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# Overview of High Energy String Scattering Amplitudes and Symmetries of String Theory

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**Abstract:** In this paper, we studied symmetries of string scattering amplitudes in the high energy limits of both the fixed angle or Gross regime (GR) and the fixed momentum transfer or Regge regime (RR). We calculated high energy string scattering amplitudes (SSA) at arbitrary mass levels for both regimes. We discovered the infinite linear relations among fixed angle string amplitudes and the infinite recurrence relations among Regge string amplitudes. The linear relations we obtained in the GR corrected the saddle point calculations by Gross, Gross and Mende. In addition, for the high energy closed string scatterings, our results differ from theirs by an oscillating prefactor which was crucial to recover the KLT relation valid for all energies. We showed that all the high energy string amplitudes can be solved using the linear or recurrence relations, so that all the string amplitudes can be expressed in terms of a single string amplitude. We further found that, at each mass level, the ratios among the fixed angle amplitudes can be extracted from the Regge string scattering amplitudes. Finally, we reviewed the recent developments on the discovery of infinite number of recurrence relations valid for all energies among Lauricella SSA. The symmetries or relations among SSA at various limits obtained previously can be exactly reproduced. It leads us to argue that the known  $SL(K + 3, C)$  dynamical symmetry of the Lauricella function may be crucial to probe spacetime symmetry of string theory.

**Keywords:** symmetries of strings; hard string scattering amplitudes; regge string scattering amplitudes; high energy limits; zero norm states; linear relations; recurrence relations

## 1. Introduction

Quantum Field Theory (QFT) is a powerful theory in modern physics. The microcosmic physics is successfully described by using standard model of particle physics based on QFT. Various experiments have confirmed all important predictions by standard model at a rather imprecise level. Nevertheless, the crucial procedure to solve the UV divergence problem in QFT, i.e., renormalization, is mathematically complicated and has not been fully understood. Furthermore, the renormalization procedure does not work for gravity, so that a consistent quantum gravity theory is impossible to be constructed by using the conventional QFT. It is usually believed that the divergence in QFT is due to the topological structure of point-like particles that cannot be removed without modifying its topological structure. In string theory, a point-like particle is extended to a small piece of a string, which changes the topological structure of the theory. The Feynman diagram of strings interaction becomes a smooth world-sheet instead of the QFT world-line with singularity at interacting points.

To clarify the UV divergence problem in QFT, let us briefly examine the high energy behavior of a Feynman diagram by a simple power counting. In high energy hard limit, the tree diagram of particles

scattering by interchanging a spin- $J$  particle behaves as  $A_{tree}^{(J)} \sim E^{-2(1-J)}$ , so that a one-loop diagram behaves as

$$A_{1-loop}^{(J)} \sim \int d^4p \frac{(A_{tree}^{(J)})^2}{(p^2)^2} \sim \int E^{-4(2-J)} d^4E, \quad (1)$$

which is manifestly finite for scalar particles ( $J = 0$ ), renormalizable for vector particles ( $J = 1$ ), but nonrenormalizable for particles with  $J \geq 2$ , including graviton ( $J = 2$ ). Nevertheless, there is a loophole in this simple argument. Assuming that the interchanging states could have different spins, we thus should sum them all and the final amplitude becomes

$$A_{tree} = \sum_J A_{tree}^{(J)} \sim \sum_J a_J E^{-2(1-J)}, \quad (2)$$

which has a essential singularity as  $E \rightarrow \infty$  and could behave rather softly, so that loop amplitudes would be finite, provided the following two conditions are satisfied simultaneously [1]:

1. there are infinite higher spin  $J$  particles
2. the coefficients  $a_J$ 's are precisely related to each other.

In string theory, the behavior of the string scattering amplitudes was known to have a very soft exponential fall-off in the fixed angle high energy limit compared to the power law behavior of a local quantum field theory. Therefore, it is natural for us to believe that string theory satisfies the above two conditions.

String theory trivially satisfied the first condition because a string has infinite oscillation modes and each of them corresponds to a state with different spin. However, the second condition is highly nontrivial. We thus conjecture that it corresponds to a huge hidden symmetry in string theory.

Indeed, there was evidence which showed that the huge hidden symmetries proposed was closely related to the softness of string scattering amplitudes in the hard scattering limit. In [2,3], for string scatterings in the compact space, the authors discovered the existence of a power-law regime at fixed angle and simultaneously the linear relations (symmetry) break down in this regime.

To determine the form of the interaction in a local quantum field theory, a symmetry principle was postulated beforehand; while, in string theory, on the contrary, it was the interaction, which was prescribed by the very tight quantum consistency conditions due to the extendedness of one dimensional string, which determines the form of the symmetry.

It is well known that, in local gauge field theories, symmetries are spontaneously broken at low energy, but are restored at high energies. Motivated by this high energy behavior, historically, the first key progress to uncover symmetries of string theory was to study the high energy, fixed angle behavior of the hard string scattering (HSS) amplitudes [4–8] instead of the low energy field theory ones. In the pioneer work of Gross in 1988 [6,7], he made two main conjectures on this subject. The first one was that at high energies, fixed angle regime or Gross regime (GR) of the theory, there existed an infinite number of linear relations among the string scattering amplitudes of different string states at each order in string perturbation theory. The other one was that the scattering amplitudes of all the infinite number of string states can be determined in terms of one single dilaton (tachyon for the case of open string) scattering amplitudes by this symmetry. Nevertheless, the symmetry charges of his proposed high energy stringy symmetries were not understood and the ratios among scattering amplitudes of different string states were not calculated.

The second key progress to uncover symmetries of string theory was the realization of the importance of zero norm states (ZNS) in the old covariant first quantized (OCFQ) string spectrum. In the works

of [9–11], it was proposed that spacetime symmetry charges of string theory were originated from an infinite number of ZNS with arbitrary high spins in the old covariant quantized string spectrum.

This review is organized as following. In Section 2, we discuss and calculate stringy symmetries which were calculated to be valid for all energies. These calculations include stringy symmetries calculated by (1)  $\sigma$ -model approach of string theory in the first order weak field approximation, (2) Discrete ZNS and  $w_\infty$  symmetry of 2D string theory and (3) Soliton ZNS and the corresponding enhanced stringy gauge symmetries. We will concentrate on using the idea of ZNS and its applications to various calculations of stringy symmetries.

In Section 3, we will calculate high energy, fixed angle HSS amplitudes. The hard stringy Ward identities derived from the decoupling of ZNS in the HSS limit will be used to explicitly prove Gross's two conjectures [12–17]. An infinite number of linear relations among hard string scattering amplitudes of different string states were then derived. Remarkably, these linear relations can be used to determine the proportionality constants or ratios among HSS amplitudes of different string states algebraically at each fixed mass level.

In Section 4, we discuss the hard closed string scatterings. We will point out and clarify the inconsistency of the calculation of Gross and Mende [4,5] by using the KLT relation [18] which is valid for all energies. The first “string BCJ relation” was then discovered [19] in 2006, and was independent of the discovery of field theory BCJ relation [20] in 2008. We will also discuss hard string scattered from D-branes/O-planes and in compact space, and study their high energy behaviors.

In Section 5, we will calculate another high energy string scattering amplitude, the Regge string scattering (RSS) amplitude. We will see that, in contrast to the linear relations in the GR, there exists an infinite number of “recurrence relations” among RSS amplitudes of different string states. These recurrence relations can be used to solve all RSS amplitudes and express them in terms of one single amplitude. Moreover, an interesting link between the HSS and the RSS amplitudes was discovered, and the ratios among fixed angle amplitudes can be extracted from RSS amplitudes [21,22].

In Section 6, we briefly review the recent developments on the discovery of an infinite number of recurrence relations which were valid for *all* energies among the *exact* open bosonic string scattering amplitudes of three tachyons and one arbitrary string state, or the so-called Lauricella string scattering (LSS) amplitudes [23]. These infinite number of recurrence relations can be used to solve [24] all the LSS amplitudes algebraically and express them in terms of one single four tachyon amplitude.

Moreover, string scattering amplitudes and symmetries or relations among string scattering amplitudes among different string states at various scattering limits calculated previously can be rederived. These include the stringy linear relations conjectured by Gross and proved by Taiwan group [12–17] in the hard scattering limit, the recurrence relations in the Regge scattering limit [25,26] and the extended recurrence relations in the nonrelativistic scattering limit discovered recently [27]. Finally, a conclusion is given in Section 7.

For a more detailed review of high energy string scattering amplitudes and symmetries of string theory, see the recent long review paper [28].

## 2. Zero Norm States and Enlarged Stringy Symmetries

In the calculation of  $\sigma$ -model approach of string theory, one turns on background fields on the worldsheet energy momentum tensor  $T$ . The conformal invariance of the worldsheet then requires, in addition to  $D = 26$ , the cancellation of various kinds of  $q$ -number anomalies and results to equations of motion of the background fields [29]. It was then demonstrated that [9] for each *spacetime* ZNS, one can systematically construct a *worldsheet*  $(1, 1)$  primary field  $\delta T_\Phi$  such that

$$T_\Phi + \delta T_\Phi = T_{\Phi+\delta\Phi} \quad (3)$$

is satisfied to some order of weak field approximation in the  $\sigma$ -model background fields  $\beta$  function calculation. In the above equation,  $T_\Phi$  is the worldsheet energy momentum tensor with background fields  $\Phi$  and  $T_{\Phi+\delta\Phi}$  is the new energy momentum tensor with the new background fields  $\Phi + \delta\Phi$ . As a result, for each ZNS one can construct a spacetime symmetry transformation for the corresponding string background fields.

In addition to the positive norm physical propagating modes, there are two types of physical ZNS in the old covariant first quantized (OCFQ) open bosonic string spectrum: [29]

$$\text{Type I : } L_{-1} |x\rangle, \text{ where } L_1 |x\rangle = L_2 |x\rangle = 0, L_0 |x\rangle = 0; \quad (4)$$

$$\text{Type II : } \left( L_{-2} + \frac{3}{2} L_{-1}^2 \right) |\tilde{x}\rangle, \text{ where } L_1 |\tilde{x}\rangle = L_2 |\tilde{x}\rangle = 0, (L_0 + 1) |\tilde{x}\rangle = 0. \quad (5)$$

while type I states have zero-norm at any spacetime dimension, type II states have zero-norm *only* at  $D = 26$ . For example, among other stringy symmetries, an *inter-particle* symmetry transformation for two propagating states at mass level  $M^2 = 4$  of open bosonic string can be generated [9]

$$\delta C_{(\mu\nu\lambda)} = \frac{1}{2} \partial_{(\mu} \partial_\nu \theta_\lambda^2 - 2\eta_{(\mu\nu} \theta_\lambda^2), \delta C_{[\mu\nu]} = 9\partial_{[\mu} \theta_{\nu]}^2, \quad (6)$$

where  $\partial^\mu \theta_\mu^2 = 0, (\partial^2 - 4)\theta_\mu^2 = 0$  which are the on-shell conditions of the  $D_2$  vector ZNS with polarization  $\theta_\mu^2$  [9]

$$|D_2\rangle = \left[ \left( \frac{1}{2} k_\mu k_\nu \theta_\lambda^2 + 2\eta_{\mu\nu} \theta_\lambda^2 \right) \alpha_{-1}^\mu \alpha_{-1}^\nu \alpha_{-1}^\lambda + 9k_\mu \theta_\nu^2 \alpha_{-2}^{[\mu} \alpha_{-1}^{\nu]} - 6\theta_\mu^2 \alpha_{-3}^\mu \right] |0, k\rangle, \quad k \cdot \theta^2 = 0, \quad (7)$$

and  $C_{(\mu\nu\lambda)}$  and  $C_{[\mu\nu]}$  are the two background fields of the symmetric spin-three and antisymmetric spin-two propagating states respectively.

In the higher mass levels,  $M^2 = 6$  for example, a new phenomenon begins to show up. There are ambiguities in defining positive-norm spin-two and scalar states due to the existence of ZNS in the same Young representations [11]. As a result, the degenerate spin two and scalar positive-norm states can be gauged to the higher rank fields, the symmetric spin four  $D_{\mu\nu\alpha\beta}$  and mixed-symmetric spin three  $D_{\mu\nu\alpha}$  in the first order weak field approximation. In fact, it was shown [30] that the scattering amplitude involving the positive-norm spin-two state could be expressed in terms of the of spin-four and mixed-symmetric spin-three states due to the existence of a *degenerate* type I and a type II spin-two ZNS. Presumably, this stringy phenomenon seems to persist to higher mass levels.

This calculation is consistent with the result in the HSS limit. In fact, it can be shown that in the HSS limit all the scattering amplitudes of leading order in energy at each fixed mass level can be expressed in terms of those of the leading trajectory string states with transverse polarizations on the scattering plane. See the calculations of Equations (13), (26) and (34) in Section 3. One can also justify this decoupling phenomenon by WSFT [31]. Finally one expects this decoupling to persist even if one includes the higher order corrections in the calculation of weak field approximation, as there will be even stronger relations among background fields order by order through iteration.

The calculation of Equation (6) obtained in the first order weak field approximation is valid for all energies or all orders in  $\alpha'$ . The second order calculation in the weak field approximation implies an even more interesting spontaneously broken inter-mass level symmetry in string theory [32,33]. Some implications of the corresponding stringy Ward identities on the string scattering amplitudes were discussed in [32,34]. On the other hand, it was then realized that [31,35] the symmetry in Equation (6) can be reproduced from the off-shell gauge transformations of Witten string field theory (WSFT) [36] by

imposing the no ghost conditions. It is important to note that this stringy symmetry exists only for  $D = 26$  due to the type II ZNS in the OCFQ string spectrum, which only exists for  $D = 26$ .

