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A New Identity Involving Balancing Polynomials and Balancing Numbers

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Abstract: In this paper, a second-order nonlinear recursive sequence $M(h, i)$ is studied. By using this sequence, the properties of the power series, and the combinatorial methods, some interesting symmetry identities of the structural properties of balancing numbers and balancing polynomials are deduced.

Keywords: balancing numbers; balancing polynomials; combinatorial methods; symmetry sums

1. Introduction

For any positive integer $n \geq 2$, we denote the balancing number by B_n and the balancer corresponding to it by $r(n)$ if

$$1+2+\cdots+(B_n-1)=(B_n+1)+(B_n+2)+\cdots+(B_n+r(n))$$

holds for some positive integer $r(n)$ and B_n. It is clear that $r(n)=\frac{B_n-B_{n-1}-1}{2}$, for example, $r(2)=2$, $r(3)=14$, $r(4)=84$, $r(5)=492\ldots$

It is found that the balancing numbers satisfy the second order linear recursive sequence $B_{n+1}=6B_n-B_{n-1}$ $(n \geq 1)$, providing $B_0=0$ and $B_1=1$ [1].

The balancing polynomials $B_n(x)$ are defined by $B_0(x)=1$, $B_1(x)=6x$, $B_2(x)=36x^2-1$, $B_3(x)=216x^3-12x$, $B_4(x)=1296x^4-108x^2+1$, and the second-order linear difference equation:

$$B_{n+1}(x)=6xB_n(x)-B_{n-1}(x), n \geq 1,$$

where x is any real number. While $n \geq 1$, we get $B_{n+1}=6B_n-B_{n-1}$ with $B_n(1)=B_{n+1}$. Such balancing numbers have been widely studied in recent years. G. K. Panda and T. Komatsu [2] studied the reciprocal sums of the balancing numbers and proved the following inequation holds for any positive integer n:

$$\frac{1}{B_n-B_{n-1}}<\sum_{k=n}^{\infty}\frac{1}{B_k}<\frac{1}{B_n-B_{n-1}-1}.$$

G. K. Panda [3] studied some fascinating properties of balancing numbers and gave the following result for any natural numbers $m>n$:

$$(B_m+B_n)(B_m-B_n)=B_{m+n}\cdot B_{m-n}.$$

Other achievements related to balancing numbers can be found in [4–7].

It is found that the balancing polynomials $B_n(x)$ can be generally expressed as

$$B_n(x)=\frac{1}{2\sqrt{9x^2-1}}\left[\left(3x+\sqrt{9x^2-1}\right)^{n+1}-\left(3x-\sqrt{9x^2-1}\right)^{n+1}\right],$$

and the generating function of the balancing polynomials $B_n(x)$ is given by

$$\frac{1}{1-6xt+t^2} = \sum_{n=0}^{\infty} B_n(x)\cdot t^n. \tag{1}$$

Recently, our attention was drawn to the sums of polynomials calculating problem [8–11], which is important in mathematical application. We are going to study the computational problem of the symmetry summation:

$$\sum_{a_1+a_2+\cdots+a_{h+1}=n} B_{a_1}(x)B_{a_2}(x)\cdots B_{a_{h+1}}(x),$$

where h is any positive integer. We shall prove the following theorem holds.

Theorem 1. *For any specific positive integer h and any integer $n \geq 0$, the following identity stands:*

$$\begin{aligned}
&\sum_{a_1+a_2+\cdots+a_{h+1}=n} B_{a_1}(x)B_{a_2}(x)\cdots B_{a_{h+1}}(x)\\
=\ &\frac{1}{2^h\cdot h!}\cdot\sum_{j=1}^{h}\frac{M(h,j)}{(3x)^{2h-j}}\sum_{i=0}^{n}\frac{(n-i+j)!}{(n-i)!}\cdot\frac{B_{n-i+j}(x)}{(3x)^i}\cdot\binom{2h+i-j-1}{i},
\end{aligned}$$

where $M(h,i)$ is defined by $M(h,0)=0$, $M(h,i)=\frac{(2h-i-1)!}{2^{h-i}\cdot(h-i)!\cdot(i-1)!}$ for all positive integers $1\leq i\leq h$.

