\mathcal{Q}^4, \hat{g}\}$ endowed with a corresponding background symmetric metric tensor field $\hat{g} \equiv \{\hat{g}_{\mu\nu}\}$, in such a way that the Hamiltonian theory of GR remains self-consistently defined. Hence, GCC is realized by the construction of a manifestly-covariant Hamiltonian structure $\{x, H\}$ formed respectively by a suitable canonical state $x = \{g, \pi\}$ and Hamiltonian density H . Here, g and π are real tensor fields respectively identified with the variational metric tensor, i.e., the generalized continuum Lagrangian coordinates, and the corresponding canonical momentum. These are defined upon appropriate configuration and extended momentum spaces U_g and U_π . The Hamiltonian structure is defined in such a way that:

- Its Hamiltonian density and canonical variables are respectively identified with a 4-scalar and 4-tensor with respect to the group of local point transformations which preserve the differential-manifold structure of $\{\mathcal{Q}^4, \hat{g}\}$;
- $g \equiv \{g_{\mu\nu}\} = \{g_{\nu\mu}\}$ is a symmetric second-order tensor, generally different from the background metric tensor $\hat{g} \equiv \{\hat{g}_{\mu\nu}\} \equiv \{\hat{g}_{\nu\mu}\}$, while

$$\pi \equiv \{\Pi_{\mu\nu}^\alpha\} \equiv \{\Pi_{\nu\mu}^\alpha\} \quad (2)$$

is a third-order 4-tensor, symmetric in the contravariant indexes, referred to as extended canonical momentum.

- It is frame-independent; i.e., it holds for arbitrary coordinate systems, and therefore does not require a preliminary foliation of the space-time $\{\mathbf{Q}^4, \hat{g}\}$;
- It is constraint-free, in the sense that the continuum canonical variables are treated as independent.

Given these premises, we therefore consider here the problem of the physical conjecture that, in the context of classical GR, gravitational signals or general perturbations of the metric field tensor propagate with a well-definite invariant speed, and in particular, that they can be characterized by a suitable front surface (namely, a surface associated with the front-propagation of gravitational signals) which is endowed with a maximum speed of propagation coinciding with the speed of light c . Rather than investigating per se solutions of the Einstein field equations in their customary representation as a set of 2nd-order nonlinear PDEs for the components of the metric tensor, our aim was to carry out the study in the context of the manifestly-covariant Hamiltonian theory of GR previously developed in references [4,5]. This was realized in terms of a two-step approach. More precisely, first it is shown that the same Hamiltonian structure $\{x, H\}$ holding for the variational tensor field x is uniquely associated with the Einstein field equations holding on the manifold $\{\mathbf{Q}^4, \hat{g}\}$. As a result, a set of dynamical evolution equations (which identify the continuous Hamilton equations) are reached for the canonical state, with the stationary solution $x = \{g \equiv \hat{g}, \pi \equiv 0\}$ satisfying identically the Einstein field equations. Notably, as in conventional Hamiltonian theories, a related Hamilton–Jacobi theory is induced on the configuration space $U_{\hat{g}}$ spanned by the variational metric tensor g . This permits us to determine an appropriate Hamilton–Jacobi equation which is fully equivalent to same canonical equations. Second, it is shown that the Hamilton–Jacobi equation determined in this way retains its customary meaning as the equation of wave-front dynamics [5].

However, as discussed below, the same physical interpretation is preserved also in validity of the “collapse condition” $g = \hat{g}, \pi = \hat{\pi} \equiv 0$, which means that it actually prescribes also the wave-front surface dynamics which is associated with the background metric tensor field \hat{g} . Since the same surface is necessarily associated with a 4-scalar equation, it follows, as explained below, that in validity of the collapse condition the same equation can be cast in the form of a 4-scalar Klein–Gordon wave equation for a real 4-scalar field S . Such an equation identifies, therefore, the surface wave-front of the background gravitational metric-tensor \hat{g} ; i.e., the dynamics associated with the wave-front propagation of in principle arbitrary solutions $\hat{g} \equiv \{\hat{g}_{\mu\nu}\}$ of the Einstein field equations. These wave-fronts are referred to here as “gravitational signals”. Notice that their meaning is not purely mathematical, but their actual physical occurrence might in principle be tested.

The result, it must be stressed, does not require, however, that such solutions have necessarily a wave-like form. In other words, it does not mean that gravitational waves, besides the linearized theory, should exist also in a “fully non-linear” regime of the Einstein field equations (i.e., for arbitrary solutions of the same equations). The proof instead concerns, here, the propagation of the said front surfaces; i.e., “gravitational signals” only (and not the actual determination of the background field \hat{g}). Nevertheless, although a surface of this type could propagate in a given background space-time as a wave at speed of light, the metric tensor generated by its passage can only be obtained by solving the Einstein field equations.

Concerning the speed of propagations of wave fronts, different cases with contrasting solutions have been treated in the literature. For example, in the same work cited above, Temchin asserted that the characteristic manifold of the equation for the free gravitational field is compatible with the existence of wave fronts whose velocity may differ from the speed of light c [34]. On the other hand, L.Ya Arifov [42] concluded that the velocity of a gravitational wave front in a synchronous frame of reference (of which inertial systems in a Minkowski space-time are a special case) is constant and equal to the fundamental velocity c . Furthermore, it is well-known that in the framework of linearized theory of GR, and in the geometric-optic limit, gravitational waves are characterized by null wave vector k^μ ; namely, such that in this approximation

$$g_{\mu\nu}k^\mu k^\nu = 0. \quad (3)$$

By introducing the representation $k_\mu = \frac{\partial S_{GW}}{\partial x^\mu}$, one obtains the fundamental equation for gravitational waves in the geometric optic limit, which reads

$$g^{\mu\nu} \frac{\partial S_{GW}}{\partial x^\mu} \frac{\partial S_{GW}}{\partial x^\nu} = 0, \quad (4)$$

where S_{GW} is the so-called eikonal of the gravitational wave [1]. In particular, in such a framework, the surfaces defined by the condition $S_{GW} = \text{const.}$ are customarily identified with wave-fronts of the gravitational wave. However, a general unambiguous treatment of the problem holding for the complete GR equations without restrictions to particular types of space-time solutions or choice of reference frames and yielding a satisfactory proof of the physical properties of gravitational wave-front dynamics is actually missing.

2. Lagrangian Formalism

In order to address the subject of the paper, the construction of suitable manifestly-covariant Lagrangian and corresponding Hamiltonian representations of GR is required. The setting of the theory is based here on a synchronous and manifestly-covariant variational formulation of the Einstein field equations [35], which belongs to the class of so-called deDonder-Weyl approaches to continuum field theory [43–49].

The functional is characterized by the adoption of superabundant variables. This is realized by the distinction between a continuum background metric tensor $\hat{g} = \{\hat{g}_{\mu\nu}\}$ and an independent variational field $g = \{g_{\mu\nu}\}$, both represented by symmetric 4-tensor fields with respect to the background space-time $\{\mathbf{Q}^4, \hat{g}\}$. In particular, this means that while performing the variations in the variational functional the components of $g_{\mu\nu} \equiv g_{\nu\mu}$ (and the corresponding conjugate momenta) are considered all independent, while their extremal values (appearing in the Euler–Lagrange equations) are then required to coincide with $\hat{g}_{\mu\nu} \equiv \hat{g}_{\nu\mu}$. Concerning the notation, in the paper all hatted quantities refer to functions of the background tensor \hat{g} . According to this picture, the covariance properties of the theory are defined with respect to the background metric tensor \hat{g} , which therefore raises/lowers tensor indices and has vanishing covariant derivatives. Instead, the variational tensor g identifies the physical properties of gravitational field expressed through kinetic and potential contributions in the corresponding Lagrangian function. In this sense, g has no geometrical interpretation, and therefore it does not raise or lower indices, and its dynamical equations are determined by the Euler–Lagrange equations following from the synchronous variational principle.

The manifestly-covariant variational formulation is based on the adoption of the variational functional

$$S_L(g, \hat{\nabla}g, \hat{g}) = \int d\Omega L(g, \hat{\nabla}g, \hat{g}), \quad (5)$$

with $L \equiv L(g, \hat{\nabla}g, \hat{g})$ being a smooth real 4-scalar Lagrangian, together with the related synchronous variational principle

$$\delta S_L(g, \hat{\nabla}g, \hat{g}) = 0. \quad (6)$$

Here, δ is the synchronous variation operator indicated above, which for an arbitrary functional variation $\delta g = g - g_1 \equiv \{g_{\mu\nu} - g_{1\mu\nu}\}$, with $g = \{g_{\mu\nu}\}$ and $g_1 = \{g_{1\mu\nu}\}$ being two arbitrary symmetric variational fields, coincides with the Lagrangian Frechet derivative; i.e.,

$$\delta \int d\Omega L(g, \hat{\nabla}g, \hat{g}) = \lim_{\alpha \rightarrow 0} \int d\Omega L(g + \alpha \delta g, \hat{\nabla}(g + \alpha \delta g), \hat{g}), \quad (7)$$

while denoting by construction $\delta \hat{g} = \hat{g} - \hat{g}_1$, one requires $\delta \hat{g} \equiv 0$. As far as the Lagrangian density L is concerned, which appears in the variational functional, this is assumed:

- A smooth real function of the generalized coordinate-field $g \equiv \{g_{\mu\nu}\}$ and its covariant derivative $\widehat{\nabla}g \equiv \{\widehat{\nabla}^\alpha g_{\mu\nu}\}$, with $\widehat{\nabla}^\alpha$ denoting the covariant derivative operator in which the Christoffel symbols are associated to the background field \widehat{g} ;
- Coordinate-independent, in the sense that it does not explicitly depend on the 4-position $r \equiv \{r^\mu\}$ (however, the implicit dependence through g and $\widehat{\nabla}g$ still remains).