Incidentally, it was well known in toy 2D string theory that the operator products of the discrete positive norm states [37]

$$\psi_{J,M}^{\pm} \sim \begin{vmatrix} S_{2J-1} & S_{2J-2} & \cdots & S_{J+M} \\ S_{2J-2} & S_{2J-3} & \cdots & S_{J+M-1} \\ \vdots & \vdots & \ddots & \vdots \\ S_{J+M} & S_{J+M-1} & \cdots & S_{2M+1} \end{vmatrix} \cdot \exp \left[ \sqrt{2}(iMX(0) + (-1 \pm J)\phi(0)) \right] \quad (8)$$

form a  $w_{\infty}$  algebra [38–40]

$$\int \frac{dz}{2\pi i} \psi_{J_1, M_1}^+ \psi_{J_2, M_2}^+ = (J_2 M_1 - J_1 M_2) \psi_{J_1+J_2-1, M_1+M_2}^+ \quad (9)$$

This is in analogy to the work of Reference [41,42] where the ground ring structure of ghost number zero operators was identified in the BRST quantization. In Equation (8),  $S_k = S_k \left( \left\{ \frac{-i\sqrt{2}}{k!} \partial^k X(0) \right\} \right)$  is the Schur polynomial, which is a function of  $\{a_k\} = \{a_i : i \in \mathbb{Z}_k\}$  where  $X$  is the conformal matter and  $\phi$  is the scalar Liouville field. We will denote the rank  $(J - M)$  determinant as  $\Delta(J, M, -i\sqrt{2}X)$  below.

It was remarkable that a set of discrete ZNS  $G_{J,M}^+$  with Polyakov momenta can be constructed [37]

$$\begin{aligned} G_{J,M}^+ &= (J + M + 1)^{-1} \int \frac{dz}{2\pi i} \left[ \psi_{1,-1}^+(z) \psi_{J,M+1}^+(0) + \psi_{J,M+1}^+(z) \psi_{1,-1}^+(0) \right] \\ &\sim (J - M)! \Delta(J, M, -i\sqrt{2}X) \text{Exp} \left[ \sqrt{2}(iMX + (J - 1)\phi) \right] \\ &\quad + (-1)^{2J} \sum_{j=1}^{J-M} (J - M - 1)! \int \frac{dz}{2\pi i} \mathcal{D}(J, M, -i\sqrt{2}X(z), j) \\ &\quad \cdot \text{Exp} \left[ \sqrt{2}(i(M + 1)X(z) + (J - 1)\phi(z) - X(0)) \right] \end{aligned} \quad (10)$$

and moreover  $G_{J,M}^+$  were also shown [37,43] to carry the spacetime  $w_{\infty}$  symmetry [38–40] charges of 2D string theory [37,43]

$$\int \frac{dz}{2\pi i} G_{J_1, M_1}^+(z) G_{J_2, M_2}^+(0) = (J_2 M_1 - J_1 M_2) G_{J_1+J_2-1, M_1+M_2}^+(0). \quad (11)$$

In Equation (10) above,  $\mathcal{D}(J, M, -i\sqrt{2}X(z), j)$  is defined to be the same as  $\Delta(J, M, -i\sqrt{2}X(z))$  except that the  $j^{\text{th}}$  row is replaced by  $\{(-z)^{j-1-2J}, (-z)^{j-2J} \dots\}$ . The calculation above can be generalized to 2D superstring theory presented in [43].

Moreover, it was shown that [16] the high energy limit of the discrete ZNS  $G_{J,M}^+$  in 2D string theory constructed in Equation (10) approaches  $\psi_{J,M}^+$  in Equation (8) and, as a result, they form a high energy  $w_{\infty}$  symmetry of 2D string theory. This result seems to strongly suggest that the linear relations obtained from decoupling of ZNS in 26D string theory are indeed closely related to the hidden symmetry for the 26D string theory.

One can also use ZNS to calculate spacetime symmetries of string on compact backgrounds. The existence of soliton ZNS at some special moduli points was shown to be responsible for the enhanced Kac-Moody symmetry of closed bosonic string theory. As a simple example, for the case of 26D bosonic closed string compactified on a 2-dimensional torus  $T^2 \equiv \frac{R^2}{2\pi\Lambda^2}$ , it was pointed out that massless ZNS

(including soliton ZNS) form a representation of enhanced Kac-Moody  $SU(3)_R \otimes SU(3)_L$  symmetry at the moduli point [44]

$$R_1 = R_2 = \sqrt{2}, B = \frac{1}{2}, \vec{e}_1 = (\sqrt{2}, 0), \vec{e}_2 = \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{3}{2}}\right) \quad (12)$$

where  $\Lambda^2$  is a 2-dimensional lattice with a basis  $\left\{R_1 \frac{\vec{e}_1}{\sqrt{2}}, R_2 \frac{\vec{e}_2}{\sqrt{2}}\right\}$ , and  $B$  is the antisymmetric tensor  $B_{ij} = B\epsilon_{ij}$ . In the above calculation one obtained four moduli parameters  $R_1, R_2, B$  and  $\vec{e}_1 \cdot \vec{e}_2$  with  $|\vec{e}_i|^2 = 2$ . Moreover, an infinite number of massive soliton ZNS at arbitrary higher massive level of the spectrum were constructed in [44]. Presumably, these massive soliton ZNS were responsible for enhanced stringy symmetries of the bosonic string theory.

For the case of open string compactification, unlike the closed string case discussed above, it was pointed out that [45] the soliton ZNS only exist at massive levels. These Chan-Paton soliton ZNS correspond to the existence of the enhanced massive stringy symmetries with transformation parameters containing both Einstein and Yang-Mills indices for the case of Heterotic string [32]. On the other hand, in the T-dual picture, these symmetries exist only at some discrete values of compactified radii when  $N$   $D$ -branes are coincident [45].

### 3. Stringy Symmetries of Hard String Scattering Amplitudes

In this Section, we will review high energy, fixed angle calculations of HSS amplitudes. The high energy, fixed angle Ward identities derived from the decoupling of ZNS in the HSS limit, which combines the previous two key ideas of probing stringy symmetry, can be used to explicitly prove Gross's two conjectures [12–17].

An infinite number of linear relations among high energy stringy scattering amplitudes of different string states can be derived. Remarkably, the algebraical constraints from these linear relations were just good enough (no more and no less) to determine the ratios among HSS amplitudes of different string states at each fixed mass level.

The first simple example that has been shown was the ratios among HSS amplitudes at mass level  $M^2 = 4$  [12,14] (see the definition of polarizations  $e^T$  and  $e^L$  after Equation (20) below)

$$\mathcal{T}_{TTT} : \mathcal{T}_{LLT} : \mathcal{T}_{(LT)} : \mathcal{T}_{[LT]} = 8 : 1 : -1 : -1 \quad (13)$$

which corresponds to stringy symmetries in the  $\sigma$ -model calculation discussed from Equation (3) to Equation (7). Equation (13) is valid at any order in string perturbation theory since we expect the decoupling of ZNS to be valid for arbitrary string loop amplitudes [46].

One of the three methods to calculate Equation (13) is to use the method of decoupling of ZNS. We first note that there are four ZNS at mass level  $M^2 = 4$  in the old covariant first quantized string spectrum. For type I ZNS, there is one symmetric spin two tensor, one vector and one scalar ZNS. In addition, it is important to note that there exists one vector type II ZNS. The corresponding Ward identities for these four ZNS were calculated to be [34]



$$k_\mu \theta_{\nu\lambda} \mathcal{T}_\chi^{(\mu\nu\lambda)} + 2\theta_{\mu\nu} \mathcal{T}_\chi^{(\mu\nu)} = 0, \quad (14)$$

$$\left(\frac{5}{2}k_\mu k_\nu \theta'_\lambda + \eta_{\mu\nu} \theta'_\lambda\right) \mathcal{T}_\chi^{(\mu\nu\lambda)} + 9k_\mu \theta'_\nu \mathcal{T}_\chi^{(\mu\nu)} + 6\theta'_\mu \mathcal{T}_\chi^\mu = 0, \quad (15)$$

$$\left(\frac{1}{2}k_\mu k_\nu \theta_\lambda + 2\eta_{\mu\nu} \theta_\lambda\right) \mathcal{T}_\chi^{(\mu\nu\lambda)} + 9k_\mu \theta_\nu \mathcal{T}_\chi^{[\mu\nu]} - 6\theta_\mu \mathcal{T}_\chi^\mu = 0, \quad (16)$$

$$\left(\frac{17}{4}k_\mu k_\nu k_\lambda + \frac{9}{2}\eta_{\mu\nu} k_\lambda\right) \mathcal{T}_\chi^{(\mu\nu\lambda)} + (9\eta_{\mu\nu} + 21k_\mu k_\nu) \mathcal{T}_\chi^{(\mu\nu)} + 25k_\mu \mathcal{T}_\chi^\mu = 0, \quad (17)$$

where the polarization  $\theta_{\mu\nu}$  is a transverse and traceless tensor,  $\theta'_\lambda$  and  $\theta_\lambda$  are transverse vectors.  $\mathcal{T}'_\chi$ s in the above equations are the mass level  $M^2 = 4$ ,  $\chi$ -th order string-loop amplitudes. In the above equations,  $v_2(k_2)$  is chosen to be the physical vertex operators constructed from ZNS and  $k_\mu \equiv k_{2\mu}$ . Please note that Equation (16) is the “inter-particle Ward identity” corresponding to  $D_2$  vector ZNS in Equation (7) obtained by antisymmetrizing the terms containing  $\alpha_{-1}^\mu \alpha_{-2}^\nu$  [9]. We will use 1 and 2 to represent the incoming particles and 3 and 4 for the scattered particles. In the Ward identities calculated above, the vertices can be any string states and for simplicity we have omitted their tensor indices for the cases of excited string states for the vertices 1, 3 and 4.

In the HSS limit, one enjoys many simplifications in the calculation. First of all, all polarizations of the scattering amplitudes that are orthogonal to the scattering plane are of subleading order in energy, and one needs only consider polarizations on the scattering plane. Secondly, to the leading order in energy,  $e^P \simeq e^L$  in the HSS calculation. In the end of the calculation, one ends up with the simple linear equations for leading order HSS amplitudes [12,14]

$$\mathcal{T}_{LLT}^{5 \rightarrow 3} + \mathcal{T}_{(LT)}^3 = 0, \quad (18)$$

$$10\mathcal{T}_{LLT}^{5 \rightarrow 3} + \mathcal{T}_{TTT}^3 + 18\mathcal{T}_{(LT)}^3 = 0, \quad (19)$$

$$\mathcal{T}_{LLT}^{5 \rightarrow 3} + \mathcal{T}_{TTT}^3 + 9\mathcal{T}_{[LT]}^3 = 0 \quad (20)$$

where  $e^P = \frac{1}{M_2}(E_2, k_2, 0) = \frac{k_2}{M_2}$  the momentum polarization,  $e^L = \frac{1}{M_2}(k_2, E_2, 0)$  the longitudinal polarization and  $e^T = (0, 0, 1)$  the transverse polarization are three polarizations on the scattering plane. In Equations (18)–(20), each scattering amplitude has been assigned a relative energy power. For each longitudinal  $L$  component, the energy order is  $E^2$  while for each transverse  $T$  component, the energy order is  $E$ . This is due to the definitions of  $e_L$  and  $e_T$  above, where  $e_L$  got one energy power more than that of  $e_T$ . By Equation (19), the naive leading order  $E^5$  term of the energy expansion for  $\mathcal{T}_{LLT}$  is forced to be zero. As a result, the real leading order term is  $E^3$ . A similar rule also applies to  $\mathcal{T}_{LLT}$  in Equations (18) and (20). The solution of these three linear relations gives Equation (13). Equation (13) gives the first evidence of Gross conjecture [6,7] on HSS amplitudes.

To confirm the validity of the calculation of decoupling of ZNS above, a sample calculation of HSS amplitudes for mass level  $M^2 = 4$  [14] justified the ratios calculated in Equation (13). Since the ratios in Equation (13) are independent of the choices of string states at the vertices 1, 3 and 4, for simplicity, we will choose them to be all tachyons. At the string-tree level  $\chi = 1$ , with a tensor string state at vertex 2 and three tachyons at the vertices 1, 3 and 4, all HSS amplitudes of mass level  $M_2^2 = 4$  can be calculated to be ( $s - t$  channel)

$$\mathcal{T}_{TTT} = -8E^9 \mathcal{T}(3) \sin^3 \phi_{CM} \left[ 1 + \frac{3}{E^2} + \frac{5}{4E^4} - \frac{5}{4E^6} + O\left(\frac{1}{E^8}\right) \right], \quad (21)$$

$$\mathcal{T}_{LT} = -E^9 \mathcal{T}(3) \left[ \sin^3 \phi_{CM} + (6 \sin \phi_{CM} \cos^2 \phi_{CM}) \frac{1}{E^2} - \sin \phi_{CM} \left( \frac{11}{2} \sin^2 \phi_{CM} - 6 \right) \frac{1}{E^4} + O\left(\frac{1}{E^6}\right) \right], \quad (22)$$

$$\mathcal{T}_{[LT]} = E^9 \mathcal{T}(3) \left[ \sin^3 \phi_{CM} - (2 \sin \phi_{CM} \cos^2 \phi_{CM}) \frac{1}{E^2} + \sin \phi_{CM} \left( \frac{3}{2} \sin^2 \phi_{CM} - 2 \right) \frac{1}{E^4} + O\left(\frac{1}{E^6}\right) \right], \quad (23)$$

$$\mathcal{T}_{(LT)} = E^9 \mathcal{T}(3) \left[ \sin^3 \phi_{CM} + \sin \phi_{CM} \left( \frac{3}{2} - 10 \cos \phi_{CM} - \frac{3}{2} \cos^2 \phi_{CM} \right) \frac{1}{E^2} - \sin \phi_{CM} \left( \frac{1}{4} + 10 \cos \phi_{CM} + \frac{3}{4} \cos^2 \phi_{CM} \right) \frac{1}{E^4} + O\left(\frac{1}{E^6}\right) \right], \quad (24)$$

where

$$\mathcal{T}(N) = \sqrt{\pi} (-1)^{N-1} 2^{-N} E^{-1-2N} \left( \sin \frac{\phi_{CM}}{2} \right)^{-3} \left( \cos \frac{\phi_{CM}}{2} \right)^{5-2N} \cdot \exp \left( -\frac{s \ln s + t \ln t - (s+t) \ln(s+t)}{2} \right), \quad (25)$$

is the high energy limit of  $\frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)}$  with  $s+t+u = 2N-8$ . We thus have justified the ratios calculated in Equation (13) with  $\mathcal{T}_{TTT}^3 = -8E^9 \mathcal{T}(3) \sin^3 \phi_{CM}$ .