In particular, for $n=0$, the following corollary can be deduced.

Corollary 1. *For any positive integer $h\geq 1$, the following formula holds:*

$$\sum_{j=1}^{h} M(h,j)\cdot j!\cdot(3x)^j\cdot B_j(x) = 2^h\cdot h!\cdot(3x)^{2h}.$$

The formula in Corollary 1 shows the close relationship among the balancing polynomials. For $h=2$, the following corollary can be inferred by Theorem 1.

Corollary 2. *For any integer $n\geq 0$, we obtain*

$$\begin{aligned}
\sum_{a+b+c=n} B_a(x)\cdot B_b(x)\cdot B_c(x) &= \frac{1}{216x^3}\sum_{i=0}^{n}(n-i+1)(i+1)(i+2)\cdot\frac{B_{n-i+2}}{(3x)^i}\\
&+\frac{1}{72x^2}\sum_{i=0}^{n}(n-i+1)(n-i+2)(i+1)\cdot\frac{B_{n-i+3}}{(3x)^i}.
\end{aligned}$$

For $x=1$, $h=2$ *and* 3, according to Theorem 1 we can also infer the following corollaries:

Corollary 3. *For any integer $n\geq 0$, we obtain*

$$\begin{aligned}
\sum_{a+b+c=n} B_{a+1}\cdot B_{b+1}\cdot B_{c+1} &= \frac{1}{216}\sum_{i=0}^{n}(n-i+1)(i+1)(i+2)\cdot\frac{B_{n-i+2}}{3^i}\\
&+\frac{1}{72}\sum_{i=0}^{n}(n-i+1)(n-i+2)(i+1)\cdot\frac{B_{n-i+3}}{3^i}.
\end{aligned}$$

Corollary 4. *For any integer $n \geq 0$, we obtain:*

$$\begin{aligned}
&\sum_{a+b+c+d=n} B_{a+1} \cdot B_{b+1} \cdot B_{c+1} \cdot B_{d+1} \\
= \;& \frac{1}{3888} \sum_{i=0}^{n} (n-i+1)(i+1)(i+2)(i+3)(i+4) \cdot \frac{B_{n-i+2}}{3^i} \\
+ \;& \frac{1}{1296} \sum_{i=0}^{n} (n-i+1)(n-i+2)(i+1)(i+2)(i+3) \cdot \frac{B_{n-i+3}}{3^i} \\
+ \;& \frac{1}{1296} \sum_{i=0}^{n} (n-i+1)(n-i+2)(n-i+3)(i+1)(i+2) \cdot \frac{B_{n-i+4}}{3^i}.
\end{aligned}$$

Corollary 5. *For any odd prime p, we have the congruence $M(p,i) \equiv 0(mod p)$, $0 \leq i \leq p-1$.*

Corollary 6. *The balancing polynomials are essentially Chebyshev polynomials of the second kind, specifically $B_n(x) = U_n(3x)$. Taking $x = \frac{1}{3}x$ in Theorem 1, we can get the following:*

$$\begin{aligned}
&\sum_{a_1+a_2+\cdots+a_{h+1}=n} U_{a_1}(x)U_{a_2}(x)\cdots U_{a_{h+1}}(x) \\
= \;& \frac{1}{2^h \cdot h!} \cdot \sum_{j=1}^{h} \frac{(2h-j-1)!}{2^{h-j} \cdot (h-j)! \cdot (j-1)! \cdot x^{2h-j}} \sum_{i=0}^{n} \frac{(n-i+j)!}{(n-i)!} \cdot \frac{U_{n-i+j}(x)}{x^i} \cdot \binom{2h+i-j-1}{i}.
\end{aligned}$$

Compared with [8], we give a more precise result for $\sum_{a_1+a_2+\cdots+a_{h+1}=n} U_{a_1}(x)U_{a_2}(x)\cdots U_{a_{h+1}}(x)$ with the specific expressions of $M(h,i)$. This shows our novelty.