Therefore, it can be conveniently represented as

$$L(g, \widehat{\nabla}g, \widehat{g}) = L_G(g, \widehat{\nabla}g, \widehat{g}) + L_\Lambda(g, \widehat{g}) + L_F(g, \widehat{g}), \quad (8)$$

where all contributions are considered coordinate-independent. In particular, the contributions L_G and L_Λ refer respectively to the so-called gravitational and cosmological-constant Lagrangian terms, defined as follows (see also reference [50]):

$$L_G(g, \widehat{\nabla}g, \widehat{g}) = -\kappa g^{\mu\nu} \widehat{R}_{\mu\nu} h(g, \widehat{g}) + \kappa \frac{1}{2} \widehat{\nabla}^\alpha g_{\mu\nu} \widehat{\nabla}_\alpha g^{\mu\nu}, \quad (9)$$

$$L_\Lambda(g, \widehat{g}) = -2\kappa \Lambda h(g, \widehat{g}). \quad (10)$$

Instead, the term L_F in Equation (5) identifies the Lagrangian contribution carried by external fields different from the gravitational one, which can be prescribed in non-vacuum configurations and generates the stress-energy tensor $T_{\mu\nu}$ according to the customary variational derivation [1]. The notations in the previous equations are standard ones; namely, $\widehat{R}_{\mu\nu}$ is the Ricci tensor defined with respect to the background metric tensor $\widehat{g}_{\mu\nu}$, κ identifies the constant $\kappa = \frac{c^3}{16\pi G}$, Λ is the cosmological constant, and $h(Z, \widehat{Z})$ is the 4-scalar multiplicative factor.

$$h(g, \widehat{g}) = \left(2 - \frac{1}{4} g^{\alpha\beta} g_{\alpha\beta}\right), \quad (11)$$

where by definition $g^{\alpha\beta} g_{\alpha\beta} \neq \delta^\alpha_\alpha$ for variational curves. We notice, in particular, that the function $h(g, \widehat{g})$ originates from the invariance property of the Lagrangian and the use of superabundant variables. Its synchronous variation warrants the manifest covariance property of the variational theory and replaces the contribution arising from the variation of $\sqrt{-g}$ in asynchronous principles. From the physical point of view the function $h(g, \widehat{g})$ can be interpreted as a measure of the discrepancy between the virtual (i.e., variational) fields $g_{\mu\nu}$ from the background tensor $\widehat{g}_{\mu\nu}$ for which instead, identically, $\widehat{g}_{\mu\nu} \widehat{g}^{\mu\nu} = 4$, and is such that $h(\widehat{g}, \widehat{g}) = 1$.

A property to be mentioned is the inclusion in the Lagrangian function of the kinetic term proportional to the square of the generalized field velocity $\widehat{\nabla}g$. Such types of contributions are necessarily ruled out in asynchronous principles, where by definition the variational metric tensor shares the same properties of the extremal one, and in particular it is allowed to raise/lower indices, so that identically, its covariant derivatives expressed in terms of Christoffel symbols are vanishing. In contrast, the kinetic term in the synchronous principle reveals itself to be crucial since: (a) it permits us to reach a representation of the Lagrangian for the gravitational field which has a structure analogous to that of other classical continuum fields, pointing out the role of the customary Ricci contribution to be a potential term; (b) it affords the derivation of corresponding classical Hamiltonian and Hamilton–Jacobi theories for the Einstein field equations, according to the developments reported in references [4,5]; (c) it is essential for the construction of a manifestly-covariant quantum gravity theory with canonical quantization method [50,51].

3. Hamilton and Hamilton–Jacobi Representations

Based on the Lagrangian variational setting presented above, in this section both the Hamilton (H) and Hamilton–Jacobi (HJ) theories holding for the variational tensor fields g and π are detailed. Notice

that due to the choice (2) of the canonical momentum, these theories in reference [4] are referred to as extended H and HJ representations of GR (a label here omitted for brevity). This refers to the adoption of a representation in which the canonical state takes the form $x \equiv \{g_{\mu\nu}, \Pi_{\mu\nu}^\alpha\}$, with the canonical momentum $\pi \equiv \{\Pi_{\mu\nu}^\alpha\}$ being a third-order 4-tensor. This allows one to display the relationship of the synchronous approach with the background space-time metric tensor $\hat{g}_{\mu\nu}$ and the extremal field equations represented by the classical Einstein equations.

The Hamiltonian representation is realized by the classical Hamiltonian structure $\{x, H\}$ which is formed by an appropriate *extended* 4-tensor canonical state $x \equiv \{g_{\mu\nu}, \Pi_{\mu\nu}^\alpha\}$ and a suitable 4-scalar Hamiltonian density $H(x, \hat{x})$, where $g_{\mu\nu}$ represents the Lagrangian coordinate expressed by the second-order variational field 4-tensor of the gravitational field, and $\Pi_{\mu\nu}^\alpha$ is its conjugate third-order momentum 4-tensor. By construction the state $\hat{x} \equiv \{\hat{g}_{\mu\nu}, \hat{\Pi}_{\mu\nu}^\alpha \equiv 0\}$ denotes the extremal state associated with the background space-time. This Hamiltonian representation, due to the prescription (2), is characterized by a canonical momentum (conjugate to the Lagrangian coordinates g) given by

$$\Pi_{\mu\nu}^\alpha = \kappa \hat{\nabla}^\alpha g_{\mu\nu}, \quad (12)$$

which is non-vanishing, since $g_{\mu\nu} \neq \hat{g}_{\mu\nu}$, and must therefore be considered non-extremal. On the other hand, by construction, it follows that the extremal value of $\Pi_{\mu\nu}^\alpha$, namely, $\hat{\Pi}_{\mu\nu}^\alpha = \kappa \hat{\nabla}^\alpha \hat{g}_{\mu\nu}$ vanishes identically. The Hamiltonian density $H = H(x, \hat{x})$ associated with the Lagrangian $L(g, \hat{\nabla}g, \hat{g})$ is then provided by the Legendre transform

$$L(g, \hat{\nabla}g, \hat{g}) \equiv \Pi_{\mu\nu}^\alpha \hat{\nabla}_\alpha g^{\mu\nu} - H(x, \hat{x}), \quad (13)$$

and is found to be

$$H(x, \hat{x}) = H_G(x, \hat{x}) - L_\Lambda(g, \hat{g}) - L_F(g, \hat{g}), \quad (14)$$

where in particular,

$$H_G(x, \hat{x}) \equiv \frac{1}{2\kappa} \Pi_{\mu\nu}^\alpha \Pi_{\alpha}^{\mu\nu} + \kappa g^{\mu\nu} \hat{R}_{\mu\nu} h. \quad (15)$$

Thus, introducing the Hamiltonian action functional

$$S_H(x, \hat{x}) = \int d\Omega L(x, \hat{x}), \quad (16)$$

with $L(x, \hat{x})$ identifying now the Lagrangian density (13) expressed in canonical variables, the synchronous Hamilton variational principle becomes

$$\delta S_H(x, \hat{x}) = 0, \quad (17)$$

for which the variation is performed with respect to the canonical state; i.e., in terms of arbitrary independent synchronous variations $\delta x = \{\delta g_{\mu\nu}, \delta \Pi_{\mu\nu}^\alpha\}$. Hence, this means that now the synchronous variation operator δ in Equation (17) is identified with the Hamiltonian Frechet derivative, so that:

$$\delta \int d\Omega L(x, \hat{x}) = \lim_{\alpha \rightarrow 0} \int d\Omega L(x + \alpha \delta x, \hat{x}). \quad (18)$$

The corresponding variational derivatives yield the so-called extended continuum Hamilton equations

$$\widehat{\nabla}_\alpha \Pi_{\mu\nu}^\alpha = -\frac{\partial H(x, \widehat{x})}{\partial g^{\mu\nu}}, \quad (19)$$

$$\widehat{\nabla}_\alpha g^{\mu\nu} = \frac{1}{\kappa} \Pi_\alpha^{\mu\nu}, \quad (20)$$

so that the second one recovers as usual the definition of the canonical momentum. These equations must be supplemented by suitable boundary conditions for the extended canonical state $x \equiv \{g_{\mu\nu}, \Pi_{\mu\nu}^\alpha\}$, so that the Hamiltonian problem defined in this way is equivalent to the Einstein field equations. More precisely, the connection is established provided $g_{\mu\nu}$ is identified with the extremal solution $\widehat{g}_{\mu\nu}$, so that respectively $\widehat{\Pi}_{\mu\nu}^\alpha$ and $\widehat{g}_{\alpha\beta}$ satisfy, identically, the equation $\widehat{\Pi}_{\mu\nu}^\alpha = 0$ and the Einstein equations.