The calculations based on ZNS thus relate [35] gauge transformation of WSFT to high energy string symmetries of Gross. However, in the sample calculation of [8], two of the four high energy amplitudes in Equation (13) were missing, and as a result the decoupling of ZNS or unitarity was violated there. This is due to the unawareness of the importance of ZNS in the saddle-point calculation of [4–8].

The calculations for  $M^2 = 4$  above can be generalized to  $M^2 = 6$  [14]. To the leading order in the hard scattering limit, one ended up with 8 constraint equations and 9 HSS amplitudes. A calculation showed that [14]

$$\mathcal{T}_{TTTT}^4 : \mathcal{T}_{TTLL}^4 : \mathcal{T}_{LLLL}^4 : \mathcal{T}_{TTT}^4 : \mathcal{T}_{LLL}^4 : \tilde{\mathcal{T}}_{LT,T}^4 : \tilde{\mathcal{T}}_{LP,P}^4 : \mathcal{T}_{LL}^4 : \tilde{\mathcal{T}}_{LL}^4 = 16 : \frac{4}{3} : \frac{1}{3} : -\frac{4\sqrt{6}}{9} : -\frac{\sqrt{6}}{9} : -\frac{2\sqrt{6}}{3} : 0 : \frac{2}{3} : 0. \quad (26)$$

The results for mass level  $M^2 = 8$  [16] can also be obtained with more lengthy calculation.

Remarkably, the above results for up to mass level  $M^2 = 8$  can be generalized to arbitrary higher mass levels. From the calculations of Equation (18) to Equation (20), one first observes that only string states of the following form [16,17]

$$|N, 2m, q\rangle \equiv (\alpha_{-1}^T)^{N-2m-2q} (\alpha_{-1}^L)^{2m} (\alpha_{-2}^L)^q |0, k\rangle \quad (27)$$

are of leading order in energy in the HSS limit. The request of only even power  $2m$  in  $\alpha_{-1}^L$  is the result from the observation that the naive energy order of the HSS amplitudes will drop by even number of energy power as can be seen in Equations (18)–(20). It can be shown that the HSS amplitudes for states



with  $(\alpha_{-1}^L)^{2m+1}$  are of subleading order in energy and can be ignored at the beginning of the calculation. The Ward identities could be simplified a lot if we only consider the high energy states in Equation (27) in the HSS limit. First, consider the decoupling of type I high energy ZNS

$$L_{-1}|N-1, 2m-1, q\rangle \simeq M|N, 2m, q\rangle + (2m-1)|N, 2m-2, q+1\rangle \quad (28)$$

where the terms that are not in the form of Equation (27) can be omitted. This implies that

$$\mathcal{T}^{(N, 2m, q)} = -\frac{2m-1}{M} \mathcal{T}^{(N, 2m-2, q+1)}. \quad (29)$$

Using this recurrent relation, one can easily obtain

$$\mathcal{T}^{(N, 2m, q)} = \frac{(2m-1)!!}{(-M)^m} \mathcal{T}^{(N, 0, m+q)} \quad (30)$$

where the double factorial is defined to be  $(2m-1)!! = \frac{(2m)!}{2^m m!}$ .

Next, we consider the decoupling of type II high energy ZNS

$$L_{-2}|N-2, 0, q\rangle \simeq \frac{1}{2}|N, 0, q\rangle + M|N, 0, q+1\rangle. \quad (31)$$

Similarly, the irrelevant terms of the subleading order in energy can be omitted, and it implies that

$$\mathcal{T}^{(N, 0, q+1)} = -\frac{1}{2M} \mathcal{T}^{(N, 0, q)}, \quad (32)$$

which leads to the relation

$$\mathcal{T}^{(N, 0, q)} = \frac{1}{(-2M)^q} \mathcal{T}^{(N, 0, 0)}. \quad (33)$$

Finally, the ratios for arbitrary mass levels  $M^2 = 2(N-1)$  is an immediate deduction of the above two equations, Equations (30) and (33), [16,17]

$$\frac{\mathcal{T}^{(N, 2m, q)}}{\mathcal{T}^{(N, 0, 0)}} = \left(-\frac{1}{M}\right)^{2m+q} \left(\frac{1}{2}\right)^{m+q} (2m-1)!!. \quad (34)$$

To justify the ratios calculated in Equation (34) by the method of decoupling of high energy ZNS, it was shown that exactly the same results can also be consistently obtained by two other calculations, the Virasoro constraint calculation and the saddle-point calculation. Here we review the saddle-point calculation. Since the result in Equation (34) is valid for all string loop order, we need only do saddle-point calculation of the string tree level scattering amplitudes. Without loss of generality, we will choose the vertices 1, 3 and 4 to be tachyons, and the vertex 2 to be in the form of Equation (27). The  $t-u$  channel contribution to the stringy amplitude at tree level is

$$\begin{aligned} \mathcal{T}^{(N, 2m, q)} = & \int_1^\infty dx x^{(1,2)} (1-x)^{(2,3)} \left[ \frac{e^T \cdot k_1}{x} - \frac{e^T \cdot k_3}{1-x} \right]^{N-2m-2q} \\ & \cdot \left[ \frac{e^P \cdot k_1}{x} - \frac{e^P \cdot k_3}{1-x} \right]^{2m} \left[ -\frac{e^P \cdot k_1}{x^2} - \frac{e^P \cdot k_3}{(1-x)^2} \right]^q \end{aligned} \quad (35)$$

where we have defined the notation  $(1, 2) = k_1 \cdot k_2$  etc.

In the saddle-point calculation, we transform the above scattering amplitude into the following form [16,17]

$$\mathcal{T}^{(N,2m,q)}(K) = \int_1^\infty dx u(x) e^{-Kf(x)} \quad (36)$$

where various quantities above are defined to be

$$K \equiv -(1,2) \rightarrow 2E^2, \quad (37)$$

$$\tau \equiv -\frac{(2,3)}{(1,2)} \rightarrow \sin^2 \frac{\phi}{2}, \quad (38)$$

$$f(x) \equiv \ln x - \tau \ln(1-x), \quad (39)$$

$$u(x) \equiv \left[ \frac{(1,2)}{M} \right]^{2m+q} (1-x)^{-N+2m+2q} (f')^{2m} (f'')^q (-e^T \cdot k_3)^{N-2m-2q}. \quad (40)$$

The saddle-point for the integration,  $x = x_0$ , is defined to be

$$f'(x_0) = 0, \quad (41)$$

and we have

$$x_0 = \frac{1}{1-\tau}, \quad f''(x_0) = (1-\tau)^3 \tau^{-1}. \quad (42)$$

It is very crucial to note that

$$u(x_0) = u'(x_0) = \dots = u^{(2m-1)}(x_0) = 0, \quad (43)$$

and the leading term can be calculated to be

$$u^{(2m)}(x_0) = \left[ \frac{(1,2)}{M} \right]^{2m+q} (1-x_0)^{-N+2m+2q} (2m)! (f_0'')^{2m+q} (-e^T \cdot k_3)^{N-2m-2q}. \quad (44)$$

One can now calculate the Gaussian integral associated with the four-point HSS amplitudes

$$\begin{aligned} & \int_1^\infty dx u(x) e^{-Kf(x)} \\ &= \sqrt{\frac{2\pi}{Kf_0''}} e^{-Kf_0} \left[ \frac{u_0^{(2m)}}{2^m m! (f_0'')^m K^m} + O\left(\frac{1}{K^{m+1}}\right) \right] \\ &= \sqrt{\frac{2\pi}{Kf_0''}} e^{-Kf_0} \left[ (-1)^{N-q} \frac{2^{N-2m-q} (2m)!}{m! M^{2m+q}} \tau^{-\frac{N}{2}} (1-\tau)^{\frac{3N}{2}} E^N + O(E^{N-2}) \right]. \end{aligned} \quad (45)$$

This result explicitly shows that with one higher spin tensor and three tachyons, the four-point HSS amplitudes have the same dependence on the scattering angle at each mass level  $N$ . One can then extract the ratios

$$\begin{aligned} \lim_{E \rightarrow \infty} \frac{\mathcal{T}^{(N,2m,q)}}{\mathcal{T}^{(N,0,0)}} &= \frac{(-1)^q (2m)!}{m! (2M)^{2m+q}} \\ &= \left( -\frac{2m-1}{M} \right) \dots \left( -\frac{3}{M} \right) \left( -\frac{1}{M} \right) \left( -\frac{1}{2M} \right)^{m+q}, \end{aligned} \quad (46)$$

which is consistent with the calculation of decoupling of high energy ZNS obtained in Equation (34).

We conclude that at each fixed mass level there is only one independent HSS amplitude. As a result, one can then express the general four-point HSS amplitude for four arbitrary string states in terms of four tachyon scattering amplitude. This completes the proof [12–17] of Gross's two conjectures on high energy symmetry of string theory stated above.

All the above calculations can be generalized to the case of the NS-sector of hard superstring scattering amplitudes. However, it was interesting to find that [47] there were new HSS amplitudes for the superstring case. The worldsheet fermion exchange in the correlation functions induces new contributions to the high energy scattering amplitudes of string states with polarizations orthogonal to the scattering plane. This is presumably related to the high energy massive spacetime fermionic scattering amplitudes in the R-sector of superstring theory.

Incidentally, it was important to discover [12–15] that the result of saddle-point calculation in Refs. [4–8] was inconsistent with the high energy ZNS calculation in Refs. [12–15]. A simple example is that two of the four amplitudes in Equation (13) were missed as was pointed out in [12,14]. The *corrected saddle-point calculation* was given in [15] where the missing terms in Refs. [4–8] were remedied to recover the stringy Ward identities.

Indeed, it was found [15] that saddle point calculation in [4–8] is only valid for the four tachyon amplitude. In general, the results calculated in [4–8] gave the right energy exponent in the scattering amplitudes, but not the energy power factors in front of the exponential for the cases of the massive *excited string states*. These energy power factors are subleading terms ignored in [4–8] but they are crucial if one wants to get the linear relations among HSS amplitudes conjectured by Gross.

Interestingly, the inconsistency of the saddle point calculation discussed above for the excited string states was also pointed out by the authors of [48]. The source of disagreement in their group theoretic approach of stringy symmetries stems from the proper choice of local coordinates for the worldsheet saddle points to describe the behavior of the excited string states at high energy limit. It seems that both the ZNS calculation and the calculation based on group theoretic approach agree with tachyon amplitudes obtained in [4–8] (ignore the possible phase factors in the amplitudes to be discussed in the next section), but disagree with amplitudes for other excited string states.

#### 4. Hard Closed String Scatterings, KLT and Hard String BCJ Relations

The next interesting issues are the calculation of *closed* string scattering amplitudes and their symmetries in the HSS limit [19]. Historically, the open string four tachyon amplitude in the HSS limit was first calculated in the original paper of Veneziano in 1968. On the other hand, the  $\mathcal{N}$ -loop closed HSS amplitudes were calculated by the saddle-point method in 1988 [4,5]. Both open and closed HSS amplitudes exhibit the very soft exponential fall-off behaviors in contrast to the power law behavior of the scattering amplitudes of quantum field theory.

However, an inconsistency would arise if one simply plugs, for example, the tree level four tachyon open and closed string HSS amplitudes calculated by these authors into the KLT relation [18]

$$A_{\text{closed}}^{(4)}(s, t, u) = \sin(\pi k_2 \cdot k_3) A_{\text{open}}^{(4)}(s, t) \bar{A}_{\text{open}}^{(4)}(t, u) \quad (47)$$

which is valid for *all* kinematic regimes and for *all* string states. This inconsistency is due to the phase factor  $\sin(\pi k_2 \cdot k_3)$  in the above equation which was missing in the closed string saddle-point calculation in [4,5]. One simple clue to see the origin of this inconsistency is to note that the saddle-point  $x_0 = \frac{1}{1-\tau} \prec 1$  identified for the open string calculation in Equation (42) is in the regime  $[1, \infty)$ . So only saddle point calculation for  $\bar{A}_{\text{open}}^{(4)}(t, u)$  is reliable, but not that of  $A_{\text{open}}^{(4)}(s, t)$  and neither that of closed string amplitude  $A_{\text{closed}}^{(4)}(s, t, u)$  [19] by the KLT relation.