Here, we list the first several terms of $M(h,i)$ in Table 1 in order to demonstrate the properties of the sequence $M(h,i)$ clearly.

Table 1. Values of $M(h,i)$.

$M(h,i)$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$
$h=1$	1							
$h=2$	1	1						
$h=3$	3	3	1					
$h=4$	15	15	6	1				
$h=5$	105	105	45	10	1			
$h=6$	945	945	420	105	15	1		
$h=7$	10,395	10,395	4725	1260	210	21	1	
$h=8$	135,135	135,135	62,370	17,325	3150	378	28	1

2. Several Lemmas

For the sake of clarity, several lemmas that are necessary for proving our theorem will be given in this section.

Lemma 1. *For the sequence $M(n,i)$, the following identity holds for all $1 \leq i \leq n$:*

$$M(n,i) = \frac{(2n-i-1)!}{2^{n-i} \cdot (n-i)! \cdot (i-1)!}.$$

Proof. We present a straightforward proof of this lemma by using mathematical introduction. It is obvious that

$$M(1,1) = \frac{0!}{1 \cdot 0! \cdot 0!} = 1.$$

This means Lemma 1 is valid for $n = 1$. Without loss of generality, we assume that Lemma 1 holds for $1 \le n = h$ and all $1 \le i \le h$. Then, we have

$$M(h,i) = \frac{(2h-i-1)!}{2^{h-i} \cdot (h-i)! \cdot (i-1)!},$$

$$M(h,i+1) = \frac{(2h-i-2)!}{2^{h-i-1} \cdot (h-i-1)! \cdot i!}.$$

According to the definitions of $M(n,i)$, it is easy to find that

$$\begin{aligned} M(h+1,i+1) &= (2h-1-i) \cdot M(h,i+1) + M(h,i) \\ &= (2h-1-i) \cdot \frac{2(h-i)}{(2h-i-1)i} \cdot M(h,i) + M(h,i) \\ &= \frac{2h-i}{i} M(h,i) = \frac{(2h-i)!}{2^{h-i} \cdot (h-i)! \cdot i!} \\ &= \frac{(2(h+1)-(i+1)-1)!}{2^{h-i} \cdot (h-i)! \cdot i!}. \end{aligned}$$

Thus, Lemma 1 is also valid for $n = h+1$. From now on, Lemma 1 has been proved. □

Lemma 2. *If we have a function $f(t) = \frac{1}{1-6xt+t^2}$, then for any positive integer n, real numbers x and t with $|t| < |3x|$, the following identity holds:*

$$2^n \cdot n! \cdot f^{n+1}(t) = \sum_{i=1}^{n} M(n,i) \cdot \frac{f^{(i)}(t)}{(3x-t)^{2n-i}},$$

where $f^{(i)}(t)$ denotes the i-th order derivative of $f(t)$, with respect to variable t and $M(n,i)$, which is defined in the theorem.

Proof. Similarly, Lemma 2 will be proved by mathematical induction. We start by showing that Lemma 2 is valid for $n = 1$. Using the properties of the derivative, we have:

$$f'(t) = (6x-2t) \cdot f^2(t),$$

or

$$2f^2(t) = \frac{f'(t)}{3x-t} = M(1,1) \cdot \frac{f'(t)}{3x-t}.$$

This is in fact true and provides the main idea to show the following steps. Without loss of generality, we assume that Lemma 2 holds for $1 \le n = h$. Then, we have

$$2^h \cdot h! \cdot f^{h+1}(t) = \sum_{i=1}^{h} M(h,i) \cdot \frac{f^{(i)}(t)}{(3x-t)^{2h-i}}. \tag{2}$$