In terms of the extended Hamiltonian structure it is possible to develop a theory of canonical transformations yielding a corresponding manifestly-covariant extended Hamilton–Jacobi theory. To this aim, we denote by $x \equiv \{g^{\beta\gamma}, \Pi_{\beta\gamma}^\alpha\}$ and $X \equiv \{Q^{\beta\gamma}, P_{\beta\gamma}^\alpha\}$ the set of canonical conjugate variables of the two canonical states related by a canonical transformation, whereby $\widehat{x} \equiv \{\widehat{g}_{\alpha\beta}, \widehat{\Pi}_{\mu\nu}^\alpha = 0\}$ and the functional dependence of X is taken to be of the general form $X = X(x, \widehat{x}, r)$.

The canonical map transforms the Lagrangian density $L(x, \widehat{x}) \equiv L(g, \widehat{\nabla}g, \widehat{g})$ given by Equation (13) in terms of the equation

$$L(x, \widehat{x}) = L_T(X, \widehat{X}, r) + \widehat{\nabla}_\alpha S^\alpha, \quad (21)$$

where $L_T(X, \widehat{X}, r)$ is the transformed Lagrangian which is taken to be of the form

$$L_T(X, \widehat{X}, r) = P_{\mu\nu}^\alpha \widehat{\nabla}_\alpha Q^{\mu\nu} - K(X, \widehat{X}, r). \quad (22)$$

Here, $K(X, \widehat{X}, r)$ is the transformed Hamiltonian function, while $S^\alpha = S^\alpha(x, X, r)$ can be interpreted as an arbitrary 4-vector mixed-variable generating function (gauge function) of the canonical transformation. In particular, it is possible to prescribe S^α in terms of the Legendre transformation

$$S^\alpha = -Q^{\mu\nu} P_{\mu\nu}^\alpha + S_2^\alpha(g, P, r), \quad (23)$$

where in the second term on the rhs the internal repeated index α is dummy. As a consequence of the previous equation, Equation (21) requires that the transformed Hamiltonian density K must be allowed to be also explicitly dependent on the 4-position r . Then, the following transformation equations are obtained:

$$\Pi_{\mu\nu}^\alpha = \frac{\partial S_2^\alpha(g, P, r)}{\partial g^{\mu\nu}}, \quad (24)$$

$$Q^{\mu\nu} = \frac{\partial S_2^\alpha(g, P, r)}{\partial P_{\mu\nu}^\alpha}, \quad (25)$$

$$K(X, \widehat{X}, r) = H(x, \widehat{x}) + \widehat{\nabla}_\alpha S_2^\alpha(g, P, r) \Big|_{(g, P)}, \quad (26)$$

where in the second equation the summation on the index α is understood, while in the last term of the third equation the condition on the covariant derivative must hold for each index α . Equations (24) and (25) identify respectively half of the inverse and direct canonical transformations.

Among the possible canonical transformations produced by a mixed-variable generating function S^α that can be introduced, we consider the particular one which brings to transformed fields X which are constant, in the sense that

$$\widehat{\nabla}_\alpha P^\alpha_{\mu\nu} = [P^\alpha_{\mu\nu}, K]_\alpha \equiv 0, \quad (27)$$

$$\widehat{\nabla}_\alpha Q^{\mu\nu} = [Q^{\mu\nu}, K]_\alpha \equiv 0, \quad (28)$$

where use is made of canonical Poisson bracket formalism (see reference [4]). The previous constraint equations require that the transformed Hamiltonian K cannot depend on the transformed state X . Hence, it can only be a function of the type $K = K(\widehat{x}, r)$, and in particular, it can be proven that K can always be defined in such a way that $K(\widehat{x}, r) = 0$; namely, it vanishes identically.

A particular realization of Equations (27) and (28) is provided by the identification $X = \widehat{x}$. As a consequence, both $Q^{\mu\nu}$ and $P^\alpha_{\mu\nu}$ are necessarily functions of the background metric tensor $\widehat{g}^{\alpha\beta}$ only. Therefore, from Equation (26), invoking also Equation (24), it follows that

$$H\left(g, \frac{\partial S_2(g, P, r)}{\partial g}\right) + \widehat{\nabla}_\alpha S_2^\alpha(g, P, r)\Big|_{(g, P)} = 0, \quad (29)$$

to be denoted as *continuum extended Hamilton–Jacobi equation* for the *Hamilton 4-vector generating function* $S_2^\alpha(g, P, r)$ applying to the variational state $g_{\mu\nu}$. Notice that the solution of Equation (29) must be determined letting initially $P^\alpha_{\beta\gamma} \neq 0$. From the previous equation, the 4-vector field S_2^α can then be prescribed by imposing the constraint equation as well:

$$Q^{\mu\nu} = \frac{\partial S_2^\alpha(g, P, r)}{\partial P^\alpha_{\mu\nu}}\Big|_{P=0}. \quad (30)$$

Provided the condition on the Hessian determinant $\det\left|\frac{\partial^2 S_2^\alpha(g, P, r)}{\partial g^{\mu\nu} \partial P^\alpha_{\mu\nu}}\right| \neq 0$ is satisfied, Equation (30) realizes an implicit equation for $g^{\beta\gamma}$. This condition warrants the existence of the inverse canonical transformation. In fact, once the generating function S_2^α is determined by solving the Hamilton–Jacobi equation, the vector Equation (30) together with Equation (24) yields the implicit function

$$g^{\beta\gamma} = g^{\beta\gamma}(\widehat{g}, r), \quad (31)$$

whereby the canonical momenta are provided by Equation (24). Finally, as shown in references [4,5], it can be proven that Equation (29) is equivalent to the set of extended Hamilton equations, which means that the fields $g^{\beta\gamma}$ and Π^β_α satisfy continuum canonical equations as well. This means that the continuum extended Hamilton–Jacobi equation is equivalent to the continuum extended Hamilton equations, and the solutions of the Hamilton–Jacobi problem are equivalent from a mathematical point of view to the solutions of the set of canonical equations. Both equations hold in principle for arbitrary canonical fields x , in which the variational 4-tensor $g_{\mu\nu}$ is still considered different from the background metric tensor $\widehat{g}_{\mu\nu}$.

4. Induced Hamilton–Jacobi Equation on $\{\widehat{g}\}$

The extended Hamilton and Hamilton–Jacobi theories presented in previous sections apply to the variational tensor field $g_{\mu\nu}$ defined in the corresponding 10-dimensional space U_g spanned by the real symmetric tensor $g = \{g_{\mu\nu}\}$, which in the synchronous variational approach is kept distinguished and independent from the background metric tensor $\widehat{g}_{\mu\nu}$. Nevertheless, classical Euler–Lagrange or Hamilton equations can be shown to recover the Einstein field equations when the “projection” on the space-time metric tensor $\widehat{g} = \{\widehat{g}_{\mu\nu}\}$ is performed. For the continuum extended Hamilton

equations, this is realized by setting that the following “collapse conditions” hold everywhere in the space-time $\{\mathbf{Q}^4, \hat{g}\}$:

$$g_{\mu\nu} = \hat{g}_{\mu\nu}, \quad (32)$$

$$\Pi_{\mu\nu}^\alpha = 0, \quad (33)$$

to be referred to in the following as the $\{\hat{g}\}$ set, where the second equation follows from the identification of canonical momentum in terms of the covariant derivative of $g_{\mu\nu}$, which vanishes for $g_{\mu\nu} = \hat{g}_{\mu\nu}$ due to metric conservation and Christoffel symbols definition. Notice that when the fields $g_{\mu\nu}$ and $\Pi_{\mu\nu}^\alpha$ are “collapsed” according to Equations (32) and (33), the solution of the continuous Hamilton Equations (19) and (20) coincides identically with the same fields (32) and (33). In fact, Equations (32) and (33) then imply:

$$-\frac{\partial H(x, \hat{x})}{\partial g^{\mu\nu}} = \left[\frac{\partial}{\partial g^{\mu\nu}} \left(\kappa g^{\mu\nu} \hat{R}_{\mu\nu} h - L_\Lambda(g, \hat{g}) - L_F(g, \hat{g}) \right) \right]_{g=\hat{g}} = 0, \quad (34)$$

which can be shown [50] to recover exactly the classical Einstein field equations for $\{\hat{g}\}$.

Given these premises, it is therefore meaningful to consider the analogous projection on $\{\hat{g}\}$ of the Hamilton–Jacobi equation and to study the implications for the wave-like description of the dynamics of $\hat{g}_{\mu\nu}$. To this aim, we start from the extended Hamilton–Jacobi Equation (29). When the condition (33) applies, from Equation (24) one has that

$$\Pi_{\mu\nu}^\alpha = \frac{\partial S_2^\alpha}{\partial g^{\mu\nu}} = 0, \quad (35)$$

$$S_2^\alpha(\hat{g}, P = 0, r) = \hat{S}^\alpha(\hat{g}, r). \quad (36)$$

Hence, when both conditions (32) and (33) apply simultaneously, the two 4-scalar contributions entering the Hamilton–Jacobi Equation (29) reduce respectively according to the following relations:

$$H\left(g, \frac{\partial S_2(g, P, r)}{\partial g}\right) \rightarrow \hat{H} \equiv \hat{H}(\hat{g}), \quad (37)$$

$$\hat{\nabla}_\alpha S_2^\alpha(g, P, r) \Big|_{(g,P)} \rightarrow \hat{\nabla}_\alpha \hat{S}^\alpha(\hat{g}, r). \quad (38)$$