Instead of using saddle-point calculation for the closed HSS amplitudes, the above considerations led the authors of [19] to study the relationship between  $A_{\text{open}}^{(4)}(s, t)$  and  $\bar{A}_{\text{open}}^{(4)}(t, u)$  for arbitrary string states in the HSS limit. With the help of the *infinite linear relations in Equation (34)*, one needs only calculate relationship between  $s - t$  and  $t - u$  channel HSS amplitudes for the leading trajectory string states which were much easier to calculate. They ended up with the following result in the HSS limit (2006) [19]

$$A_{\text{open}}^{(4)}(s, t) = \frac{\sin(\pi k_2 \cdot k_4)}{\sin(\pi k_1 k_2)} \bar{A}_{\text{open}}^{(4)}(t, u) \quad (\text{HSS}), \quad (48)$$

which is valid for four arbitrary string states. It is now clear that due to the phase factor in the above equation, the saddle-point calculation of  $A_{\text{open}}^{(4)}(s, t)$  is not reliable, neither for the closed one  $A_{\text{closed}}^{(4)}(s, t, u)$  in view of the KLT relation in Equation (47). One can now use the reliable saddle-point calculation of  $\bar{A}_{\text{open}}^{(4)}(t, u)$

$$A_{\text{open}}^{(4-\text{tachyon})}(t, u) \simeq (stu)^{-\frac{3}{2}} \exp\left(-\frac{s \ln s + t \ln t + u \ln u}{2}\right), \quad (49)$$

and Equation (48) to calculate  $A_{\text{open}}^{(4)}(s, t)$  in the HSS limit. The consistent closed string four-tachyon HSS amplitudes can then be calculated by using the KLT relation in Equation (47) to be [19]

$$A_{\text{closed}}^{(4-\text{tachyon})}(s, t, u) \simeq \frac{\sin(\pi t/2) \sin(\pi u/2)}{\sin(\pi s/2)} (stu)^{-3} \exp\left(-\frac{s \ln s + t \ln t + u \ln u}{4}\right). \quad (50)$$

The exponential factor in Equation (49) was first discussed by Veneziano [49]. The result for the high energy closed string four-tachyon amplitude in Equation (50) differs from the one calculated in the literature [4,5] by an oscillating factor  $\frac{\sin(\pi t/2) \sin(\pi u/2)}{\sin(\pi s/2)}$ . It is important to note that the results of Equations (50), (49) and Equation (48) are consistent with the KLT formula, while the calculation in [4,5] is NOT.

Indeed, one might try to use the saddle-point method to calculate the high energy closed string scattering amplitude. The closed string four-tachyon scattering amplitude is

$$\begin{aligned} A_{\text{closed}}^{(4-\text{tachyon})}(s, t, u) &= \int dx dy \exp\left(\frac{k_1 \cdot k_2}{2} \ln|z| + \frac{k_2 \cdot k_3}{2} \ln|1-z|\right) \\ &\equiv \int dx dy (x^2 + y^2)^{-2} [(1-x)^2 + y^2]^{-2} \exp[-Kf(x, y)] \end{aligned} \quad (51)$$

where  $K = s/8$  and  $f(x, y) = \ln(x^2 + y^2) - \tau \ln[(1-x)^2 + y^2]$  with  $\tau = -t/s$ . The “saddle-point” of  $f(x, y)$  can then be calculated to be

$$\nabla f(x, y) \big|_{x_0=\frac{1}{1-\tau}, y_0=0} = 0. \quad (52)$$

The HSS limit of the closed string four-tachyon scattering amplitude is calculated to be

$$\begin{aligned} A_{\text{closed}}^{(4-\text{tachyon})}(s, t, u) &\simeq \frac{2\pi}{K \sqrt{\det \frac{\partial^2 f(x_0, y_0)}{\partial x \partial y}}} \exp[-Kf(x_0, y_0)] \\ &\simeq (stu)^{-3} \exp\left(-\frac{s \ln s + t \ln t + u \ln u}{4}\right), \end{aligned} \quad (53)$$

which is consistent with the result calculated in the literature [4,5], but is different from the one in Equation (50). However, one notes that

$$\frac{\partial^2 f(x_0, y_0)}{\partial x^2} = \frac{2(1-\tau)^3}{\tau} = -\frac{\partial^2 f(x_0, y_0)}{\partial y^2}, \quad \frac{\partial^2 f(x_0, y_0)}{\partial x \partial y} = 0, \quad (54)$$

which means that  $(x_0, y_0)$  is NOT the local minimum of  $f(x, y)$ , and one should not trust this saddle-point calculation. There was other evidence pointed out by authors of [19] to support this conclusion. Finally, the ratios of closed HSS amplitudes turned out to be the tensor products of two open string ratios

$$\frac{T(N; 2m, 2m'; q, q')}{T(N; 0, 0; 0, 0)} = \left(-\frac{1}{M_2}\right)^{2(m+m') + q + q'} \left(\frac{1}{2}\right)^{m+m' + q + q'} (2m-1)!!(2m'-1)!! \quad (55)$$

The relationship between  $s-t$  and  $t-u$  channels HSS amplitudes in Equation (48) was later argued to be valid for *all* kinematic regime (2009) [50]

$$A_{\text{open}}^{(4)}(s, t) = \frac{\sin(\pi k_2 \cdot k_4)}{\sin(\pi k_1 k_2)} \bar{A}_{\text{open}}^{(4)}(t, u), \quad (56)$$

the so-called “string BCJ relation”, based on the monodromy of integration in the string amplitudes calculation in 2009 [50]. An explicit proof of Equation (48) for arbitrary four string states and all kinematic regimes was given recently in [23,27].

The motivation for the author in [50] to calculate Equation (56) was different from the result in Equation (48) which was motivated by the calculation of hard closed string scattering amplitudes, while the motivation in [50] was based on the field theory BCJ relation [20] for the scattering amplitudes  $A$  in Yang-Mills theory, which was first pointed out and calculated in 2008 [20] to be

$$sA(k_1, k_2, k_3, k_4) - uA(k_1, k_4, k_2, k_3) = 0. \quad (57)$$

Please note that for the supersymmetric case, there is no tachyon and the low energy massless limit of Equation (56) reproduces Equation (57).

Recently, the stringy generalization of the massless field theory BCJ to the higher spin string states was calculated to be [23,27]

$$\frac{A_{st}^{(p,r,q)}}{A_{tu}^{(p,r,q)}} = (-1)^N \frac{B\left(-M_1 M_2 + 1, \frac{M_1 M_2}{2}\right)}{B\left(\frac{M_1 M_2}{2}, \frac{M_1 M_2}{2}\right)} \simeq \frac{\sin \pi(k_2 \cdot k_4)}{\sin \pi(k_1 \cdot k_2)} \quad (58)$$

by taking the *nonrelativistic* limit  $|\vec{k}_2| \ll M_S$  of Equation (48). In Equation (58),  $B$  was the beta function, and  $k_1, k_3$  and  $k_4$  were taken to be tachyons, and  $k_2$  was the following tensor string state

$$V_2 = (i\partial X^T)^p (i\partial X^L)^r (i\partial X^P)^q e^{ik_2 X} \quad (59)$$

where

$$N = p + r + q, \quad M_2^2 = 2(N-1), \quad N \geq 2. \quad (60)$$

Moreover, many *reduced recurrence relations in the nonrelativistic string scattering (NSS) limit* can be derived [23,27]. These recurrence relations relates nonrelativistic string scattering amplitudes of different higher spin particles at both  $s-t$  and  $t-u$  channels for each fixed mass level.

The generalization of the four point function relation in Equation (48) to higher point string amplitudes can be found in [50]. It is interesting to see that historically the four point (high energy) “string BCJ relations” Equation (48) [19] were discovered even earlier than the field theory BCJ relations in Equation (57)! Ref. [20] (and string BCJ relations in Equation (56) [50]).

The ratios calculated in Equation (55) persist for the case of closed string D-particle scatterings in the HSS limit. For the simple case of  $m = 0 = m'$ , the ratios were first calculated to be  $\left(-\frac{1}{2M}\right)^{q+q'}$  [51]. The complete ratios were then calculated through the open string HSS ratios Equation (34), and were found to be *factorized* [52]

$$\frac{T_{SD}^{(N;2m,2m';q,q')}}{T_{SD}^{(N;0,0;0,0)}} = \left(-\frac{1}{M_2}\right)^{2(m+m')+q+q'} \left(\frac{1}{2}\right)^{m+m'+q+q'} (2m-1)!!(2m'-1)!! \quad (61)$$

It has been known for a long time that closed strings scattering amplitudes can be factorized into two open strings scattering amplitudes due to the existence of the KLT formula [18]. On the contrary, there is no physical picture for the amplitude of open strings scattered from a D-particle and thus no factorization for the amplitude of closed strings scattered from a D-particle into two channels of open strings scattered from a D-particle, and hence no KLT-like formula in that case. Therefore, the factorized ratios in HSS regime calculated above came as a surprise. However, these ratios are consistent with the decoupling of high energy ZNS calculated previously in [12–17,19,47,53]. It will be helpful to clarify this paradox if one can calculate the complete HSS amplitudes and see how the *non-factorized amplitudes* can give the result of factorized ratios.

On the other hand, it was shown that the HSS amplitudes of closed string scattered from D24-brane, or D-domain-wall, behave as *power-law with Regge-pole structure* [54] instead of the exponential fall-off behavior. This is to be compared with the well-known power law behavior of the D-instanton scatterings.

This discovery makes D-domain-wall scatterings an unique example of a hybrid of string and field theory scatterings. Moreover, it was discovered that [54] the usual linear relations of HSS amplitudes of Equation (61) breaks down for the D-domain-wall scatterings. This gives a strong evidence that the existence of the infinite linear relations, or stringy symmetries, of HSS amplitudes is responsible for the softer, exponential fall-off behavior of HSS scatterings than the power-law field theory scatterings.

Being a consistent theory of quantum gravity, string theory is remarkable for its soft ultraviolet structure. This is mainly due to *three* closely related fundamental characteristics of HSS amplitudes. The first is the exponential fall-off behavior of the HSS amplitudes in contrast to the power-law field theory scatterings amplitudes. The second is the existence of infinite Regge poles in the string scattering amplitudes. The existence of infinite linear relations discussed in Section 3 of this review is the *third* fundamental characteristics of HSS amplitudes.

It will be important to study more string scatterings to justify the above three fundamental characteristics in the HSS limit. The scatterings of massless states from Orientifold planes have been studied in the literature by using the boundary states formalism [55–58], and on the worldsheet of real projected plane  $RP_2$  [59]. Many speculations have been made about the scatterings of *massive* string states from the O-domain-wall scatterings. It is one of the purposes of reference [60] to clarify these speculations and to discuss their relations with the above three fundamental characteristics of HSS scatterings. In [60], the authors studied closed strings scattered from O-planes. In particular they calculated massive closed string states at arbitrary mass levels scattered from Orientifold planes in the HSS limit. Except for O-domain-wall, as we explained above, one expects the infinite linear relations HSS for the generic Op-planes with  $p \geq 0$ . For simplicity, one considered only the case of O-particle HSS [60], and confirmed



that there exist only  $t$ -channel closed string Regge poles in the form factor of the O-particle scatterings amplitudes as expected.

Like the well-known D-instanton scatterings, one found that the amplitudes of O-domain-wall scatterings behave like field theory scatterings, namely *UV power-law without Regge pole*. In addition, only finite number of  $t$ -channel closed string poles in the O-domain-wall scatterings was found with the masses of the poles being bounded by the masses of the external legs [60]. One thus confirmed that all massive closed string states do couple to the O-domain-wall as was conjectured previously [59,61]. This is also consistent with the boundary state descriptions of O-planes.

For both cases of O-particle and O-domain-wall scatterings, one concluded that no  $s$ -channel open string Regge poles existed since O-planes were known to be not dynamical. Therefore, because the UV behavior of its scatterings is power-law instead of exponential fall-off, the usual claim that there is a thickness of order  $\sqrt{\alpha'}$  for the O-domain-wall is misleading.

We summarize the pole structures of closed string states scattered from various D-branes and O-planes in Table 1. Since O-planes are not dynamical, the  $s$ -channel open string Regge poles are not allowed. Furthermore, for both cases of Domain-wall scatterings, the  $t$ -channel closed string Regge poles are not allowed because there is only single kinematic variable instead of two as in the usual cases.

**Table 1.** Poles structure in various branes.

Diminsion	$p = -1$	$1 \leq p \leq 23$	$p = 24$
Dp-branes	X	C + O	O
Op-planes	X	C	X

In this table, “C” and “O” represent infinite Closed string Regge poles and Open string Regge poles respectively. “X” means that there are no infinite Regge poles.

Following the suggestion of Mende [62], the authors In [2,3] calculated high energy massive scattering amplitudes of bosonic string with some coordinates compacted on the torus [2,3]. Infinite linear relations among high energy scattering amplitudes of different string states in the Hard scattering limit were obtained. Furthermore, all possible power-law and exponential fall-off regimes of high energy compacted bosonic string scatterings were analyzed and classified by comparing the scatterings with their 26D noncompactified counterparts.

Interestingly, it was discovered in [3] that there exist a power-law regime at fixed angle and an exponential fall-off regime at small angle for high energy compacted open string scatterings [3]. The linear relations break down as expected in all power-law regimes. The analysis can be generaliiized to the high energy scatterings of the compactified closed string, which corrects and extends the results in [2].

At this point, one may ask an important question for the results of Equations (13), (26) and (34) above, namely, is there any group theoretical structure of the ratios of these scattering amplitudes? There is indeed a simple analogy from the ratios of the nucleon-nucleon scattering processes in particle physics,

$$\begin{aligned}
 (a) \quad & p + p \rightarrow d + \pi^+, \\
 (b) \quad & p + n \rightarrow d + \pi^0, \\
 (c) \quad & n + n \rightarrow d + \pi^-,
 \end{aligned} \tag{62}$$

which can be calculated to be (ignore the mass difference between proton and neutron)

$$T_a : T_b : T_c = 1 : \frac{1}{\sqrt{2}} : 1 \tag{63}$$

from  $SU(2)$  isospin symmetry. Is there any symmetry structure which can be used to calculate ratios in Equations (13), (26), and (34)? It turned out that part of the answer can be addressed by studying another high energy regime of string scattering amplitudes, i.e., the fixed momentum transfer or Regge regime (RR) which will be the main subject of the next section [21,22,63–70].

