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Around the Model of Infection Disease: The Cauchy Matrix and Its Properties

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Abstract: In this paper the model of infection diseases by Marchuk is considered. Mathematical questions which are important in its study are discussed. Among them there are stability of stationary points, construction of the Cauchy matrices of linearized models, estimates of solutions. The novelty we propose is in a distributed feedback control which affects the antibody concentration. We use this control in the form of an integral term and come to the analysis of nonlinear integro-differential systems. New methods for the study of stability of linearized integro–differential systems describing the model of infection diseases are proposed. Explicit conditions of the exponential stability of the stationary points characterizing the state of the healthy body are obtained. The method of the paper is based on the symmetry properties of the Cauchy matrices which allow us their construction.

Keywords: integro–differential systems; Cauchy matrix; exponential stability; distributed control

1. Introduction

In this paper we consider the Marchuk model of infection diseases

$$\begin{cases} \frac{dV}{dt} = \beta V(t) - \gamma F(t) V(t) \\ \frac{dC}{dt} = \zeta(m(t)) \alpha F(t) V(t) - \mu_c (C(t) - C^*) \\ \frac{dF}{dt} = \rho C(t) - \eta\gamma F(t) V(t) - \mu_f F(t) \\ \frac{dm}{dt} = \sigma V(t) - \mu_m m(t) \end{cases}, \tag{1}$$

proposed in the book [1].

Here t is time, $V(t)$ is antigen concentration rate, $C(t)$ is the plasma cell concentration rate, $F(t)$ is the antibody concentration rate, $m(t)$ is relative features of the body, $m = 0$ for the healthy body, $\zeta(m)$ takes into account the destruction of the normal functioning of the immune system, $\zeta(0) = 1$. $\alpha, \beta, \gamma, \rho, \eta, \mu_f, \mu_m, \mu_c, C^*$ are corresponding coefficients obtained as results of laboratory experiments. Let us note their biological sense of the coefficients: β—coefficient describing the antigen activity, γ—the antigen neutralizing factor, α—stimulation factor of the immune system, ρ—rate of production of antibodies by one plasma cell, μ_f—coefficient inversely proportional to the decay time of the antibodies, μ_m—coefficient inversely proportional to the organ recovery time, i.e., the coefficient μ_m characterizes the rate of regeneration of the target organ, μ_c—coefficient of reduction of plasma cells due to ageing (inversely proportional to the lifetime), σ—constant related with a particular disease, C^*—the plasma cell concentration of the healthy body. Let us describe now the structure of the model (1). The first equation presents the block of the virus dynamics. It describes the changes in the antigen concentration rate and includes the amount of the antigen in the blood. The antigen concentration decreases as a result of the interaction with the antibodies. The immune process is characterized by the antibodies, whose concentration changes with time (destruction rate) and is described by the third equation. The amount of the antibody cells decreases as a result of interaction with antigen and also as a result of the natural destruction. However, the plasma restores the antibodies and therefore the plasma state plays an important role in the immune process. Thus, the change in the concentration rate of the plasma cell is included in several differential equations describing this model. Taking into account the healthy body level of plasma cells and their natural ageing, the term $\mu_c\,(C(t) - C^*)$ is included in the second equation of system (1). The second and third equations present the immune response dynamics. Concerning the last equation of system (1), the following can be noted: (1) the value of m increases with the antigen's concentration rate $V(t)$; (2) the maximum value of m is one, in the case of 100% organ damage or zero for a fully healthy organ.

This model was studied in many works, note, for example, the recent papers [2–6] and the bibliography therein. The adding control to stabilize the system in the neighborhood of a stationary point was proposed, for example, in [5–8]. In the works [4,9,10], the basic mathematical model that takes into account the concentrated control of the immune response is proposed.

Let us discuss a motivation and novelty of our approach. In constructing every model, the influences of various additional factors that have seemed to be nonessential were neglected. The influence effect of choosing nonlinear terms by their linearization in neighborhood of stationary solution is also neglected. Even in the frame of linearized model, only approximate values of coefficients instead of exact ones are used. Changes of these coefficients with respect to time are not usually taken into account. It looks important to estimate an influence of all these factors.