The first one determines the Hamiltonian function on $\{\hat{g}\}$, in which the quadratic contribution of the canonical momenta characteristic of the Hamiltonian theory vanishes identically in such a limit. This limit quantity is denoted with the symbol \hat{H} and for shortness of notation we only indicate the functional dependence on the space-time metric tensor $\hat{g}_{\mu\nu}$ to stress that there is no more dependence on $g_{\mu\nu}$ here. It is nevertheless understood that the same Hamiltonian function can still depend on external field contributions and on possible classical cosmological constant term according to the definition (14). Concerning instead the relation (38), again we have introduced a short notation by denoting the Hamilton principal function on $\{\hat{g}\}$ as \hat{S}^α . In conclusion, the projected extended Hamilton–Jacobi equation on the configuration space-time becomes

$$\hat{H}(\hat{g}) + \hat{\nabla}_\alpha \hat{S}^\alpha(\hat{g}, r) = 0. \quad (39)$$

This represents an equation for the 4-vector field \hat{S}^α which depends only on space-time contributions on \hat{g} due to the source Hamiltonian term \hat{H} . By definition, the latter coincides with the extremal value of the original Lagrangian function, since the kinetic term proportional to covariant derivatives is null. The previous equation must be intended as a boundary-value problem (in the domain D^4 with boundary ∂D^4) associated with the same PDE for the unknown real 4-vector field \hat{S}^α , with $\hat{g}_{\mu\nu}$ to be considered prescribed (i.e., a solution of the Einstein equations). In particular, here we

shall identify the boundary ∂D^4 either with the improper set of D^4 , requiring that the same boundary is asymptotically-flat, or with the boundary of the bounded subset of D^4 , denoted as D_5 where the source term $\hat{H}(\hat{g})$ is non-zero and bounded. In contrast, in vacuum one expects

$$\hat{H}(\hat{g}) = 0 \quad (40)$$

identically, so that the vector field $\hat{S}^\alpha(\hat{g}, r)$ is necessarily divergence-free:

$$\hat{\nabla}_\alpha \hat{S}^\alpha(\hat{g}, r) = 0. \quad (41)$$

Notice that Equation (41) is identically fulfilled if $\hat{S}^\alpha = \hat{S}^\alpha(\hat{g})$, thanks to the chain rule, and since by construction $\hat{\nabla}_\alpha \hat{g}_{\mu\nu} = 0$. The physical meaning of \hat{H} is therefore that of carrying the set of invariant 4-scalars of the Lagrangian fields, which in the case of the gravitational field is given by the curvature Ricci 4-scalar $\hat{R} \equiv \hat{g}^{\mu\nu} \hat{R}_{\mu\nu}$. Hence, according to Equation (39), the dynamics of the field \hat{S}^α is determined by such invariant field densities, which are by definition observable quantities. It follows that necessarily also \hat{S}^α must be related to a suitable observable field, to be properly identified. In particular, since Equation (39) is generated by a Hamilton–Jacobi equation for the wave-front 4-vector S_2^α on $\{\hat{g}\}$, one can expect that its projection on $\{\hat{g}\}$ identifies the wave-front 4-vector field \hat{S}^α associated with the space-time dynamical tensor $\hat{g}_{\mu\nu}$. In fact, the projection of the extended Hamilton–Jacobi cannot generate any new additional physical field besides those originally present in the Hamiltonian function H . On the other hand, while external field contributions (e.g., due to the electromagnetic field) can be included or not, the space-time metric term is always present, even in a flat space-time limit through index saturation. Therefore, in view of these reasons for the Equation (39) to hold also in the case of vacuum external fields, the wave-front 4-vector field \hat{S}^α must be in some sense suitably related to the wave-front dynamics associated with the gravitational space-time metric tensor \hat{g} . However, as a final consideration, we notice that the same projected Hamilton–Jacobi equation inherits the arbitrariness originally present in Equation (29) for the determination of the 4-vector field S_2^α . In fact, the 4-scalar Equation (39) cannot uniquely determine the four components of the 4-vector \hat{S}^α , which therefore, at this stage, remains undetermined. The issue of its unique prescription will be sorted out in the next section.

5. Reduced Hamilton–Jacobi Theory on $\{\hat{g}\}$

The manifestly-covariant Hamilton–Jacobi theory determined by Equations (29) and (30) yields a wave-like description for the dynamics of the continuum tensorial field $g_{\mu\nu}$ with respect to the background metric tensor $\hat{g}_{\mu\nu}$ whose solution is equivalent to that of underlying Hamilton or Euler–Lagrange equations. This requires the determination of the Hamilton 4-vector generating function S_2^α , which in such a framework preserves the meaning of being a tensor related to the field wave-front. Properties of such Hamilton–Jacobi wave theory and implications for its manifestly-covariant quantization were studied in reference [5].

More precisely, in that work the problem of constructing a Hamilton–Jacobi theory for the gravitational field $g_{\mu\nu}$ which takes a form suitable for its quantum representation and the development of a corresponding manifestly-covariant quantum gravity theory was addressed. However, because of the arbitrariness in the determination of the 4-vector function S_2^α by the 4-scalar Hamilton–Jacobi equation, it turns out that the extended formulation (of the type indicated above) is not appropriate for this task, since a requisite to satisfy is that the canonical variables should have the same tensorial dimension, with the Hamilton principal function being represented by a single 4-scalar real function. The mathematical procedure identified in reference [5] to achieve this kind of representation is based on the introduction of a suitable projection operator Σ , which is prescribed to act on the Hamilton principal function in such a way to lower its tensorial order, while leaving unchanged the underlying canonical structure of the theory. As a result, the action of Σ can generate a corresponding *reduced Hamilton–Jacobi theory* which proves to be equivalent to the extended one. The operator Σ can be

realized in terms of the 4-vector $\Sigma \equiv \Sigma^\alpha$, so that the projection is interpreted as occurring along a prescribed tensorial direction given by the same Σ^α . It must be stressed that the arbitrariness which characterizes the Hamilton and Hamilton–Jacobi theories developed above for the determination of the canonical tensorial momenta and Hamilton principal function appears as a characteristic feature of the manifestly-covariant deDonder–Weyl type of approach to continuum field dynamics. As a consequence, in such a framework the search for reduced-dimensional canonical theories arises as an unavoidable step in order to gain further physical insight of the theory.

Given these considerations, we now revert our attention to the projected Hamilton–Jacobi Equation (39), which includes the particular case of Equation (41), both obtained in the previous section and holding on the configuration space-time of metric tensors $\hat{g}_{\mu\nu}$. This equation inherits from the Hamilton–Jacobi Equation (29) holding for $g_{\mu\nu}$ the same type of arbitrariness in the definition of the 4-vector Hamilton principal function. It is therefore necessary to identify a procedure to reduce the tensorial order of the function \hat{S}^α associated with the wave dynamics of the classical tensor $\hat{g}_{\mu\nu}$. This amounts to assume a representation for \hat{S}^α , in terms of a single 4-scalar unknown function S , that must be of the type

$$\hat{S}^\alpha = \Sigma^\alpha S, \quad (42)$$

with Σ^α being a suitable 4-vector field. This is realized either in terms of a real function or covariant linear and local differential operator to be specified, while $S = S(\hat{g}, r)$ is an unknown 4-scalar field to be determined via the projected Hamilton–Jacobi Equation (39). Due to the arbitrariness of Σ^α and S it follows that both functions are actually prescribed up to an arbitrary multiplicative scalar gauge and hence can be equivalently represented as

$$\hat{S}^\alpha = \kappa_1 \Sigma^\alpha \frac{1}{\kappa_2} S, \quad (43)$$

where for generality $\kappa_1 = \kappa_1(\hat{g}, r)$ and $\kappa_2 = \kappa_2(\hat{g}, r)$ identify in principle arbitrary 4-scalar real functions. Thus, if \hat{S}^α is gauge invariant (i.e., it is an observable) it follows that $\kappa_1(\hat{g}, r) = \kappa_2(\hat{g}, r)$, and in particular, in the case where Σ^α is a linear operator, it must commute with $\kappa_1(\hat{g}, r)$, so that necessarily $\Sigma^\alpha \kappa_1(\hat{g}, r) = 0$.

In order to reach an appropriate identification of Σ^α in Equation (42) we introduce a number of physical and mathematical requirements that such a 4-vector should satisfy:

(1) In the framework of the projected Hamilton–Jacobi equation, the Hamilton principal function \hat{S}^α must be related to the wave-front dynamics of $\hat{g}_{\mu\nu}$, as discussed after Equation (39). This means that the prescription of Σ^α must preserve the character of Equation (39) to be a wave-equation, in the sense of the classification detailed in Section 1.

(2) For greater generality it is assumed that the function \hat{S}^α defined on the set $\{\hat{g}\}$ is not continuously related to the Hamilton principal function $S_2^\alpha(g, P, r)$ so that actually

$$\lim_{g \rightarrow \hat{g}} S_2^\alpha(g, P, r) \neq \hat{S}^\alpha(\hat{g}, r). \quad (44)$$

(3) One notices that in the case of validity of the vacuum field Equation (41), a particular possible realization for Σ^α is achieved by identifying it with a null constant 4-vector; namely, such that $\Sigma^\alpha = k^\alpha$ and $\nabla_\alpha k^\alpha = 0$ with $k^\alpha k_\alpha = 0$. From the physical standpoint this corresponds to the particular case in which \hat{S}^α describes a plane wave propagating with the speed of light c . However, the consistency of such a particular solution should be verified *a posteriori* in terms of the more general collapsed Equation (39).