## 5. Stringy Symmetries of Regge String Scattering Amplitudes

In addition to the hard limit, i.e., the Gross regime (GR), which we have reviewed in the previous sections, another important regime is the high energy limit with a small scattered angle, i.e., the Regge regime (RR). In this section, we are going to review the Regge string scattering (RSS) amplitudes and their relations to the HSS amplitudes. We will show that the number of RSS amplitudes is much more numerous than that of HSS amplitudes. For example, there are only 4 HSS amplitudes as we discussed in Section 3, while 22 RSS amplitudes at mass level  $M^2 = 4$  [21]. This is one of the reasons why decoupling of ZNS in the RR, in contrast to the GR, is not good enough to solve RSS amplitudes in terms of one single amplitude at each mass level.

The RR is defined as

$$s \rightarrow \infty, \sqrt{-t} = \text{fixed (but } \sqrt{-t} \neq \infty). \quad (64)$$

The relevant kinematics are

$$e^P \cdot k_1 = -\frac{1}{M_2} \left( \sqrt{p^2 + M_1^2} \sqrt{p^2 + M_2^2} + p^2 \right) \simeq -\frac{s}{2M_2}, \quad (65a)$$

$$e^L \cdot k_1 = -\frac{p}{M_2} \left( \sqrt{p^2 + M_1^2} + \sqrt{p^2 + M_2^2} \right) \simeq -\frac{s}{2M_2}, \quad (65b)$$

$$e^T \cdot k_1 = 0 \quad (65c)$$

and

$$e^P \cdot k_3 = \frac{1}{M_2} \left( \sqrt{q^2 + M_3^2} \sqrt{p^2 + M_2^2} - pq \cos \theta \right) \simeq -\frac{\tilde{t}}{2M_2} \equiv -\frac{t - M_2^2 - M_3^2}{2M_2}, \quad (66a)$$

$$e^L \cdot k_3 = \frac{1}{M_2} \left( p \sqrt{q^2 + M_3^2} - q \sqrt{p^2 + M_2^2} \cos \theta \right) \simeq -\frac{\tilde{t}'}{2M_2} \equiv -\frac{t + M_2^2 - M_3^2}{2M_2}, \quad (66b)$$

$$e^T \cdot k_3 = -q \sin \phi \simeq -\sqrt{-t}. \quad (66c)$$

Please note that in contrast to the identification  $e^P \simeq e^L$  in the HSS limit, we *cannot* identify  $e^P$  with  $e^L$  in the RSS limit. However, to compare with the HSS in the GR, we will consider the “relevant” RSS amplitudes which contain polarizations ( $e^T, e^L$ ) only.

For illustration and to identify the ratios in Equation (13) from RSS amplitudes, we will first show an example at mass level  $M^2 = 4$  in the RR. In this case, there are eight high energy string amplitudes with polarizations ( $e^T, e^L$ ) in the RR,

$$\begin{aligned} & \alpha_{-1}^T \alpha_{-1}^T \alpha_{-1}^T |0\rangle, \alpha_{-1}^L \alpha_{-1}^T \alpha_{-1}^T |0\rangle, \alpha_{-1}^L \alpha_{-1}^L \alpha_{-1}^T |0\rangle, \alpha_{-1}^L \alpha_{-1}^L \alpha_{-1}^L |0\rangle, \\ & \alpha_{-1}^T \alpha_{-2}^T |0\rangle, \alpha_{-1}^T \alpha_{-2}^L |0\rangle, \alpha_{-1}^L \alpha_{-2}^T |0\rangle, \alpha_{-1}^L \alpha_{-2}^L |0\rangle. \end{aligned} \quad (67)$$

Among them only four of the above amplitudes are relevant here and can be calculated to be [21]

$$\begin{aligned}
 A^{TTT} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left( \frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right)^3 \\
 &\simeq -i \left( \sqrt{-t} \right)^3 \frac{\Gamma\left(-\frac{s}{2}-1\right) \Gamma\left(-\frac{t}{2}-1\right)}{\Gamma\left(\frac{u}{2}+3\right)} \cdot \left( -\frac{1}{8}s^3 + \frac{1}{2}s \right),
 \end{aligned} \tag{68}$$

$$\begin{aligned}
 A^{LLT} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left( \frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right) \left( \frac{ie^L \cdot k_1}{x} - \frac{ie^L \cdot k_3}{1-x} \right)^2 \\
 &\simeq -i \left( \sqrt{-t} \right) \left( -\frac{1}{2M_2} \right)^2 \frac{\Gamma\left(-\frac{s}{2}-1\right) \Gamma\left(-\frac{t}{2}-1\right)}{\Gamma\left(\frac{u}{2}+3\right)} \\
 &\quad \cdot \left[ \left( \frac{1}{4}t - \frac{9}{2} \right) s^3 + \left( \frac{1}{4}t^2 + \frac{7}{2}t \right) s^2 + \frac{(t+6)^2}{2}s \right],
 \end{aligned} \tag{69}$$

$$\begin{aligned}
 A^{TL} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left( \frac{ie^T \cdot k_1}{x} - \frac{ie^T \cdot k_3}{1-x} \right) \left[ \frac{e^L \cdot k_1}{x^2} + \frac{e^L \cdot k_3}{(1-x)^2} \right] \\
 &\simeq i \left( \sqrt{-t} \right) \left( -\frac{1}{2M_2} \right) \frac{\Gamma\left(-\frac{s}{2}-1\right) \Gamma\left(-\frac{t}{2}-1\right)}{\Gamma\left(\frac{u}{2}+3\right)} \\
 &\quad \left[ -\left( \frac{1}{8}t + \frac{3}{4} \right) s^3 - \frac{1}{8} (t^2 - 2t) s^2 - \left( \frac{1}{4}t^2 - t - 3 \right) s \right],
 \end{aligned} \tag{70}$$

and

$$\begin{aligned}
 A^{LT} &= \int_0^1 dx \cdot x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left( \frac{ie^L \cdot k_1}{x} - \frac{ie^L \cdot k_3}{1-x} \right) \left[ \frac{e^T \cdot k_1}{x^2} + \frac{e^T \cdot k_3}{(1-x)^2} \right] \\
 &\simeq i \left( \sqrt{-t} \right) \left( -\frac{1}{2M_2} \right) \frac{\Gamma\left(-\frac{s}{2}-1\right) \Gamma\left(-\frac{t}{2}-1\right)}{\Gamma\left(\frac{u}{2}+3\right)} \cdot \left[ \frac{3}{4}s^3 - \frac{t}{4}s^2 - \left( \frac{t}{2} + 3 \right) s \right].
 \end{aligned} \tag{71}$$

where the kinematic variables  $(s, t)$  were used for convenient instead of  $(E, \theta)$  used in the GR. The conversion between the kinematic variables is straightforward. From the above expressions, it is easy to show that all the string amplitudes in the RR have the same leading order ( $\sim s^3$ ) in energy. On the other hand, we note that, some terms. E.g.,  $\sqrt{-t}t^2s^2$  in  $A^{LLT}$  and  $A^{TL}$ , are of the leading order in the GR, but subleading order in the RR; while some other terms, e.g.  $\sqrt{-t}s^3$  in  $A^{LLT}$  and  $A^{TL}$ , are of the subleading order in the GR, but leading order in the RR. This implies that the high energy string scattering amplitudes in the GR and RR are complementary to each other.

To compare with the string amplitudes in the GR, we consider the high energy string amplitudes in the RR with the same structure as those in the GR in Equation (27). The amplitudes  $A^{TTT}$ ,  $A^{LLT}$ ,  $A^{TL}$  and  $A^{LT}$  at mass level  $M^2 = 4$  are the examples. For these string amplitudes, the coefficients of the highest power of  $t$  in the leading order amplitudes of the RR are proportional to [21]

$$A^{TTT} = -i \left( \sqrt{-t} \right) \frac{\Gamma \left( -\frac{s}{2} - 1 \right) \Gamma \left( -\frac{\tilde{t}}{2} - 1 \right)}{\Gamma \left( \frac{u}{2} + 3 \right)} \cdot \left( \frac{1}{8} t s^3 \right) \sim \frac{1}{8}, \quad (72)$$

$$A^{LLT} = -i \left( \sqrt{-t} \right) \left( -\frac{1}{2M_2} \right)^2 \frac{\Gamma \left( -\frac{s}{2} - 1 \right) \Gamma \left( -\frac{\tilde{t}}{2} - 1 \right)}{\Gamma \left( \frac{u}{2} + 3 \right)} \left( \frac{1}{4} t s^3 \right) \sim \frac{1}{64}, \quad (73)$$

$$A^{TL} = i \left( \sqrt{-t} \right) \left( -\frac{1}{2M_2} \right) \frac{\Gamma \left( -\frac{s}{2} - 1 \right) \Gamma \left( -\frac{\tilde{t}}{2} - 1 \right)}{\Gamma \left( \frac{u}{2} + 3 \right)} \cdot \left( -\frac{1}{8} t s^3 \right) \sim -\frac{1}{32}, \quad (74)$$

$$A^{LT} = i \left( \sqrt{-t} \right) \left( -\frac{1}{2M_2} \right) \frac{\Gamma \left( -\frac{s}{2} - 1 \right) \Gamma \left( -\frac{\tilde{t}}{2} - 1 \right)}{\Gamma \left( \frac{u}{2} + 3 \right)} \cdot \left( \frac{3}{4} \frac{1}{t} t s^3 \right) \sim 0, \quad (75)$$

which produces the exactly same ratios in the GR in Equation (13). Here we defined the symmetrized and anti-symmetrized amplitudes as

$$A^{(TL)} = \frac{1}{2} \left( A^{TL} + A^{LT} \right) \sim \frac{1}{2} A^{TL}, \quad (76)$$

$$A^{[TL]} = \frac{1}{2} \left( A^{TL} - A^{LT} \right) \sim \frac{1}{2} A^{TL}. \quad (77)$$

It is interesting to see that  $A^{LT} \sim (\alpha_{-1}^L)(\alpha_{-2}^T)|0\rangle$  is in the subleading energy order in the GR, while it is in the leading energy order in the RR, and it will not affect the ratios calculated above.

>From the above example of  $M^2 = 4$ , it was therefore believed that there exist intimate connections between high energy string amplitudes in the GR and RR. To study this link and to reproduce the ratios in Equation (34) in particular, one was led to calculate RSS amplitudes at arbitrary mass levels. Using the fact that  $e^T \cdot k_1 = 0$  in Equation (34) and the energy power counting, we obtain the following rules,

$$\alpha_{-n}^T : \quad 1 \text{ term (contraction of } ik_3 \cdot X \text{ with } \varepsilon_T \cdot \partial^n X), \quad (78)$$

$$\alpha_{-n}^L : \quad \begin{cases} n > 1, & 1 \text{ term} \\ n = 1 & 2 \text{ terms (contraction of } ik_1 \cdot X \text{ and } ik_3 \cdot X \text{ with } \varepsilon_L \cdot \partial^n X). \end{cases} \quad (79)$$

The open string states with polarizations  $(e^T, e^L)$  in the leading order of the RR at each mass level  $N = \sum_{n,l>0} np_n + lr_l$  are

$$|p_n, r_l\rangle = \prod_{n>0} (\alpha_{-n}^T)^{p_n} \prod_{l>0} (\alpha_{-l}^L)^{r_l} |0, k\rangle. \quad (80)$$

The scattering string amplitudes of this state with three tachyonic states in the  $s - t$  channel can be calculated to be [21]

$$A^{(p_n, q_m)} = \left( -\frac{i}{M_2} \right)^{q_1} U \left( -q_1, \frac{t}{2} + 2 - q_1, \frac{\tilde{t}'}{2} \right) B \left( -1 - \frac{s}{2}, -1 - \frac{t}{2} \right) \cdot \prod_{n=1} \left[ i \sqrt{-t} (n-1)! \right]^{p_n} \prod_{m=2} \left[ i \tilde{t}' (m-1)! \left( -\frac{1}{2M_2} \right) \right]^{q_m}. \quad (81)$$

where  $U(a, c, x)$  is the Kummer function of the second kind. It is crucial to note that, in our formula, the parameter  $c = t/2 + 2 - q_1$  is not a constant independent of the variable  $x = \tilde{t}'/2$ , so that the function  $U$  in the above amplitude does not satisfy the Kummer equation. On the other hand, the parameter  $a = -q_1$  is an integer, which causes that the Kummer function in Equation (81) is truncated to a finite sum.

It can be seen from Equation (81) that the RSS amplitudes with spin polarizations corresponding to Equation (27) at each mass level are not proportional to each other. Their ratios depend on  $t$ , so does the scattering angle, and can be calculated to be [21]

$$\frac{A^{(N,2m,q)}(s,t)}{A^{(N,0,0)}(s,t)} = (-1)^m \left(-\frac{1}{2M_2}\right)^{2m+q} (\tilde{t}' - 2N)^{-m-q} (\tilde{t}')^{2m+q} \cdot \sum_{j=0}^{2m} (-2m)_j \left(-1 + N - \frac{\tilde{t}'}{2}\right)_j \frac{(-2/\tilde{t}')^j}{j!} + O\left\{\left(\frac{1}{\tilde{t}'}\right)^{m+1}\right\}, \quad (82)$$

where  $(x)_j = x(x+1)(x+2)\cdots(x+j-1)$  is the Pochhammer symbol.