As an immediate consequence, we can tell by (2), the properties of $M(n,i)$, and the derivative, we get

$$\begin{aligned} &2^h \cdot (h+1)! \cdot f^h(t) \cdot f'(t) = 2^{h+1} \cdot (h+1)! \cdot (3x-t) \cdot f^{h+2}(t) \\ = &\sum_{i=1}^{h} \frac{M(h,i)}{(3x-t)^{2h-i}} \cdot f^{(i+1)}(t) + \sum_{i=1}^{h} \frac{(2h-i)M(h,i)}{(3x-t)^{2h-i+1}} \cdot f^{(i)}(t) \end{aligned}$$

$$
\begin{aligned}
&= \frac{M(h,h)}{(3x-t)^h}\cdot f^{(h+1)}(t)+\sum_{i=1}^{h-1}\frac{M(h,i)}{(3x-t)^{2h-i}}\cdot f^{(i+1)}(t)+\frac{(2h-1)M(h,1)}{(3x-t)^{2h}}\cdot f'(t)\\
&\quad+\sum_{i=1}^{h-1}\frac{(2h-i-1)M(h,i+1)}{(3x-t)^{2h-i}}\cdot f^{(i+1)}(t)\\
&= \frac{M(h+1,h+1)}{(3x-t)^h}\cdot f^{(h+1)}(t)+\frac{M(h+1,1)}{(3x-t)^{2h}}\cdot f'(t)+\sum_{i=1}^{h-1}\frac{M(h+1,i+1)}{(3x-t)^{2h-i}}\cdot f^{(i+1)}(t)\\
&= \frac{M(h+1,h+1)}{(3x-t)^h}\cdot f^{(h+1)}(t)+\frac{M(h+1,1)}{(3x-t)^{2h}}\cdot f'(t)+\sum_{i=2}^{h}\frac{M(h+1,i)}{(3x-t)^{2h+1-i}}\cdot f^{(i)}(t)\\
&= \sum_{i=1}^{h+1}M(h+1,i)\cdot\frac{f^{(i)}(t)}{(3x-t)^{2h+1-i}}. \qquad (3)
\end{aligned}
$$

Then, it is deduced that

$$2^{h+1}\cdot(h+1)!\cdot(3x-t)\cdot f^{h+2}(t)=\sum_{i=1}^{h+1}M(h+1,i)\cdot\frac{f^{(i)}(t)}{(3x-t)^{2h+1-i}},$$

or

$$2^{h+1}\cdot(h+1)!\cdot f^{h+2}(t)=\sum_{i=1}^{h+1}M(h+1,i)\cdot\frac{f^{(i)}(t)}{(3x-t)^{2h+2-i}}.$$

Thus, Lemma 2 is also valid for $n=h+1$. From now on, Lemma 2 has been proved. □

Lemma 3. *The following power series expansion holds for arbitrary positive integers h and k:*

$$\frac{f^{(h)}(t)}{(3x-t)^k}=\frac{1}{(3x)^k}\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\frac{(n-i+h)!}{(n-i)!}\cdot\frac{B_{n-i+h}(x)}{(3x)^i}\cdot\binom{i+k-1}{i}\right)t^n,$$

where t and x are any real numbers with $|t|<|3x|$.

Proof. According to the definition of the balancing polynomials $B_n(x)$, we have:

$$f(t)=\frac{1}{1-6xt+t^2}=\sum_{n=0}^{\infty}B_n(x)\cdot t^n.$$

For any positive integer h, from the properties of the power series, we can obtain

$$
\begin{aligned}
f^{(h)}(t) &= \sum_{n=0}^{\infty}(n+h)(n+h-1)\cdots(n+1)\cdot B_{n+h}(x)\cdot t^n\\
&= \sum_{n=0}^{\infty}\frac{(n+h)!}{n!}\cdot B_{n+h}(x)\cdot t^n. \qquad (4)
\end{aligned}
$$