(4) The prescription of Σ^α , however, should apply in both cases of vacuum and non-vacuum sources in Equation (39); namely, respectively when $\hat{H} = 0$ and $\hat{H} \neq 0$. This means, therefore, that Σ^α can only depend on \hat{g} and not on possible external field contributions.

(5) The representation of Σ^α should permit the identification of the product $\Sigma^\alpha S$ in terms of the 4-vector field determined by the propagation of the wave-front surface associated with $\hat{g}_{\mu\nu}$, should be consistent with the small amplitude wave equation associated with the classical Einstein field equations, and should include, in particular, the case of flat space-time given by the Minkowski tensor $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$.

Collecting all these requirements, and in particular, the last one, it is possible to verify that the only admissible representation for Σ^α is provided by the choice

$$\Sigma^\alpha \equiv \hat{\nabla}^\alpha, \quad (45)$$

with $\hat{\nabla}^\alpha$ denoting again the covariant derivative; i.e., defined with respect to the same background metric tensor $\hat{g}_{\mu\nu}$ via the related Christoffel symbols. Substituting Equation (45) into Equation (42) yields, therefore,

$$\hat{S}^\alpha = \hat{\nabla}^\alpha S(\hat{g}, r), \quad (46)$$

so that effectively the initial arbitrariness of \hat{S}^α is sorted out and the unknown function remains now only the 4-scalar S , as is required for the consistency of Equation (39). Recalling the physical meaning of the Hamilton–Jacobi equation in classical mechanics, in the following the function S will be referred to as the space-time 4-scalar wave-front. The result (46) represents the sought solution to the problem of the search for a reduced Hamilton–Jacobi theory on $\hat{g}_{\mu\nu}$. This outcome will be implemented in next section to obtain a 4-scalar wave-front equation for the propagation of classical gravitational signals on the space-time $\hat{g}_{\mu\nu}$.

6. The Klein–Gordon Gravitational Wave-Front Equation

In this section we derive the final form of the reduced Hamilton–Jacobi equation induced on $\hat{g}_{\mu\nu}$, showing that upon resolving the indeterminacy on the representation of the Hamilton principal function \hat{S}^α according to the previous considerations, this can be realized by a generally non-homogeneous Klein–Gordon type of differential equation for a 4-scalar field in curved space-time. In this regard, the following proposition holds.

Theorem 1 (Klein–Gordon gravitational wave-front equation). *Given validity of Equation (39) and the representation (46), it follows that the generating function $\hat{S}^\alpha \equiv \hat{S}^\alpha(\hat{g}, r)$ can be taken of the general form*

$$\hat{S}^\alpha(\hat{g}, r) = \hat{\nabla}^\alpha S(\hat{g}, r) + \sigma^\alpha(\hat{g}, r), \quad (47)$$

where, respectively, $S(\hat{g}, r)$ and $\sigma^\alpha(\hat{g}, r)$ satisfy the equations

$$\hat{H}(\hat{g}, r) + \square S(\hat{g}, r) = 0, \quad (48)$$

$$\hat{\nabla}_\alpha \sigma^\alpha(\hat{g}, r) = 0. \quad (49)$$

Here, $\hat{H} \equiv \hat{H}(\hat{g}, r)$ and $\square \equiv \hat{g}_{\alpha\beta} \hat{\nabla}^\alpha \hat{\nabla}^\beta$ denotes the D'Alembertian differential operator in the background curved space-time $\hat{g}_{\alpha\beta}$. Then, provided $\sigma^\alpha(\hat{g}, r)$ satisfies the boundary condition

$$\sigma^\alpha(\hat{g}, r)|_{\partial D^4} = 0, \quad (50)$$

necessarily, it coincides with the null solution; namely, $\sigma^\alpha(\hat{g}, r) \equiv 0$. The Equation (48) is referred to as the non-homogeneous Klein–Gordon gravitational wave-front equation for the space-time 4-scalar wave-front S with source term given by the Hamiltonian $\hat{H}(r)$.

Proof. First, Equation (49) follows directly from the validity of Equation (48), when the decomposition (47) is introduced in Equation (39). Second, since Equation (49) is a first-order PDE, Equation (50)

is sufficient to warrant the uniqueness of the null solution in the whole domain. As a consequence, the initial Hamilton–Jacobi Equation (39) reduces to Equation (48). \square

Notice that the term $\sigma^\alpha(\hat{g}, r)$ can represent in principle any function obtained by saturation of higher-order tensors; for example, of the type $\sigma^\alpha(\hat{g}, r) = \hat{\nabla}_\beta h^{\beta\alpha}(r)$, with $h^{\beta\alpha}(r)$ being an arbitrary smooth symmetric real tensor field. All contributions of this type are automatically ruled out by the boundary condition indicated above, although it is obvious that such a choice is in principle non-unique. The virtue of the decomposition (47) and of the boundary condition (50) is twofold: (1) first it introduces the dependence in terms of the 4-scalar $S(\hat{g}, r)$, to be denoted as the gravitational wave-front function, as required by the considerations displayed in Section 5; (2) second, the related hypersurfaces $S(\hat{g}, r) = \text{const.}$ can in principle be associated with a wave-front surface propagating gravitational signals in a background space-time $\hat{g}_{\mu\nu}(r)$. Indeed, we notice that from the previous considerations, it follows that only Equation (48) survives. The wave equation obtained in this way must be supplied by suitable boundary conditions, which can be chosen to be the same ones prescribed by Trautman in references [27,28] to extract radiation-field solutions from the Einstein field equations. Accordingly, Equation (48) is identified with the non-homogeneous Klein–Gordon wave-like equation for the real 4-scalar field $S(\hat{g}, r)$. This determines the dynamics of the hypersurface $S(\hat{g}, r) = \text{const.}$ and its propagation direction, given by the 4-vector $\hat{\nabla}^\alpha S(\hat{g}, r)$. Therefore, in this picture, $S(\hat{g}, r)$ can be interpreted as a physical observable associated with gravitational wave-dynamics generated by the “potential energy density” $\hat{H}(\hat{g}, r)$.

A remark, however, must be made concerning the correct physical meaning of Equation (48), in order to avoid misunderstandings. First, the 4-scalar field $S(r)$ is uniquely related to the same space-time metric tensor $\hat{g}_{\mu\nu}(r)$ and expresses a property of its dynamics. Hence, it must not be understood as an external scalar field dynamically evolving over a prescribed space-time according to a linear wave equation and such that for it a linearity superposition principle applies. For the same reason, it would be wrong to believe that the Klein–Gordon equation thus determined can be complementary to the GR equations and/or can solve for the metric tensor itself. The physical meaning of Equation (48) is as follows. First, the background space-time $\hat{g}_{\mu\nu}(r)$ is provided as a solution of the Einstein field equations. This allows for the definition of the D’Alembertian operator \square . Then, if a gravitational signal is generated and propagates in such a space-time, Equation (48) tells us that the dynamical propagation of the wave-front associated with such a gravitational disturbance evolves in the given background space-time with a wave character according to the Klein–Gordon wave equation supplied by appropriate initial/boundary conditions [52]. In view of the definition adopted in the present treatment for the wave-front surface (see the Introduction), intended as a discontinuity surface separating two different space-times, one has that necessarily the propagation of the wave-front $S(\hat{g}, r)$ leaves behind its passage a new space-time metric tensor. Its precise determination can only be obtained by solving the full set of non-linear Einstein field equations and the associated initial-value mathematical problem. In this sense the superposition principle does not apply in such a case for the gravitational 4-scalar wave-front $S(r)$, consistent with the non-linear character of the Einstein field equations. In fact, let us suppose that two wave-front surfaces of different origin cross each other on some crossing domain. Each of them brings a new space-time solution, and the Einstein equations must be consistently solved in order to calculate the resulting space-time metric tensor. The latter one in fact is required in order to define the D’Alembertian operator. Hence, after crossing, the information carried by each wave-front cannot simply sum-up, but can only be propagated according to the Klein–Gordon equation after definition of the new differential operator and prescription of novel initial/boundary conditions. From the physical point of view, unless particular perturbative schemes or linearization techniques in weak field regimes can apply to the GR equations, the superposition principle fails to hold in the gravitational case of Equation (48). Indeed, every 4-scalar field $S(r)$, although propagating with a wave differential equation, modifies the space-time and this in turn affects the differential operator propagating its wave dynamics. This type of self-field interaction

thus inhibits the possibility of realizing in the general case a linear superposition principle for the gravitational wave-front surfaces.

Finally, as a note added in proof, it is interesting to report here an alternative derivation of the Klein–Gordon Equation (48). Although apparently conceptually different, the new derivation is shown to be equivalent to the one indicated above. For definiteness, let us consider here the general case of non-vacuum fields with position-dependent sources for which the Hamiltonian density takes the form:

$$H = H\left(g, \frac{\partial S_2(g, P, r)}{\partial g}, r\right); \quad (51)$$

namely, it can also be dependent explicitly on the 4-position. Let us prove that in such a case the Klein–Gordon Equation (48) follows directly from the H-J Equation (29). For this purpose we intend to investigate the form of the same H-J Equation (29) under the action of suitable gauge transformations which, instead, leave invariant the continuous Hamilton Equations (19) and (20).