To reproduce the ratio in Equation (34) from the RSS for the general mass levels, suggested by the explicit calculation at the mass level  $M_2^2 = 4$  [21], one needs to use the following identity,

$$\sum_{j=0}^{2m} (-2m)_j \left(-L - \frac{\tilde{t}'}{2}\right)_j \frac{(-2/\tilde{t}')^j}{j!} = 0 \cdot (-\tilde{t}')^0 + 0 \cdot (-\tilde{t}')^{-1} + \cdots + 0 \cdot (-\tilde{t}')^{-m+1} + \frac{(2m)!}{m!} (-\tilde{t}')^{-m} + O\left\{\left(\frac{1}{\tilde{t}'}\right)^{m+1}\right\} \quad (83)$$

where  $L = 1 - N$  is an integer. The identity was proved to be valid for any non-negative integer  $m$  and any *real* number  $L$  by using technique of combinatorial number theory [71]. It was remarkable to first predict [21] the mathematical identity above provided by string theory, and then a rigorous mathematical proof followed [71]. It was also interesting to see that the validity of the above identity includes non-integer values of  $L$  which were later shown to be realized by Regge string scatterings in compact space [72]. We thus have shown that the ratios among HSS amplitudes calculated in Equations (13) and (34) can be deduced and extracted from Kummer functions [21,73,74],

$$\frac{T^{(N,2m,q)}}{T^{(N,0,0)}} = \lim_{t \rightarrow \infty} \frac{A^{(N,2m,q)}}{A^{(N,0,0)}} = \left(-\frac{1}{2M}\right)^{2m+q} 2^{2m} \lim_{t \rightarrow \infty} (-t)^{-m} U\left(-2m, \frac{t}{2} + 2 - 2m, \frac{t}{2}\right). \quad (84)$$

The above calculations can be generalized to the four classes of superstring Regge scattering amplitudes [22].

The next interesting issue is to study relations among RSS amplitudes for different string states. To achieve this, one considers the more general high energy open string states with all three polarizations  $(e^T, e^P, e^L)$  in the RR at the mass level  $N = \sum_{n,m,l>0} np_n + mq_m + lr_l$

$$|p_n, q_m, r_l\rangle = \prod_{n>0} (\alpha_{-n}^T)^{p_n} \prod_{m>0} (\alpha_{-m}^P)^{q_m} \prod_{l>0} (\alpha_{-l}^L)^{r_l} |0, k\rangle. \quad (85)$$

The string scattering amplitudes of the above state with three tachyonic states in  $s - t$  channel can be calculated to be

$$A(p_n; q_m; r_l) = \int_0^1 dx x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} \cdot \left[\frac{e^P \cdot k_1}{x} - \frac{e^P \cdot k_3}{1-x}\right]^{q_1} \left[\frac{e^L \cdot k_1}{x} + \frac{e^L \cdot k_3}{1-x}\right]^{r_1} \cdot \prod_{n=1} \left[\frac{(n-1)! e^T \cdot k_3}{(1-x)^n}\right]^{p_n} \prod_{m=2} \left[\frac{(m-1)! e^P \cdot k_3}{(1-x)^m}\right]^{q_m} \prod_{l=2} \left[\frac{(l-1)! e^L \cdot k_3}{(1-x)^l}\right]^{r_l}. \quad (86)$$

Finally, the string amplitudes can be expressed in two equivalent forms [26]

$$A^{(p_n; q_m; r_l)} = \prod_{n>0} \left[ (n-1)! \sqrt{-t} \right]^{p_n} \cdot \prod_{m>0} \left[ -(m-1)! \frac{\tilde{t}}{2M_2} \right]^{q_m} \cdot \prod_{l>1} \left[ (l-1)! \frac{\tilde{t}'}{2M_2} \right]^{r_l} \\ \cdot B \left( -\frac{s}{2} - 1, -\frac{t}{2} + 1 \right) \left( \frac{1}{M_2} \right)^{r_1} \\ \cdot \sum_{i=0}^{q_1} \binom{q_1}{i} \left( \frac{2}{\tilde{t}} \right)^i \left( -\frac{t}{2} - 1 \right)_i U \left( -r_1, \frac{t}{2} + 2 - i - r_1, \frac{\tilde{t}'}{2} \right) \quad (87)$$

$$= \prod_{n>0} \left[ (n-1)! \sqrt{-t} \right]^{p_n} \cdot \prod_{m>1} \left[ -(m-1)! \frac{\tilde{t}}{2M} \right]^{q_m} \cdot \prod_{l>0} \left[ (l-1)! \frac{\tilde{t}'}{2M} \right]^{r_l} \\ \cdot B \left( -\frac{s}{2} - 1, -\frac{t}{2} + 1 \right) \left( -\frac{1}{M_2} \right)^{q_1} \\ \cdot \sum_{j=0}^{r_1} \binom{r_1}{j} \left( \frac{2}{\tilde{t}'} \right)^j \left( -\frac{t}{2} - 1 \right)_j U \left( -q_1, \frac{t}{2} + 2 - j - q_1, \frac{\tilde{t}}{2} \right). \quad (88)$$

It is worth noting that, for  $q_1 = 0$  or  $r_1 = 0$ , the RSS amplitudes can be expressed in terms of a single Kummer function  $U \left( -r_1, \frac{t}{2} + 2 - i - r_1, \frac{\tilde{t}'}{2} \right)$  or  $U \left( -q_1, \frac{t}{2} + 2 - j - q_1, \frac{\tilde{t}}{2} \right)$ . In general the RSS amplitudes can be expressed in terms of a finite sum of Kummer functions, then one can use the recurrence relations of Kummer functions to derive recurrence relations among RSS amplitudes [26].

For example, at mass level  $M^2 = 4$ , the recurrence relation

$$U \left( -3, \frac{t}{2} - 1, \frac{t}{2} - 1 \right) + \left( \frac{t}{2} + 1 \right) U \left( -2, \frac{t}{2} - 1, \frac{t}{2} - 1 \right) - \left( \frac{t}{2} - 1 \right) U \left( -2, \frac{t}{2}, \frac{t}{2} - 1 \right) = 0 \quad (89)$$

induces a recurrence relation among RSS amplitudes

$$M\sqrt{-t}A^{PPP} - 4A^{PPT} + M\sqrt{-t}A^{PPL} = 0. \quad (90)$$

Furthermore, the addition theorem of Kummer functions [75]

$$U(a, c, x + y) = \sum_{k=0}^{\infty} \frac{1}{k!} (a)_k (-1)^k y^k U(a + k, c + k, x) \quad (91)$$

can be used to derive the inter-mass level recurrence relation of RSS amplitudes. For example, by taking  $a = -1, c = \frac{t}{2} + 1, x = \frac{t}{2} - 1$  and  $y = 1$ , the theorem leads

$$U \left( -1, \frac{t}{2} + 1, \frac{t}{2} \right) - U \left( -1, \frac{t}{2} + 1, \frac{t}{2} - 1 \right) - U \left( 0, \frac{t}{2} + 2, \frac{t}{2} - 1 \right) = 0. \quad (92)$$

which gives to an inter-mass level recurrence relation of RSS amplitudes [26]

$$M(2)(t+6)A_2^{TP} - 2M(4)^2\sqrt{-t}A_4^{LP} + 2M(4)A_4^{LT} = 0 \quad (93)$$

where  $M(2) = \sqrt{2}$ ,  $M(4) = \sqrt{4} = 2$ , and  $A_2, A_4$  are RSS amplitudes at mass levels  $M^2 = 2, 4$ . To derive Equation (93), it is crucial to note that the power law behavior for each the RSS amplitude in Equation (93) is independent on the mass level [21].

Finally, Kummer recurrence relations can also be used to explicitly prove the Regge stringy Ward identities obtained from decoupling of ZNS in the RR, but not vice versa. Thus in the RR, recurrence



relations are more fundamental than the linear relations obtained from decoupling of ZNS. However, we should keep in mind that only Ward identities derived from the decoupling of Regge ZNS can be generalized to the string loop amplitudes. As an example, it can be shown that, in the Regge limit, the decoupling of the scalar type I Regge ZNS [26]

$$[25(\alpha_{-1}^P)^3 + 9\alpha_{-1}^P(\alpha_{-1}^L)^2 + 9\alpha_{-1}^P(\alpha_{-1}^T)^2 - 9\alpha_{-2}^L\alpha_{-1}^L - 9\alpha_{-2}^T\alpha_{-1}^T - 75\alpha_{-2}^P\alpha_{-1}^P + 50\alpha_{-3}^P] |0, k\rangle \quad (94)$$

can be demonstrated by using the following recurrence relations of Kummer functions

$$U(a-1, c, x) - (2a-c+x)U(a, c, x) + a(1+a-c)U(a+1, c, x) = 0, \quad (95)$$

$$U(a, c, x) - aU(a+1, c, x) - U(a, c-1, x) = 0, \quad (96)$$

$$(c-a-1)U(a, c-1, x) - (x+c-1)U(a, c, x) + xU(a, c+1, x) = 0. \quad (97)$$

Similarly, infinite number of recurrence relations among RSS amplitudes at arbitrary mass levels can be constructed. In general, these relations are independent of stringy Ward identities derived from the decoupling of Regge ZNS.

However, in contrast to Ward identity obtained from the decoupling of Regge ZNS like Equation (94), we have no proof at loop levels for other ward identities derived only from Kummer function recurrence relations. This is the subtle difference between linear relations obtained in the GR and the recurrence relations calculated in the RR. Similarly, one can construct recurrence relations of higher spin generalization of the BPST vertex operators [68] by using the same way [76].

In general, each RSS amplitude is a sum of Kummer functions so that it becomes complicated to derive the complete recurrence relations at higher mass levels. In a later work [25], it was shown that the  $26D$  open bosonic RSS amplitude can be expressed in terms of a *single* Appell function  $F_1$ .

In fact, the  $s-t$  channel RSS amplitudes with string state in Equation (85) and three tachyons can be calculated as [25]

$$A^{(p_n; q_m; r_1)} = \prod_{n=1} \left[ (n-1)! \sqrt{-t} \right]^{p_n} \prod_{m=1} \left[ -(m-1)! \frac{\tilde{t}}{2M_2} \right]^{q_m} \prod_{l=1} \left[ (l-1)! \frac{\tilde{t}'}{2M_2} \right]^{r_l} \\ \cdot F_1 \left( -\frac{t}{2} - 1, -q_1, -r_1, -\frac{s}{2}; \frac{s}{\tilde{t}}, \frac{s}{\tilde{t}'} \right) \cdot B \left( -\frac{t}{2} - 1, -\frac{s}{2} - 1 \right) \quad (98)$$

where the Appell function  $F_1$  is one of the four generalizations of the hypergeometric function  ${}_2F_1$  to two variables, and is defined as

$$F_1(a; b, b'; c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_m (b')_n}{m! n! (c)_{m+n}} x^m y^n \quad (99)$$

where  $(a)_n = a \cdot (a+1) \cdots (a+n-1)$  is the rising Pochhammer symbol. If  $b$  or  $b'$  is a non-positive integer, the Appell function would truncate to a finite polynomial that indeed is the case for the Appell function in the RSS amplitudes obtained above. It should be kept in mind that the expression in Equation (98) is valid only when  $s$  in the arguments of  $F_1$  goes to  $\infty$ .

Using Appell function  $F_1$ , rather than a sum of Kummer functions, makes it easier to obtain the complete recurrence relations among RSS amplitudes at arbitrary mass levels, which have been conjectured to be associated to the  $SL(5, C)$  symmetry of  $F_1$  [77]. For example, the recurrence relation among RSS amplitudes [25]

$$\sqrt{-t} \left[ A^{(N; q_1, r_1)} + A^{(N; q_1-1, r_1+1)} \right] - M A^{(N; q_1-1, r_1)} = 0 \quad (100)$$

at arbitrary mass levels  $M^2 = 2(N - 1)$  can be obtained from recurrence relations of the Appell functions. Equation (100) is a generalization of Equation (90) to arbitrary mass levels. More general recurrence relations can be obtained similarly. For example, from the leading term of  $s$  in the Regge limit, one obtain the following recurrence relation for  $b_2$

$$\begin{aligned} & cx^2 F_1(a; b_1, b_2; c; x, y) \\ & + \left[ (a - b_1 - b_2 - 1)xy^2 + cx^2 - 2cxy \right] F_1(a; b_1, b_2 + 1; c; x, y) \\ & - \left[ (a + 1)x^2y - (a - b_2 - 1)xy^2 - cx^2 + cxy \right] F_1(a; b_1, b_2 + 2; c; x, y) \\ & - (b_2 + 2)x(x - y)y F_1(a; b_1, b_2 + 3; c; x, y) = 0, \end{aligned} \quad (101)$$

which induces to a recurrence relation for RSS amplitudes at arbitrary mass levels [25]

$$\begin{aligned} & \tilde{t}'^2 A^{(N; q_1, r_1)} \\ & + \left[ \tilde{t}'^2 + \tilde{t}(t - 2\tilde{t}' - 2q_1 - 2r_1 + 4) \right] \left( \frac{\tilde{t}'}{\sqrt{-t}} \right) A^{(N; q_1, r_1 + 1)} \\ & + \left[ \tilde{t}'^2 - \tilde{t}'(\tilde{t} + t) + \tilde{t}(t - 2r_1 + 4) \right] \left( \frac{\tilde{t}'}{\sqrt{-t}} \right)^2 A^{(N; q_1, r_1 + 2)} \\ & - 2(r_1 - 2)(\tilde{t}' - \tilde{t}) \left( \frac{\tilde{t}'}{\sqrt{-t}} \right)^3 A^{(N; q_1, r_1 + 3)} = 0. \end{aligned} \quad (102)$$

More recurrence relations containing more than three Appell functions can be found in [78].

More importantly, the recurrence relations of the Appell function  $F_1$  in the Regge limit can be systematically solved, so that one can express all RSS amplitudes in terms of only one amplitude [25,26]. All these results are associated with symmetries of string scattering amplitudes in hard limit discussed in Section 3 [12–14,16,17,47,53].