In order to make this we have to obtain estimates of the elements of the Cauchy matrix of corresponding linearized (in a neighborhood of a stationary point) system. Consider the system

$$x'(t) = P(t)x(t) + G(t),$$

where $P(t)$ is a $(n \times n)$-matrix, $G(t)$ is an n-vector. Its general solution $x(t) = col\{x_1(t), ...x_n(t)\}$ can be represented in the form (see, for example, [11])

$$x(t) = \int_0^t C(t,s)G(s)ds + C(t,0)x(0),$$

where $n \times n$-matrix $C(t,s)$ is called the Cauchy matrix. Its j-th column ($j = 1, ..., n$) for every fixed s as a function of t, is a solution of the corresponding homogeneous system

$$x'(t) = P(t)x(t),$$

satisfying the initial conditions $x_i(s) = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad i = 1, ..., n,$$

(see, for example, [12]). This Cauchy matrix $C(t,s)$ satisfies the following symmetric properties $C(t,s) = X(s)X^{-1}(s)$, where $X(t)$ is a fundamental matrix, $C(t,0) = C(t,s)C(s,0)$, and in the case of constant matrix $P(t) = P$, $X(t-s) = C(t,s)$ is a fundamental matrix for every $s \geq 0$. These definition and properties allow us to construct and estimate $C(t,s)$.

It can be noted that the use of information about behaviour of a disease and the immune system for a long time (defined by distributed control, for example, in the form of an integral term) looks very natural in choosing a strategy of a possible treatment. We add a distributed control in the third equation, describing the antibody concentration rate to achieve stabilization of the process in the neighborhood of stationary solution in the form

$$u\,(t) = \int_0^t (F\,(s) - F^*)\, e^{-k(t-s)} ds. \tag{2}$$

Here F^* is the antibody concentration that we wish to achieve after the treatment. It can be noted that the influence of a correspondin average value instead of $F(t) - F^*$ at the point t looks reasonable. The kernel in (2) increases the influence of the previous moments which are closer to the current moment t. Note that this control is a reasonable one from the medical point of view. We consider a corresponding integro–differential system and construct its Cauchy matrix. This allows us to estimate the influence of all notes above factors on behavior of solutions.

Note the use of distributed control in stabilization in the papers [13,14]. The goal of this paper is to demonstrate new possibilities of distributed control in the model of infection diseases through analysis of integro-differential systems. From the medical point of view, our results could be interpreted as follows: supporting the immune system we transform infection disease to a stable state of "almost healthy" body. After getting this stable state we do not stop the use of corresponding medicine allowing to hold antibody concentration rate on the higher level than in the normal conditions of a healthy body. In all these stages it is important to estimate influence of many additional factors in order to hold the process in a corresponding zone. Going solution out of this zone can be dangerous for a patient. To give an instrument for these estimations is the main goal of this paper. We propose here a simple method of analysis and estimation based on a reduction of integro–differential systems to ones of ordinary differential equations.

Our paper consists of the following parts. In Section 2 we introduce the distributed control in the Marchuk model of infection diseases and explain how the analysis of this model of the fourth order can be reduced to the analysis of a system of ordinary differential equations of the fifth order. In Section 3 the Cauchy matrix of integro–differential system is constructed and the exponential stability of a stationary point is obtained. The case of uncertain coefficient in the control is studied in Section 4 where results on the exponential stability are proposed. The influence of changes in the right-hand side on behaviour of solutions is discussed in Section 5.