One notices preliminarily that the same Hamilton equations are necessarily invariant with respect to the action of the following two gauge transformations:

$$S_2^\alpha(g, P, r) \rightarrow \bar{S}_2^\alpha(g, P, r) = S_2^\alpha(g, P, r) + \hat{\nabla}^\alpha S(\hat{g}, r) - S_2^\alpha(\hat{g}, \hat{P} = 0, r), \quad (52)$$

$$H\left(g, \frac{\partial S_2(g, P, r)}{\partial g}, r\right) \rightarrow H\left(g, \frac{\partial S_2(g, P, r)}{\partial g}, r\right) - H\left(\hat{g}, \frac{\partial S_2(g, P, r)}{\partial g}\bigg|_{g=\hat{g}}, r\right), \quad (53)$$

where P denotes the transformed canonical momenta which are assumed independent of $g = \{g_{\mu\nu}\}$, and $S(\hat{g}, r)$ is a priori an in principle arbitrary real and suitably smooth 4-scalar function. Let us prove first the invariance of the same Equations (19)–(20) with respect to Equation (52). Indeed, by construction denoting by $\partial/\partial g^{\mu\nu}$ the partial derivative performed keeping constant $\hat{g} = \{\hat{g}_{\mu\nu}\}$, it follows that

$$\frac{\partial \hat{\nabla}^\alpha S(\hat{g}, r)}{\partial g^{\mu\nu}} \equiv 0, \quad (54)$$

$$\frac{\partial S_2^\alpha(\hat{g}, \hat{P}, r)}{\partial g^{\mu\nu}} = 0, \quad (55)$$

while it is obvious that both $\nabla^\alpha S(\hat{g}, r)$ and $S_2^\alpha(\hat{g}, \hat{P}, r)$ are also independent of $(\partial/\partial g^{\mu\nu})S_2^\alpha(g, P, r)$. This proves the invariance property for (52). Next, let us consider the Hamiltonian density transformation (53). It follows that if the collapse condition $g = \hat{g}$ holds identically, then $H\left(g, \frac{\partial S_2(g, P, r)}{\partial g}, r\right) - H\left(\hat{g}, \frac{\partial S_2(g, P, r)}{\partial g}\bigg|_{g=\hat{g}}, r\right)$ vanishes, with \hat{g} denoting the solution of the Einstein field equations. This requires for consistency that in the canonical equation for the momentum (19) the further requirement (to be referred to as collapse condition for the momentum)

$$\hat{\nabla}_\alpha \Pi_{\mu\nu}^\alpha \bigg|_{g=\hat{g}} = \hat{\nabla}_\alpha \frac{\partial S_2^\alpha(g, \hat{P}, r)}{\partial g^{\mu\nu}} \bigg|_{g=\hat{g}} = 0 \quad (56)$$

should hold too. Hence, provided Equation (56) is fulfilled, (53) indeed realizes a gauge transformation for the Hamilton Equations (19) and (20). We omit here for brevity the detailed discussion about the validity of such an equation. Next, we consider the effect of the same gauge transformations on the H-J Equation (29). Despite the gauge property indicated above, it is immediate to show that the 4-scalar function $S(\hat{g}, r)$ is actually not arbitrary; i.e., once the same H-J Equation (29) is taken into account.

In fact, thanks to the previous gauge transformations and the aforementioned invariance property, it is obvious that the same H-J equation can be equivalently rewritten as:

$$H\left(g, \frac{\partial S_2(g, P, r)}{\partial g}, r\right) - H\left(\hat{g}, \frac{\partial S_2(g, P, r)}{\partial g}\bigg|_{g=\hat{g}}, r\right) + \hat{\nabla}_\alpha \bar{S}_2^\alpha(g, P, r)\bigg|_{(g, P)} = 0. \quad (57)$$

As a consequence, upon considering the “collapse” condition, i.e., letting $g = \hat{g}$ in the previous equation, and upon noting that the second term in the previous equation reduces exactly to

$$\hat{\nabla}_\alpha \bar{S}_2^\alpha(g, P, r)\bigg|_{(g, P)} \equiv \square S(\hat{g}, r), \quad (58)$$

one concludes that the 4-scalar real function $S(\hat{g}, r)$ must necessarily satisfy the homogeneous Klein–Gordon equation

$$\square S(\hat{g}, r) = 0. \quad (59)$$

This shows that, provided Equation (53) realizes a gauge transformation (which in turn requires validity of Equation (56)), then even in the case of non-vanishing non-vacuum sources the non-homogeneous Equation (48) actually reduces to the homogeneous form (59).

Hence, excluding the trivial case $S(\hat{g}, r) \equiv 0$ and provided the same 4-scalar function $S(\hat{g}, r)$ is twice differentiable and with continuous second partial derivatives, the same function can therefore be viewed as a surface wave-function. In other words, according to such hypothesis the equation

$$S(\hat{g}, r) = \text{const.} \quad (60)$$

can be interpreted as a wave-front surface, while Equation (59) identifies the corresponding wave-front equation describing its space-time propagation in the space-time $\{\mathbf{Q}^4, \hat{g}\}$. Incidentally, what anticipated now becomes obvious, namely that, in contrast to the customary definition (see reference [34]), the front-surface defined by Equation (60) does not require in principle that the background metric tensor \hat{g} should exhibit a discontinuity on the same surface. The first circumstance occurs, for example, if the surface wave-front corresponds to the propagation of the background space-time solution in a subdomain of a different space-time, for example a space-time which, before propagation takes place, is locally flat. However, more generally even if no discontinuity occurs, the propagation of the wave-front surface (60) may still occur, as corresponds to the propagation of an arbitrary smooth and finite width “gravitational signal”.

Nevertheless, as discussed in the next section, if the space-time $\{\mathbf{Q}^4, \hat{g}\}$ is asymptotically flat one expects that *at infinity* the wave-front $S(\hat{g}, r)$ must necessarily recover the customary small-amplitude gravitational wave form typical of the Einstein LGF equations.

7. Radiation-Field Solutions

In this section we analyze some mathematical aspects of the solutions of the wave Equation (48), showing, in particular, that these include radiation fields. First we notice that the wave Equation (48) is generally non-linear in terms of the metric tensor $\hat{g}_{\mu\nu}(r)$, and it is exact, i.e., no asymptotic approximations have been invoked for its construction. Hence, in principle it can be used to describe non-linear phenomena in the gravitational field wave dynamics. In order to illustrate this point, we represent the solution of Equation (48) in the form

$$S(\hat{g}, r) = S_R(\hat{g}, r) + S_S(\hat{g}, r), \quad (61)$$

where respectively the “source” field $S_S(\hat{g}, r)$ denotes a particular solution of the inhomogeneous Equation (48) for the given source term $\hat{H}(\hat{g}, r)$, while the “radiation” field $S_R(\hat{g}, r)$ is a solution of the corresponding homogeneous equation. In fact, the general solution of the boundary-value problem

associated with the Klein–Gordon equation is necessarily of the type (61). In particular, here we shall impose different boundary conditions for $S_S(\hat{g}, r)$ and $S_R(\hat{g}, r)$.

Let us first construct explicitly the solution for $S_S(\hat{g}, r)$. In this case we require that $S_S(\hat{g}, r)$ is prescribed at infinity; namely,

$$S_S(\hat{g}, r)|_{\partial D^4} = 0. \quad (62)$$

Thanks to such a boundary condition, Equation (48) can be formally solved in terms of the Hadamard Green function $G_H(r, r')$, yielding

$$S_S(r) = - \int_{D^4} d\Omega G_H(r, r') \hat{H}(\hat{g}, r'). \quad (63)$$

We notice that from Equation (63) the wave-front at 4-position r^μ depends non-locally on $\hat{g}(r)$, and therefore, due to the Einstein causality principle, only the retarded Hadamard Green function must be retained in the integral appearing on the rhs of Equation (63). Notice that Equation (63) contains in principle the non-linear effects arising from all the prescribed sources $\hat{H}(\hat{g}, r')$ (both vacuum and non-vacuum sources).

Let us now consider the problem of determining $S_R(\hat{g}, r)$. This satisfies by construction the homogeneous real Klein–Gordon equation

$$\square S_R(\hat{g}, r) = 0. \quad (64)$$

This equation is analogous to the wave equation for the electromagnetic radiation in vacuum. Its solution must therefore be associated with gravitational wave-like signals propagating in vacuum. Notably, this means that these wave solutions must propagate at the speed of light c . To determine explicitly their precise form, one has to prescribe appropriate boundary conditions. In principle these can be defined both at infinity (on ∂D^4), and on the boundary of suitable bounded subsets ∂D_S , requiring generally that in both cases

$$S_R(\hat{g}, r)|_{\partial D^4} = S_{R, \partial D^4}(\hat{g}, r) \neq 0, \quad (65)$$

$$S_R(\hat{g}, r)|_{\partial D_S} = S_{R, \partial D_S}(\hat{g}, r) \neq 0, \quad (66)$$

where on the rhs $S_{R, \partial D^4}(\hat{g}, r)$ and $S_{R, \partial D_S}(\hat{g}, r)$ denote suitable 4-scalar fields. Then, generally, a non-trivial solution of Equation (64) may exist. On the other hand, if both boundary conditions for $S_R(\hat{g}, r)$ vanish, then only $S_S(\hat{g}, r)$ survives.