As the first step, we remind that there are two equivalent expressions [26] as was previously shown in Equation (88). It is easy to show that, for  $q_1 = 0$  or  $r_1 = 0$ , the RSS amplitudes can be expressed in terms of a single Kummer function  $U\left(-r_1, \frac{t}{2} + 2 - i - r_1, \frac{\tilde{t}'}{2}\right)$  or  $U\left(-q_1, \frac{t}{2} + 2 - j - q_1, \frac{\tilde{t}}{2}\right)$ , which are thus related to the Appell function  $F_1\left(-\frac{t}{2} - 1; 0, -r_1; \frac{s}{2}; -\frac{s}{\tilde{t}}, -\frac{s}{\tilde{t}'}\right)$  or  $F_1\left(-\frac{t}{2} - 1; -q_1, 0; \frac{s}{2}; -\frac{s}{\tilde{t}}, -\frac{s}{\tilde{t}'}\right)$  respectively,

$$\lim_{s \rightarrow \infty} F_1\left(-\frac{t}{2} - 1; 0, -r_1; \frac{s}{2}; -\frac{s}{\tilde{t}}, -\frac{s}{\tilde{t}'}\right) = \left(\frac{2}{\tilde{t}'}\right)^{r_1} U\left(-r_1, \frac{t}{2} + 2 - r_1, \frac{\tilde{t}'}{2}\right), \quad (103)$$

$$\lim_{s \rightarrow \infty} F_1\left(-\frac{t}{2} - 1; -q_1, 0; \frac{s}{2}; -\frac{s}{\tilde{t}}, -\frac{s}{\tilde{t}'}\right) = \left(\frac{2}{\tilde{t}}\right)^{q_1} U\left(-q_1, \frac{t}{2} + 2 - q_1, \frac{\tilde{t}}{2}\right). \quad (104)$$

Furthermore, the ratio of Kummer functions can be obtained as [26],

$$\frac{U(\alpha, \gamma, z)}{U(0, z, z)} = f(\alpha, \gamma, z), \alpha = 0, -1, -2, -3, \dots \quad (105)$$

where  $f(\alpha, \gamma, z)$  can be obtained from the recurrence relations of  $U(\alpha, \gamma, z)$  and  $U(0, z, z) = 1$ . In the Regge limit, one obtains that,

$$c = \frac{s}{2} \rightarrow \infty; x, y \rightarrow \infty; a, b_1, b_2 \text{ fixed}, \quad (106)$$

and the Appell functions  $F_1(a; 0, b_2; c; x, y)$  and  $F_1(a; b_1, 0; c; x, y)$  are determined up to an overall factor by recurrence relations. The next step is to obtain the recurrence relation

$$yF_1(a; b_1, b_2; c; x, y) - xF_1(a; b_1 + 1, b_2 - 1; c; x, y) + (x - y)F_1(a; b_1 + 1, b_2; c; x, y) = 0, \quad (107)$$

which can be obtained from two of the four Appell recurrence relations among contiguous functions.

We can now proceed to prove that in the Regge limit all RSS amplitudes can be expressed in terms of a single amplitude. To be concise, we will use the abbreviative notation  $F_1(a; b_1, b_2; c; x, y) = F_1(b_1, b_2)$  in the following argument. For  $b_2 = -1$ , by using Equation (107) and the known  $F_1(b_1, 0)$  and  $F_1(0, b_2)$ , one can show that  $F_1(b_1, -1)$  can be determined for all  $b_1 = -1, -2, -3, \dots$ . Similarly,  $F_1(b_1, -2)$  can be determined for all  $b_1 = -1, -2, -3, \dots$  once  $F_1(b_1, -1)$  is known. This process can be repeatedly used to determine  $F_1(b_1, b_2)$  for all  $b_1, b_2 = -1, -2, -3, \dots$ , so that all RSS amplitudes can be expressed in terms of only one amplitude.

## 6. The Lauricella String Scattering Amplitudes (LSSA)

In this section, we briefly review the Lauricella string scattering amplitudes (LSSA) discussed in a recent paper [23]. The authors considered the four-point open bosonic string scattering amplitudes with three tachyons and an arbitrary massive higher spin string state of the form,

$$\left| r_n^T, r_m^P, r_l^L \right\rangle = \prod_{n>0} (\alpha_{-n}^T)^{r_n^T} \prod_{m>0} (\alpha_{-m}^P)^{r_m^P} \prod_{l>0} (\alpha_{-l}^L)^{r_l^L} |0, k\rangle, \quad (108)$$

at the mass level  $N = \sum_{n,m,l>0} (nr_n^T + mr_m^P + lr_l^L)$ . The  $(s, t)$  channel amplitude can be exactly calculated and expressed in terms of the D-type Lauricella functions [23]

$$A_{st}^{(r_n^T, r_m^P, r_l^L)} = \prod_{n=1} \left[ -(n-1)! k_3^T \right]^{r_n^T} \cdot \prod_{m=1} \left[ -(m-1)! k_3^P \right]^{r_m^P} \prod_{l=1} \left[ -(l-1)! k_3^L \right]^{r_l^L} \cdot B \left( -\frac{t}{2} - 1, -\frac{s}{2} - 1 \right) F_D^{(K)} \left( -\frac{t}{2} - 1; R_n^T, R_m^P, R_l^L; \frac{u}{2} + 2 - N; \tilde{Z}_n^T, \tilde{Z}_m^P, \tilde{Z}_l^L \right) \quad (109)$$

where

$$R_k^X \equiv \left\{ -r_1^X \right\}^1, \dots, \left\{ -r_k^X \right\}^k \text{ with } \{a\}^n = \underbrace{a, a, \dots, a}_n \quad (110)$$

$$Z_k^X \equiv \left[ z_1^X \right], \dots, \left[ z_k^X \right] \text{ with } \left[ z_k^X \right] = z_{k0}^X, \dots, z_{k(k-1)}^X, \quad (111)$$

$$z_{kk'}^X = \left| \frac{k_1^X}{k_3^X} \right|^{\frac{1}{k}} e^{\frac{2\pi i k'}{k}} \text{ and } \tilde{z}_{kk'}^X \equiv 1 - z_{kk'}^X, k' = 0, \dots, k-1, \quad (112)$$

and the integer  $K$  depends on the spin structure as

$$K = \sum_{\{\text{for all } r_j^T \neq 0\}} j + \sum_{\{\text{for all } r_j^P \neq 0\}} j + \sum_{\{\text{for all } r_j^L \neq 0\}} j. \quad (113)$$

As a result, one can use the LSSA to reproduce SSA and the relations among SSA of different string states at various scattering limits obtained previously.

In the HSS limit  $e^P = e^L$  [13,14],  $r_1^T = N - 2m - 2q$ ,  $r_1^L = 2m$  and  $r_2^L = q$ , the LSSA can be calculated to be [23]

$$\begin{aligned} A_{st}^{(N-2m-2q, 2m, q)} &\simeq B\left(-\frac{t}{2} - 1, -\frac{s}{2} - 1\right) (E \sin \phi)^N \frac{(2m)!}{m!} \left(-\frac{1}{2M_2}\right)^{2m+q} \\ &= (2m-1)!! \left(-\frac{1}{M_2}\right)^{2m+q} \left(\frac{1}{2}\right)^{m+q} A_{st}^{(N, 0, 0)}, \end{aligned} \quad (114)$$

which gives the ratios in Equation (34), which are the same as in the previous result [13–17].

In the RSS limit, the SSA in Equation (109) reduces to [25]

$$\begin{aligned} A_{st}^{(r_n^T, r_m^P, r_l^L)} &\simeq B\left(-\frac{t}{2} - 1, -\frac{s}{2} - 1\right) \prod_{n=1}^{r_n^T} [(n-1)! \sqrt{-t}]^{r_n^T} \\ &\quad \cdot \prod_{m=1}^{r_m^P} \left[(m-1)! \frac{\tilde{t}}{2M_2}\right]^{r_m^P} \prod_{l=1}^{r_l^L} \left[(l-1)! \frac{\tilde{t}'}{2M_2}\right]^{r_l^L} \\ &\quad \cdot F_1\left(-\frac{t}{2} - 1; -q_1, -r_1; -\frac{s}{2}; \frac{s}{\tilde{t}}, \frac{s}{\tilde{t}'}\right), \end{aligned} \quad (115)$$

which agrees with the result obtained in [25].

Finally, in the NSS limit [27], for the case of  $r_1^T = N_1$ ,  $r_1^P = N_3$ ,  $r_1^L = N_2$ , and  $r_k^X = 0$  for all  $k \geq 2$ , the SSA reduces to

$$\begin{aligned} A_{st}^{(N_1, N_2, N_3)} &= \left(\frac{\epsilon}{2} \sin \phi\right)^{N_1} \left(\frac{\epsilon}{2} \cos \phi\right)^{N_2} \cdot \left(-\frac{M_1 + M_2}{2}\right)^{N_3} B\left(\frac{M_1 M_2}{2}, 1 - M_1 M_2\right) \\ &\quad \cdot {}_2F_1\left(\frac{M_1 M_2}{2}; -N_3; M_1 M_2; \frac{2M_1}{M_1 + M_2}\right), \end{aligned} \quad (116)$$

which agrees with the result obtained in [27]. The mass level dependent of the string BCJ relation presented in Equation (58) can also be obtained from Equation (109).

Moreover, in [24] it was shown that by using the following key recurrence relation of the Lauricella functions

$$x_j F_D^{(K)}(b_i - 1) - x_i F_D^{(K)}(b_j - 1) + (x_i - x_j) F_D^{(K)} = 0, \quad (117)$$

and a multiplication theorem of Gauss hypergeometry functions, one can express all the LSSA in terms of the four tachyon amplitude. This result extends the solveabilities of SSA at the HSS limit and the RSS limit discovered previously to all kinematic regimes. We expect more interesting developments on these research directions in the near future.

## 7. Conclusions

In addition to the string scatterings amplitudes at arbitrary mass level discussed in this review, there were other related approaches in the literature discussing higher spin dynamics of string theory. String theory includes infinitely many higher spin massive fields with consistent mutual interactions, and can provide useful hints on the dynamics of higher spin field theory. On the other hand, a better understanding of higher spin dynamics could also help our comprehension of string theory. It is widely believed that the tensionless limit of string [79–85] is a theory of higher spin gauge fields. An explicit and nontrivial construction of interacting higher spin gauge theory is Vasiliev's system in AdS space-time.

In [86], the spectrum of Kaluza-Klein descendants of fundamental string excitations on  $AdS_5 \times S^5$  was derived. Furthermore, in the tensionless limit, the field equations from BRST quantization of string theory provide a direct route toward local field equations for higher-spin gauge fields [87].

Recently, in [88], one conjectured that Vasiliev theory is a limit of string theory. Roughly speaking, the fundamental string of string theory is simply the flux tube string of the non-Abelian bulk Vasiliev theory. The duality between Vasiliev theory and type IIA string field theory suggests a concrete way of embedding Vasiliev theory into string theory. It is interesting to investigate whether—and in what guise—the huge bulk gauge symmetry of Vasiliev’s description survives in the bulk string sigma model description of the same system.

There existed other approaches of stringy symmetries which include other studies of string collisions in the high energy, fixed momentum transfer regime [63–69], the Hagedorn transition at high temperature [89–91], vertex operator algebra for compacted spacetime or on a lattice [92–94], group theoretical approach of string [48,95].

Another motivation of studying high energy string scattering is to investigate the gravitational effect, such as black hole formation due to high energy string collision, and to understand the nonlocal behavior of string theory. Nevertheless, in [96], it was shown that there is no evidence that the extendedness of strings produces any long-distance nonlocal effects in high energy scattering, and no grounds have been found for string effects interfering with formation of a black hole either.

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## References

1. Yi, Y. Symmetry of String Scattering Amplitudes. *J. Phys.* **2017**, *804*, 012043.
2. Lee, J.-C.; Yang, Y. Linear relations and their breakdown in high energy massive string scatterings in compact spaces. *Nucl. Phys. B* **2007**, *784*, 22–35.
3. Lee, J.-C.; Takimi, T.; Yang, Y. High-energy string scatterings of compactified open string. *Nucl. Phys. B* **2008**, *804*, 250–261.
4. Gross, D.J.; Mende, P.F. The high-energy behavior of string scattering amplitudes. *Phys. Lett. B* **1987**, *197*, 129–134.
5. Gross, D.J.; Mende, P.F. String theory beyond the Planck scale. *Nucl. Phys. B* **1988**, *303*, 407–454.
6. Gross, D.J. High-Energy Symmetries of String Theory. *Phys. Rev. Lett.* **1988**, *60*, 1229.
7. Gross, D.J. Strings at superplanckian energies: In search of the string symmetry. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1989**, *329*, 401–413.
8. Gross, D.J.; Mende, P.F. The high energy behavior of open string scattering. *Nucl. Phys. B* **1989**, *326*, 73–107.
9. Lee, J.-C. New symmetries of higher spin states in string theory. *Phys. Lett. B* **1990**, *241*, 336–342.
10. Lee, J.-C.; Ovrut, B.A. Zero-norm states and enlarged gauge symmetries of the closed bosonic string with massive background fields. *Nucl. Phys. B* **1990**, *336*, 222–244.
11. Lee, J.-C. Decoupling of degenerate positive-norm states in string theory. *Phys. Rev. Lett.* **1990**, *64*, 1636.
12. Chan, C.-T.; Lee, J.-C. Stringy symmetries and their high-energy limits. *Phys. Lett. B* **2005**, *611*, 193–198.
13. Lee, J.-C. Stringy symmetries and their high-energy limit. *arXiv* **2003**, arXiv:hep-th/0303012.
14. Chan, C.-T.; Lee, J.-C. Zero-norm states and high-energy symmetries of string theory. *Nucl. Phys. B* **2004**, *690*, 3–20.