2. Modified Model of Infection Deceases

Adding the control (2) in the right-hand side of the third equation of system (1) we come to the system of four equations

$$\begin{cases} \frac{dV}{dt} = \beta V(t) - \gamma F(t) V(t) \\ \frac{dC}{dt} = \zeta(m(t)) \alpha F(t) V(t) - \mu_c (C(t) - C^*) \\ \frac{dF}{dt} = \rho C - \eta\gamma F(t) V(t) - \mu_f F(t) - b \int\limits_0^t (F(s) - F^*) e^{-k(t-s)} ds \\ \frac{dm}{dt} = \sigma V(t) - \mu_m m(t) \end{cases}, \tag{3}$$

Let us consider the following system of five equations

$$\begin{cases} \frac{dV}{dt} = \beta V(t) - \gamma F(t) V(t) \\ \frac{dC}{dt} = \zeta(m(t)) \alpha F(t) V(t) - \mu_c (C(t) - C^*) \\ \frac{dF}{dt} = \rho C - \eta\gamma F(t) V(t) - \mu_f F(t) - b u(t) \\ \frac{dm}{dt} = \sigma V(t) - \mu_m m(t) \\ \frac{du}{dt} = F(t) - F^* - k u(t) \end{cases}. \tag{4}$$

Lemma 1. *The solution-vector $col(v(t), s(t), f(t), m(t))$ of system (3) and four first components of the solution-vector $col(v(t), s(t), f(t), m(t), u(t))$ of system (4) considered with the condition $u(0) = 0$ coincide.*

The proof of Lemma 1. follows from the formula of presentation of the general solution of the scalar linear equation $\frac{du}{dt} + ku(t) = F(t) - F^*$.

Note that a similar trick was used, for example, in papers [15,16].

Following [9] we can pass to the dimensionless case.

Substituting $V(t) = v(t)V_m$, $C(t) = s(t)C^*$, $F(t) = f(t)F^*$, $u(t) = \overline{u}(t)F^*$ into (3) we obtain.

$$\begin{cases} \frac{dv}{dt} = \beta v(t) - \gamma F^* f(t) v(t) \\ \frac{ds}{dt} = \alpha V_m \frac{F^*}{C^*} \zeta(m(t)) f(t) v(t) - \mu_c (s(t) - 1) \\ \frac{df}{dt} = \frac{\rho C^*}{F^*} s(t) - \eta\gamma V_m f(t) v(t) - \mu_f f(t) - b\overline{u}(t) \\ \frac{dm}{dt} = \sigma V_m v(t) - \mu_m m(t) \\ \frac{d\overline{u}}{dt} = f(t) - 1 - k\overline{u}(t) \end{cases}. \tag{5}$$

Substituting $a_1 = \beta$, $a_2 = \gamma F^*$, $a_3 = \alpha V_m \frac{F^*}{C^*}$, $a_4 = \mu_f = \frac{\rho C^*}{F^*}$, $a_5 = \mu_c$, $a_6 = \sigma V_m$, $a_7 = \mu_m$, $a_8 = \eta\gamma V_m$ into (6) we come to the system

$$\begin{cases} \frac{dv}{dt} = a_1 v(t) - a_2 f(t) v(t) \\ \frac{ds}{dt} = a_3 \zeta(m(t)) f(t) v(t) - a_5 (s(t) - 1) \\ \frac{df}{dt} = a_4 (s(t) - f(t)) - a_8 f(t) v(t) - b\overline{u}(t) \\ \frac{dm}{dt} = a_6 v(t) - a_7 m(t) \\ \frac{d\overline{u}}{dt} = f(t) - 1 - k\overline{u}(t) \end{cases}. \tag{6}$$

Remark 1. *It was obtained by M. Chirkov and S. Rusakov (see their method of identification of parameters, for example in [5,9]) on the basis of the laboratory data of pneumonia, that* $a_1 = 0.25$; $a_2 = 8.5000332$; $a_3 = 1.792175675 \times 10^9$; $a_4 = 1.95992344 \times 10^{-7}$; $a_5 = 0.5$; $a_6 = 10$; $a_7 = 0.4$; $a_8 = 1.7 \times 10^{-3}$.

It is clear that $v = m = \overline{u} = 0$, $s = f = 1$ is a stationary point of system (6).