To provide a physical interpretation of the vacuum wave equation, we consider the case of a solution characterized by a discontinuous wave front occurring on the hypersurface separating the domain D_W , which encloses D_S where the field S_R has propagated, from the complementary set D_{ext} , where instead S_R is assumed everywhere constant. For definiteness, we consider an eikonal representation for S_R , requiring that $\hat{\nabla}_\alpha S_R$ is given by the real part of

$$\hat{\nabla}_\alpha S_R = (\hat{\nabla}_\alpha \psi) A e^{i\psi} \Theta(\psi_0 - \psi). \quad (67)$$

Here, the notation is as follows. First, $\psi = \psi(r)$ is a real 4-scalar field, which in the domain D_{ext} takes the constant value ψ_0 . Furthermore, we assume that ψ and A are defined (i.e., are bounded) everywhere in D^4 , together with their first and second covariant derivatives which exist everywhere except on the boundary ∂D^4 (smoothness assumption). In particular, let us assume that ψ is bounded from above in D^4 , with $\sup(\psi) = \psi_0$. Therefore, by a proper choice of the constant ψ_0 , we can always assume that in the internal domain $\psi \leq \psi_0$. In addition, $A = A(\hat{g}, r)$ is generally a complex field, while $\Theta(z)$ is the weak Heaviside function, which is equal to 1 for $z \geq 0$ and 0 for $z < 0$. We notice that the eikonal representation given above warrants that in the internal domain, including ∂D_W , $\hat{\nabla}_\alpha S_R$

has the same direction of the eikonal wave-vector, consistent with the physical interpretation given above. It follows that Equation (64) requires

$$\begin{aligned}\widehat{\nabla}^\alpha \widehat{\nabla}_\alpha S_R &= \widehat{\nabla}^\alpha \left(\left(\widehat{\nabla}_\alpha \psi \right) A e^{i\psi} \Theta(\psi_o - \psi) \right) \\ &= \left(\widehat{\nabla}^\alpha \widehat{\nabla}_\alpha \psi \right) A e^{i\psi} \Theta(\psi_o - \psi) \\ &\quad + \left(\widehat{\nabla}_\alpha \psi \right) \left(\widehat{\nabla}^\alpha A \right) e^{i\psi} \Theta(\psi_o - \psi) \\ &\quad + i \left(\widehat{\nabla}_\alpha \psi \widehat{\nabla}^\alpha \psi \right) A e^{i\psi} \Theta(\psi_o - \psi) \\ &\quad - \left(\widehat{\nabla}_\alpha \psi \widehat{\nabla}^\alpha \psi \right) A e^{i\psi} \delta(\psi_o - \psi) = 0.\end{aligned}\quad (68)$$

Thanks to the assumption of smoothness of the fields A and ψ , it follows necessarily that on the boundary ∂D_W the constraint

$$\widehat{\nabla}_\alpha \psi \widehat{\nabla}^\alpha \psi = 0 \quad (69)$$

must hold. This effectively is equivalent to consider the geometric-optics limit to evaluate the wave solution of Equation (64), whereby Equation (69) represents the fundamental wave-front equation for the wave-front ψ applying in such a case. In fact, under the same smoothness assumptions, the information carried by the wave-front S_R is essentially the same as the one carried by ψ in the eikonal representation of the solution and satisfying Equation (69). This result establishes the connection with the classical theory of gravitational waves characterized by a wave-front equation provided by Equation (4) in which the 4-scalar wave-front is S_{GW} . Therefore, in the geometric-optics limit from one side the front-surfaces $S_R = \text{const.}$ are well approximated by the eikonal condition $\psi = \text{const.}$, while at the same time these correspond also to plane front-wave surfaces of gravitational waves in which $S_{GW} = \text{const.}$, so that in this limit $\psi \sim S_{GW}$. From the physical point of view, it means that in the geometric-optics limit both the wave-front and the solution of the perturbed metric tensor exhibit a wave-like behavior.

It is then immediate to determine the physical interpretation of Equation (69). In fact, let us introduce a tetrad basis of unit orthogonal 4-vectors $(a^\mu, b^\mu, c^\mu, d^\mu)$, with a^μ being time-like and (b^μ, c^μ, d^μ) space-like unit 4-vectors. Then, in the same tetrad the metric tensor $\widehat{g}_{\mu\nu}$ can be locally represented as $\eta_{\mu\nu}$, while following the customary geometric-optics interpretation $\widehat{\nabla}^\alpha \psi$ is projected as

$$\widehat{\nabla}^\mu \psi = k_0 a^\mu + k_1 b^\mu + k_2 c^\mu + k_3 d^\mu, \quad (70)$$

where we denote $k_0 \equiv \omega/c$, with ω being the mode frequency, and $\mathbf{k} \equiv (k_1, k_2, k_3)$ is the mode wave 3-vector. Thus, it follows that Equation (69) yields the dispersion relation

$$\widehat{\nabla}_\alpha \psi \widehat{\nabla}^\alpha \psi = -\frac{\omega^2}{c^2} + |\mathbf{k}|^2 = 0, \quad (71)$$

which relates the magnitude of the mode-wave 3-vector, and hence its wavelength defined by $\lambda = \frac{1}{|\mathbf{k}|}$, to the mode frequency ω . As a consequence the phase velocity is equal to the speed of light; namely, $\lambda\omega = c$. This proves that, in the geometric-optics limit, in the asymptotic case of flat space-time and in the absence of source fields, the wave-like solutions provided by the Klein–Gordon Equation (48) for the gravitational wave-front S can effectively be represented as planar waves propagating gravitational signals at the speed of light c [52].

In the internal domain, on the other hand, Equation (68) reduces to

$$\left(\widehat{\nabla}^\alpha \widehat{\nabla}_\alpha \psi \right) A e^{i\psi} + \left(\widehat{\nabla}_\alpha \psi \right) \left(\widehat{\nabla}^\alpha A \right) e^{i\psi} + i \left(\widehat{\nabla}_\alpha \psi \widehat{\nabla}^\alpha \psi \right) A e^{i\psi} = 0, \quad (72)$$

where now generally $\hat{\nabla}_\alpha \psi \hat{\nabla}^\alpha \psi \neq 0$. Hence, in such a set, the perturbation may not generally propagate with the phase velocity equal to the speed of light c , although its phase velocity cannot exceed it. On the other hand, in the geometric-optics limit, namely in which $\lambda \rightarrow 0$, the dominant term in the previous equation is the third one on the lhs. Therefore, in the same limit, one recovers again Equation (69) also in the internal domain.

From the previous considerations it follows that there are two distinct physical mechanisms which can give rise to gravitational wave-fronts propagating at the speed of light c : the first one arises when a discontinuity occurs for the wave-vector $\hat{\nabla}_\alpha S_R$, namely in the transition between internal and external domains. The second one instead is related to the asymptotic validity of geometric optics. This implies that, while in the first case the “amplitude” of the perturbation A remains finite (as the eikonal ψ), in the second one it must be considered infinitesimal. The latter case justifies the adoption of the linearized Einstein equation first introduced by Einstein himself. As a consequence, in such a case the well-known asymptotic theory of gravitational waves of this type applies. In addition, in the same geometric-optics limit the wave-front solution recovers the gravitational radiation theory provided by Trautman in terms of boundary-value to the GR equations extracting radiation fields for the metric tensor solution. In conclusion, Equations (63) and (64) are associated respectively with two different sources of gravitational wave-front solutions, where the precise nature of the perturbation carried by S_S depends intrinsically on the sources and the prescribed boundary conditions. This analysis shows that, in classical GR, the existence of underlying Hamilton and Hamilton–Jacobi theories to the Einstein field equations allows one to derive a first-principle proof that gravitational signals are characterized by wave-fronts propagating at the speed of light c .

As a final point, we address here the issue of whether in a curved space-time, i.e., in the presence of non-vanishing classical sources, gravitational signals, including, in particular, gravitational waves, still propagate with the speed of light c as in the case of flat space-time. We show that gravitational waves actually do propagate along null geodetics even in such a case of curved space-time. For definiteness, let us look for a complex, eikonal-type solution of the homogeneous Klein–Gordon Equation (59); namely, of the form

$$S(\hat{g}, r) = \phi(\hat{g}, r) \exp \{iF(\hat{g}, r)\}. \quad (73)$$

This requires therefore that the eikonal $F(\hat{g}, r)$ and the factor $\phi(\hat{g}, r)$ satisfy respectively the PDE’s

$$\hat{\nabla}_\mu \left(\frac{\hat{\nabla}^\mu \phi}{\phi} \right) + \frac{\hat{\nabla}_\mu \phi \hat{\nabla}^\mu \phi}{\phi} \equiv \frac{\hat{\nabla}_\mu \hat{\nabla}^\mu \phi}{\phi} = 0, \quad (74)$$

$$\hat{\nabla}_\mu \hat{\nabla}^\mu F + 2\hat{\nabla}^\mu F \hat{\nabla}_\mu \ln \phi = 0. \quad (75)$$

Let us therefore denote $\hat{\nabla}^\mu F = k^\mu$, so that in the curved space-time $\{\mathbf{Q}^4, \hat{g}\}$ the contra-variant component is $\hat{\nabla}_\mu F = \hat{g}_{\mu\nu} \hat{\nabla}^\nu F = \hat{g}_{\mu\nu} k^\nu = k_\mu$, and furthermore require that the eikonal solution (73) identifies a wave propagating along null geodetics. This implies that k^μ must identify a tangent 4-vector to a null geodetics and therefore it coincides itself with a tangent null 4-vector. Hence, by construction k^μ must fulfill the set of constraint equations

$$\begin{aligned} k^\mu k_\mu &= 0, \\ \nabla_\mu k^\mu &= 0. \end{aligned} \quad (76)$$