For all real t and x with $|t|<|3x|$, we have the following power series expansion:

$$\frac{1}{3x-t}=\frac{1}{3x}\cdot\sum_{n=0}^{\infty}\frac{t^n}{(3x)^n},$$

and

$$\frac{1}{(3x-t)^k}=\frac{1}{(3x)^k}\cdot\sum_{n=0}^{\infty}\binom{n+k-1}{n}\cdot\frac{t^n}{(3x)^n}, \qquad (5)$$

with any positive integer k. Then, it is found that

$$\begin{aligned}
&\frac{f^{(h)}(t)}{(3x-t)^k} \\
= \ & \frac{1}{(3x)^k}\cdot\left(\sum_{n=0}^{\infty}\frac{(n+h)!}{n!}\cdot B_{n+h}(x)\cdot t^n\right)\left(\sum_{n=0}^{\infty}\binom{n+k-1}{n}\cdot\frac{t^n}{(3x)^n}\right) \\
= \ & \frac{1}{(3x)^k}\sum_{n=0}^{\infty}\left(\sum_{i+j=n}\frac{(j+h)!}{j!}\cdot B_{j+h}(x)\cdot\binom{i+k-1}{i}\cdot\frac{1}{(3x)^i}\right)t^n \\
= \ & \frac{1}{(3x)^k}\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\frac{(n-i+h)!}{(n-i)!}\cdot B_{n-i+h}(x)\cdot\binom{i+k-1}{i}\cdot\frac{1}{(3x)^i}\right)t^n,
\end{aligned}$$

where we have used the multiplicative of the power series. Lemma 3 has been proved. □

3. Proof of Theorem

Based on the lemmas in the above section, it is easy to deduce the proof of Theorem 1. For any positive integer h, we can derive

$$\begin{aligned}
&2^h\cdot h!\cdot f^{h+1}(t) = 2^h\cdot h!\cdot\left(\sum_{n=0}^{\infty}B_n(x)\cdot t^n\right)^{h+1} \\
= \ & 2^h\cdot h!\cdot\sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\cdots+a_{h+1}=n}B_{a_1}(x)B_{a_2}(x)\cdots B_{a_{h+1}}(x)\right)\cdot t^n.
\end{aligned}\tag{6}$$

On the other hand, by the observation made in Lemma 3, it is deduced that

$$\begin{aligned}
&2^h\cdot h!\cdot f^{h+1}(t) = \sum_{j=1}^{h}M(h,j)\cdot\frac{f^{(j)}(t)}{(3x-t)^{2h-j}} \\
= \ & \sum_{j=1}^{h}\frac{M(h,j)}{(3x)^{2h-j}}\cdot\left(\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\frac{(n-i+j)!}{(n-i)!}\cdot B_{n-i+j}(x)\cdot\binom{2h+i-j-1}{i}\cdot\frac{1}{(3x)^i}\right)t^n\right) \\
= \ & \sum_{n=0}^{\infty}\left(\sum_{j=1}^{h}\frac{M(h,j)}{(3x)^{2h-j}}\sum_{i=0}^{n}\frac{(n-i+j)!}{(n-i)!}\cdot\frac{B_{n-i+j}(x)}{(3x)^i}\cdot\binom{2h+i-j-1}{i}\right)\cdot t^n.
\end{aligned}\tag{7}$$

Altogether, we obtain the identity:

$$\begin{aligned}
&2^h\cdot h!\sum_{a_1+a_2+\cdots+a_{h+1}=n}B_{a_1}(x)B_{a_2}(x)\cdots B_{a_{h+1}}(x) \\
= \ & \sum_{j=1}^{h}\frac{M(h,j)}{(3x)^{2h-j}}\sum_{i=0}^{n}\frac{(n-i+j)!}{(n-i)!}\cdot\frac{B_{n-i+j}(x)}{(3x)^i}\cdot\binom{2h+i-j-1}{i}.
\end{aligned}$$

This proves Theorem 1.

4. Conclusions

In this paper, a representation of a linear combination of balancing polynomials $B_i(x)$ (see Theorem 1) is obtained. Moreover, the specific expressions of $M(h,i)$ is given by using mathematical induction (see Lemma 1).

Theorem 1 can be reduced to various studies for the specific values of x, n, and h in the literature. For example, if $n=0$, our results reduce to Corollary 1. Taking $h=2$, our results reduce to Corollary 2. Taking $x=1$, $h=2,3$, our results reduce to Corollary 3 and Corollary 4, respectively.

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