15. Chan, C.-T.; Ho, P.-M.; Lee, J.-C. Ward identities and high energy scattering amplitudes in string theory. *Nucl. Phys. B* **2005**, *708*, 99–114.
16. Chan, C.-T.; Ho, P.-M.; Lee, J.-C.; Teraguchi, S.; Yang, Y. Solving all 4-point correlation functions for bosonic open string theory in the high-energy limit. *Nucl. Phys. B* **2005**, *725*, 352–382.
17. Chan, C.-T.; Ho, P.-M.; Lee, J.-C.; Teraguchi, S.; Yang, Y. High-energy zero-norm states and symmetries of string theory. *Phys. Rev. Lett.* **2006**, *96*, 171601.
18. Kawai, H.; Lewellen, D.C.; Tye, S.-H.H. A relation between tree amplitudes of closed and open strings. *Nucl. Phys. B* **1986**, *269*, 1–23.
19. Chan, C.-T.; Lee, J.-C.; Yang, Y. Notes on high-energy limit of bosonic closed string scattering amplitudes. *Nucl. Phys. B* **2006**, *749*, 280–290.
20. Bern, Z.; Carrasco, J.J.M.; Johansson, H. New relations for gauge-theory amplitudes. *Phys. Rev. D* **2008**, *78*, 085011.
21. Ko, S.-L.; Lee, J.-C.; Yang, Y. Patterns of high energy massive string scatterings in the Regge regime. *JHEP* **2009**, *2009*, 028.
22. He, S.; Lee, J.-C.; Takahashi, K.; Yang, Y. Massive superstring scatterings in the Regge regime. *Phys. Rev. D* **2011**, *83*, 066016.
23. Laim, S.-H.; Lee, J.-C.; Yang, Y. The exact  $sl(k+3, \mathbb{C})$  symmetry of string scattering amplitudes. *arXiv* **2016**, arXiv:1603.00396.
24. Lai, S.-H.; Lee, J.-C.; Lee, T.; Yang, Y. Solving lauricella string scattering amplitudes through recurrence relations. *JHEP* **2017**, *9*, 130.
25. Lee, J.-C.; Yi, Y. The Appell function  $F_1$  and Regge string scattering amplitudes. *Phys. Lett. B* **2014**, *739*, 370–374.
26. Lee, J.-C.; Mitsuka, Y. Recurrence relations of Kummer functions and Regge string scattering amplitudes. *JHEP* **2013**, *2013*, 1–23.
27. Lai, S.-H.; Lee, J.-C.; Yang, Y. The string bcj relations revisited and extended recurrence relations of nonrelativistic string scattering amplitudes. *arXiv* **2016**, arXiv:1601.03813.
28. Lee, J.-C.; Yang, Y. Review on high energy string scattering amplitudes and symmetries of string theory. *arXiv* **2015**, arXiv:1510.03297.
29. Green, M.B.; Schwarz, J.H.; Witten, E. *Superstring Theory, v. 1*; Cambridge University: Cambridge, UK, 1987.
30. Lee, J.-C. Reduction of stringy scattering amplitudes through massive ward identities. *Z. Phys. C* **1994**, *63*, 351–355.
31. Kao, H.-C.; Lee, J.-C. Decoupling of degenerate positive-norm states in Witten's string field theory. *Phys. Rev. D* **2003**, *67*, 086003.
32. Lee, J.-C. Heterotic massive Einstein-Yang-Mills-type symmetry and Ward identity. *Phys. Lett. B* **1994**, *337*, 69–73.
33. Lee, J.-C. Spontaneously broken symmetry in string theory. *Phys. Lett. B* **1994**, *326*, 79–83.
34. Lee, J.-C. Generalized on-shell Ward identities in string theory. *Prog. Theor. Phys.* **1994**, *91*, 353–360.
35. Chan, C.-T.; Lee, J.-C.; Yang, Y. Anatomy of zero-norm states in string theory. *Phys. Rev. D* **2005**, *71*, 086005.
36. Witten, E. Non-commutative geometry and string field theory. *Nucl. Phys. B* **1986**, *268*, 253–294.
37. Chung, T.-D.; Lee, J.-C. Discrete gauge states and  $w +$  charges in  $c = 1$  2D gravity. *Phys. Lett. B* **1995**, *350*, 22–27.
38. Avan, J.; Jevicki, A. Classical integrability and higher symmetries of collective string field theory. *Phys. Lett. B* **1991**, *266*, 35–41.
39. Avan, J.; Jevicki, A. Quantum integrability and exact eigenstates of the collective string field theory. *Phys. Lett. B* **1991**, *272*, 17–24.
40. Klebanov, I.R.; Polyakov, A.M. Interaction of discrete states in two-dimensional string theory. *Mod. Phys. Lett. A* **1991**, *6*, 3273–3281.
41. Witten, E. Ground ring of two-dimensional string theory. *Nucl. Phys. B* **1992**, *373*, 187–213.
42. Witten, E.; Zwiebach, B. Algebraic structures and differential geometry in two-dimensional string theory. *Nucl. Phys. B* **1992**, *377*, 55–112.
43. Chung, T.-D.; Lee, J.-C. Superfield form of discrete gauge states in  $\hat{c} = 1$  2d supergravity. *Z. Phys. C* **1997**, *75*, 555–558.



44. Lee, J.-C. Soliton gauge states and T-duality of closed bosonic string compactified on torus. *Eur. Phys. J. C* **1999**, *7*, 669–672.
45. Lee, J.-C. Chan–Paton soliton gauge states of the compactified open string. *Eur. Phys. J. C* **2000**, *13*, 695–697.
46. Chan, C.T.; Lee, J.-C. One-loop massive scattering amplitudes and Ward identities in string theory. *Prog. Theor. Phys.* **2006**, *115*, 229–243.
47. Chan, C.T.; Lee, J.-C.; Yang, Y. High energy scattering amplitudes of superstring theory. *Nucl. Phys. B* **2006**, *738*, 93–123.
48. Moeller, N.; West, P. Arbitrary four string scattering at high energy and fixed angle. *Nucl. Phys. B* **2005**, *729*, 1–48.
49. Veneziano, G. Construction of a crossing-symmetric, Regge-behaved amplitude for linearly rising trajectories. *Il Nuovo Cimento A* **1968**, *57*, 190–197.
50. Bjerrum-Bohr, N.E.J.; Damgaard, P.H.; Vanhove, P. Minimal basis for gauge theory amplitudes. *Phys. Rev. Lett.* **2009**, *103*, 161602.
51. Chan, C.T.; Lee, J.-C.; Yang, Y. Scatterings of massive string states from D-brane and their linear relations at high energies. *Nucl. Phys. B* **2007**, *764*, 1–14.
52. Lee, J.-C.; Mitsuka, Y.; Yang, Y. Higher spin string states scattered from d-particle in the Regge regime and factorized ratios of fixed angle scatterings. *Prog. Theor. Phys.* **2011**, *126*, 397–417.
53. Chan, C.T.; Ho, P.-M.; Lee, J.-C.; Teraguchi, S.; Yang, Y. Comments on the high energy limit of bosonic open string theory. *Nucl. Phys. B* **2006**, *749*, 266–279.
54. Chan, C.T.; Lee, J.-C.; Yang, Y. Power-law Behavior of Strings Scattered from Domain-wall at High Energies and Breakdown of their Linear Relations. *arXiv* **2006**, arXiv:hep-th/0610219.
55. Craps, B.; Roose, F. Anomalous D-brane and orientifold couplings from the boundary state. *Phys. Lett. B* **1998**, *445*, 150–159.
56. Stefański, B. Gravitational couplings of D-branes and O-planes. *Nucl. Phys. B* **1999**, *548*, 275–290.
57. Morales, J.F.; Scrucca, C.A.; Serone, M. Anomalous couplings for D-branes and O-planes. *Nucl. Phys. B* **1999**, *552*, 291–315.
58. Schnitzer, H.J.; Wyllard, N. An orientifold of  $AdS_5 \times T^{11}$  with D7-branes, the associated  $\alpha'^2$ -corrections and their role in the dual Script  $N = 1$  Sp  $(2N + 2M) \times$  Sp  $(2N)$  gauge theory. *JHEP* **2002**, *2002*, 012.
59. Garousi, M.R. Superstring scattering from O-planes. *Nucl. Phys. B* **2007**, *765*, 166–184.
60. Lee, J.-C.; Yang, Y. High-energy massive string scatterings from orientifold planes. *Nucl. Phys. B* **2008**, *798*, 198–209.
61. Garousi, M.R.; Myers, R.C. Superstring scattering from D-branes. *Nucl. Phys. B* **1996**, *475*, 193–224.
62. Mende, P.F. High energy string collisions in a compact space. *Phys. Lett. B* **1994**, *326*, 216–222.
63. Amati, D.; Ciafaloni, M.; Veneziano, G. Superstring collisions at Planckian energies. *Phys. Lett. B* **1987**, *197*, 81–88.
64. Amati, D.; Ciafaloni, M.; Veneziano, G. Classical and quantum gravity effects from Planckian energy superstring collisions. *Int. J. Mod. Phys. A* **1988**, *3*, 1615–1661.
65. Amati, D.; Ciafaloni, M.; Veneziano, G. Can spacetime be probed below the string size? *Phys. Lett. B* **1989**, *216*, 41–47.
66. Soldate, M. Partial-wave unitarity and closed-string amplitudes. *Phys. Lett. B* **1987**, *186*, 321–327.
67. Muzinich, I.J.; Soldate, M. High-energy unitarity of gravitation and strings. *Phys. Rev. D* **1988**, *37*, 359.
68. Brower, R.C.; Polchinski, J.; Strassler, M.J.; Tan, C.-I. The Pomeron and gauge/string duality. *JHEP* **2007**, *2007*, 005.
69. Brower, R.C.; Nastase, H.; Schnitzer, H.J.; Tan, C.-I. Analyticity for multi-Regge limits of the Bern–Dixon–Smirnov amplitudes. *Nucl. Phys. B* **2009**, *822*, 301–347.
70. Andreev, O.; Siegel, W. Quantized tension: Stringy amplitudes with Regge poles and parton behavior. *Phys. Rev. D* **2005**, *71*, 086001.
71. Lee, J.-C.; Yan, C.H.; Yang, Y. High-energy string scattering amplitudes and signless Stirling number identity. *SIGMA. arXiv* **2012**, arXiv:1012.5225.
72. He, S.; Lee, J.-C.; Yang, Y. Exponential fall-off behavior of Regge scatterings in compactified open string theory. *Prog. Theor. Phys.* **2012**, *128*, 887–901.

73. Ko, S.-L.; Lee, J.-C.; Yang, Y. Kummer function and high energy string scatterings. *arXiv* **2008**, arXiv:0811.4502.
74. Lee, J.-C.; Yang, Y.; Ko, S.-L. Stirling number identities and High energy String Scatterings. *arXiv* **2009**, arXiv:0909.3894.
75. Slater, L.J. *Confluent Hypergeometric Functions*; Cambridge University Press: Cambridge, UK, 1960.
76. Fu, C.-H.; Lee, J.-C.; Tan, C.-I.; Yang, Y. Recurrence relations of higher spin BPST vertex operators for open strings. *Phys. Rev. D* **2013**, *88*, 046004.
77. Miller, W., Jr. Lie theory and the Lauricella functions FD. *J. Math. Phys.* **1972**, *13*, 1393–1399.
78. Wang, X. Recursion formulas for Appell functions. *Integr. Transf. Spec. Funct.* **2012**, *23*, 421–433.
79. Sagnotti, A.; Taronna, M. String lessons for higher-spin interactions. *Nucl. Phys. B* **2011**, *842*, 299–361.
80. Isberg, J.; Lindström, U.; Sundborg, B.; Theodoridis, G. Classical and quantized tensionless strings. *Nucl. Phys. B* **1994**, *411*, 122–156.
81. Sundborg, B. Stringy gravity, interacting tensionless strings and massless higher spins. *Nucl. Phys. B* **2001**, *102*, 113–119.
82. Sezgin, E.; Sundell, P. Massless higher spins and holography. Erratum. *Nucl. Phys. B* **2003**, *660*, 403.
83. Sezgin, E.; Sundell, P. Massless higher spins and holography. *Nucl. Phys. B* **2002**, *644*, 303–370.
84. Chu, C.-S.; Ho, P.-M.; Lin, F.-L. Cubic string field theory in pp-wave background and background independent Moyal structure. *JHEP* **2002**, *2002*, 003.
85. Bonelli, G. On the tensionless limit of bosonic strings, infinite symmetries and higher spins. *Nucl. Phys. B* **2003**, *669*, 159–172.
86. Bianchi, M.; Morales, J.F.; Samtleben, H. On stringy  $AdS_5 \times S^5$  and higher spin holography. *J. High Energy Phys.* **2003**, *2003*, 062.
87. Sagnotti, A.; Tsulaia, M. On higher spins and the tensionless limit of string theory. *Nucl. Phys. B* **2004**, *682*, 83–116.
88. Chang, C.-M.; Minwalla, S.; Sharma, T.; Yin, X. Abj triality: from higher spin fields to strings. *J. Phys. A Math. Theor.* **2013**, *46*, 214009.
89. Atick, J.J.; Witten, E. The Hagedorn transition and the number of degrees of freedom of string theory. *Nucl. Phys. B* **1988**, *310*, 291–334.
90. Hagedorn, R. Statistical thermodynamics of strong interactions at high energies. *Nuovo Cim. A* **1968**, *56* (Suppl. 3), 1027.
91. Sundborg, B. The Hagedorn transition, deconfinement and  $N=4$  SYM theory. *Nucl. Phys. B* **2000**, *573*, 349–363.
92. Moore, G. Finite in all directions. *arXiv* **1993**, arXiv:hep-th/9305139.
93. Moore, G. Symmetries of the bosonic string S-matrix. *arXiv* **1993**, arXiv:hep-th/9310026.
94. Chan, C.-T.; Kawamoto, S.; Tomino, D. To see symmetry in a forest of trees. *Nucl. Phys. B* **2014**, *885*, 225–265.
95. West, P. Physical states and string symmetries. *Mod. Phys. Lett. A* **1995**, *10*, 761–771.
96. Giddings, S.B.; Gross, D.J.; Maharana, A. Gravitational effects in ultrahigh-energy string scattering. *Phys. Rev. D* **2008**, *77*, 046001.