As a further consequence, in this case Equation (75) becomes

$$k^\mu F \nabla_\mu \ln \phi = 0, \quad (77)$$

while Equation (74) remains unchanged. A particular solution of Equations (74)–(76) for (ϕ, F) is therefore provided by

$$F = \int_{r_0}^r dr^\mu k_\mu \equiv \sum_{\mu=0,3} \int_{r_0}^r dr^\mu k_\mu(\hat{g}, r), \quad (78)$$

$$\phi = \phi(F), \quad (79)$$

where $\int_{r_0}^r dr^\mu k_\mu$ denotes a path integral along the null geodetics along which the wave-front propagates. Incidentally, this path-integral solution is a characteristic non-local effect originating from the wave propagation in a curved space-time. Notice that, while in an arbitrary space-time $\{\mathbf{Q}^4, \hat{g}\}$ the tangent null 4-vector k_μ is generally of the form $k_\mu = k_\mu(\hat{g}, r)$, with $\hat{g} \equiv \hat{g}(r)$, in the case the same space-time is flat (or becomes asymptotically flat on the boundary of the space-time), k_μ reduces obviously to a constant (in the sense of Equation (76)), as the coefficient of the exponential factor ϕ . Therefore, in such a limit and with these prescriptions, Equation (73) describes the luminal wave-front of a plane wave, i.e., a wave-front which propagates at the speed of light c . Instead, in a curved space-time $\{\mathbf{Q}^4, \hat{g}\}$ and allowing also $\phi = \phi(F)$, Equation (73) identifies more generally the wave-front of a gravitational signal characterized by a non-uniform exponential factor ϕ .

The connection between the two space-time representations indicated above can be rigorously established by means of a non-local point transformation (NLPT; see reference [53]). In fact, let us introduce the NLPT $r^\mu \rightarrow R^\mu(r, [r])$ determined by prescribing a 4-displacement transformation of the form

$$dR^\mu = dr^\alpha M_\alpha^\mu(r, [r]), \quad (80)$$

with $M_\alpha^\mu(r, [r])$ denoting a non-singular and generally non-local transformation matrix and $[r]$ a suitable non-local dependence. Then, let us require that the same NLPT generates the teleparallel transformation which maps an accessible subdomain of the curved space-time $\{\mathbf{Q}^4, \hat{g}(r)\}$ into the flat Minkowski space-time $\{\mathbf{Q}_0^4, \eta\}$ (with η denoting the tensor $\eta \equiv \{\eta_{\mu\nu}\} = \text{diag}\{1, -1, -1, -1\}$). This requires necessarily the validity of the matrix constraint equation

$$\eta_{\mu\nu} = (M^{-1})_\mu^\alpha (M^{-1})_\nu^\beta \hat{g}_{\alpha\beta}(r), \quad (81)$$

a condition which warrants also the isometric condition $\hat{g}_{\alpha\beta}(r) dr^\alpha dr^\beta = \eta_{jk} dR^j dR^k$. Equation (81) then determines non-uniquely the inverse transformation matrix $(M^{-1})_j^\alpha$ (and hence in turn M_α^μ too).

Now let us consider the action of the teleparallel NLPT defined in this way on the product $dr^\mu k_\mu(\hat{g}, r)$. This is defined with respect to $\{\mathbf{Q}^4, \hat{g}(r)\}$, while dr^μ and $k_\mu(\hat{g}, r)$ respectively identify the 4-position displacement and the local tangent 4-vector to the null geodetics, both prescribed with respect to the same space-time. Then, the same scalar product can be equivalently written as

$$dr^\mu k_\mu(\hat{g}, r) = dr^\alpha M_\alpha^\nu (M^{-1})_\nu^\beta k_\beta(\hat{g}, r) = dR^\mu K_\mu, \quad (82)$$

where $dr^\alpha M_\alpha^\nu = dR^\nu$ and $(M^{-1})_\nu^\beta k_\beta(\hat{g}, r) = K_\nu$ now identify the 4-position displacement and tangent null 4-vector in the transformed (Minkowski) space-time. Thus, up to an arbitrary additive constant the identity

$$F = \int^r dr^\mu k_\mu = R^\mu K_\mu \quad (83)$$

holds. This conclusion proves therefore that:

- Provided the exponential coefficient ϕ is a constant, i.e., it is independent of F (see Equation (79)), a wave-front which in a curved space-time is of the type (73) and with eikonal (78), is represented by a plane wave in the corresponding Minkowski teleparallel representation.

- In both cases the wave-front propagates along the corresponding null geodesics, so that its speed of propagation coincides with the speed of light c .
- The non-local contribution to the wave-front solution of the Klein–Gordon equation in curved space-time can be absorbed into an analogous non-local contribution defining the teleparallel transformation to flat space-time, where the wave solution has a local character.

The analysis performed above pertains to classical gravitational disturbances and their speed of propagation. Remarkably, thanks to its general character (the homogeneous Klein–Gordon Equation (59) which holds both to vacuum and non-vacuum sources), the present derivation applies also to small-amplitude gravitational wave solutions, as predicted by the Einstein LGF equations. In particular, either the exact wave-front represented by the eikonal solution (73) or the asymptotic behavior of a generic wave-front surface still exhibits the character of a wave-like profile, as typical of non-trivial solutions of the same Klein–Gordon equation. In addition, in the geometric-optics limit, the wave-front profile is safely approximated with speed-of-light planar waves, a property which therefore marks the rigorous connection with the classical prediction of the linearized GR theory regarding the existence of gravitational waves.

8. Concluding Remarks

The identification of the Hamiltonian setting of the Einstein equations represents a crucial aspect for the proper treatment of the two theoretical issues set in the introduction; namely, first the prescription of the dynamics of the wave-front of gravitational signals, and second, the determination of the corresponding speed of propagation for arbitrary background field metric tensors $\hat{g} \equiv \{\hat{g}_{\mu\nu}\}$; i.e., solutions of the Einstein field equations. In this paper such a Hamiltonian setting has been identified with the manifestly-covariant Hamiltonian representation of GR in the so-called extended form. This means that in such a representation, the conjugate momenta are represented by the components of a third-order 4-tensor; i.e., the extended canonical momentum $\pi \equiv \{\pi_{\mu\nu}^\alpha\}$. Hence, the same tensor field π and the conjugate Lagrangian coordinates, represented by the symmetric tensor field $g \equiv \{g_{\mu\nu}\}$, are solutions of a suitable synchronous Hamiltonian variational principle. This allows also the identification of the equivalent manifestly-covariant Hamilton–Jacobi theory, equally cast in extended form. Remarkably, its validity has been shown to extend also to the collapse case $\{g, \pi\} = \{\hat{g}, \hat{\pi} = 0\}$ in which the Einstein field equations hold.

Based on these premises in this paper the two theoretical issues of GR indicated above have been answered as follows:

- First, an exact GR-non-linear gravitational wave-front equation has been determined for the gravitational field, which has been identified with the 4-scalar Klein–Gordon equation for the wave-front surface.
- Second, at the same time, the mathematical proof that for arbitrary gravitational signals these wave-fronts propagate—both in vacuum (i.e., in the absence of sources) and in an arbitrary curved space-time—at the speed of light, has been reached.
- Third, the theory developed here holds also in the case of a non-vanishing cosmological constant, to be considered as a classical universal constant 4-scalar.

The theoretical outcomes reported in this paper represent a first-principle proof of the fact that, underlying the set of non-linear Einstein field equations, wave-like dynamics exist for the propagation of gravitational disturbances governed by a Hamilton–Jacobi wave-theory. Indeed, although the same Einstein field equations for the metric tensor \hat{g} apparently formally escape any representation or classification in terms of wave equations, it has been proven here that the wave-front dynamics actually exhibit such a wave behavior, satisfying a Klein–Gordon type of wave equation, while the invariant speed of propagation of classical gravitational disturbances is the speed of light c . These conclusions hold in general for any generic background metric tensor \hat{g} and are independent of the

existence of gravitational wave solutions for the same metric tensor and the presence of a classical cosmological constant. Nevertheless, the consistency with the theory of gravitational waves, with spatial reference to the case of geometric-optics limit, has been established.

Such results have deep physical implications because they extend to arbitrary non-linear solutions \hat{g} of the Einstein field equations, while being also in agreement with the predictions based on the theory of the linearized gravitational field (LGF) formulated by Einstein himself in terms of his namesake equations. However from the mathematical viewpoint, wave fronts are characteristic manifolds of the exact Einstein equations that describe the gravitational field tensor \hat{g} . As far as the wave-front propagation is concerned, two cases can in principle be distinguished. In the first one the characteristic front-surface can separate a region of space in which there is as yet no field from one in which a field is already present. In this case there are discontinuities in \hat{g} or of its partial derivatives, so that the same front surface can separate a region of space in which there is as yet no field from one in which a field is already present. This is the classical definition of wave-front to be found in the literature. However, as pointed out here, another possible occurrence (second case) is the one in which, more generally, the front surface may still be regular for \hat{g} , so that the tensor field remains smooth together with its partial derivatives on the same surface.

These conclusions are promising from the theoretical standpoint also. In fact, they are based on a novel representation of the variational formulation of the Einstein field equations, in terms of a synchronous variational principle. Its basic feature is that of being, just like the same Einstein equations, manifestly-covariant in character. This permits us to display in a perspicuous and intuitive way the Hamiltonian character of GR, unveiling at the same time also its basic properties with particular reference to the role of gravitational signals and their speed of propagation. As such the present theory represents also a convenient pathway for the formulation of the corresponding quantum theory of GR.

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