Linearizing system in a neighborhood of this stationary point, we obtain the corresponding linear system

$$\begin{cases} \frac{dv}{dt} = (a_1 - a_2) v \\ \frac{ds}{dt} = a_3 \zeta(0) v - a_5 (s - 1) \\ \frac{df}{dt} = -a_8 v - a_4 (f - 1) + a_4 (s - 1) - b\overline{u}(t) \\ \frac{dm}{dt} = a_6 v - a_7 m \\ \frac{d\overline{u}}{dt} = f - 1 - k\overline{u} \end{cases},$$

where $\zeta(0) = 1$, as it was noted above. Denoting $x_1 = v$, $x_2 = s - 1$, $x_3 = f - 1$, $x_4 = m$, $x_5 = \overline{u}$, we obtain

$$\begin{cases} x_1' = (a_1 - a_2) x_1 \\ x_2' = a_3 x_1 - a_5 x_2 \\ x_3' = -a_8 x_1 + a_4 x_2 - a_4 x_3 - b x_5 \\ x_4' = a_6 x_1 - a_7 x_4 \\ x_5' = x_3 - k x_5 \end{cases}. \tag{7}$$

3. Constructing the Cauchy Matrix of the System (7)

In order to estimate the values of $x_1, \ldots, x_5$ and the speed of their tending to the stationary solutions we propose below a corresponding technique. Its basis is the Cauchy matrix.

The matrix of the coefficients of system (7) is following

$$A = \begin{pmatrix} a_1 - a_2 & 0 & 0 & 0 & 0 \\ a_3 & -a_5 & 0 & 0 & 0 \\ -a_8 & a_4 & -a_4 & 0 & -b \\ a_6 & 0 & 0 & -a_7 & 0 \\ 0 & 0 & 1 & 0 & -k \end{pmatrix} \tag{8}$$

Its eigenvalues are

$$\begin{gathered}\lambda_1=\frac{-a_4-k+\sqrt{(a_4-k)^2-4b}}{2},\ \ \lambda_2=\frac{-a_4-k-\sqrt{(a_4-k)^2-4b}}{2},\\ \lambda_3=-a_7,\quad \lambda_4=-a_5,\quad \lambda_5=a_1-a_2.\end{gathered}\tag{9}$$

Their negativity (negativity of the real parts in the case of complex λ_1 and λ_2) leads us to the assertion on stability of the stationary point $v=m=\overline{u}=0, s=f=1$ of system (6).

Theorem 1. *If $k>0$, $b>0$ and a_i, $1\le i\le 8$, are real positive and different and $a_1<a_2$, then system (7) is exponentially stable.*

Remark 2. *All steps can be done for the integro–differential system (3) and system of ordinary differential equations (4) also directly without needing to pass to the dimensionless case (6). The linearization will lead us to a corresponding analog of the linear system of ordinary differential Equation (7) with the matrix of the coefficients B. Let us discuss the medicine sense of our result. Let F_0 be the value of antibody concentration rate of the healthy body. The case of $F_0>\frac{\beta}{\gamma}$ is considered by G.I. Marchuk in his book. In this case the stationary point $V=0, C=C^*, F=F_0, m=0$, is stable even without control. We can try to consider the "bad" case, where $F_0<\frac{\beta}{\gamma}$. It is clear that system (1.1) could not be stable in this case in the neighborhood of this stationary point since $V(t)$ increases. It means that the immune system with the antibody concentration on the level of the healthy body cannot prevent increasing antigen concentration. Our control (2) in the third equation of system (1.1) cannot help us and makes this stationary point stable. We consider another stationary point $V=0, C=C^*, F=F^*, m=0$. Repeating the analysis of the eigenvalues of the matrix of the coefficients B, we come to the same conclusions. Let all coefficients in system (1) be positive (this is absolutely natural assumption) and $b>0, k>0$, then adding the control in the form (2), where $F^*>F_0+\frac{\beta-\gamma F_0}{\gamma}$, we can achieve the exponential stability of this new stationary point of systems (3) and (4). Actually, positivity of k, b and all coefficients $a_i (i=1,...,8)$ is preserved, to achieve the inequality $a_1-a_2<0$ we have to require the noted inequality connecting F^* and F_0. One can make a conclusion that supporting for a long time the immune system, describing by antibody concentration $F(t)$ and holding it on the level F^* can be a possible way of a treatment.*

There are three possible cases:

(1) If $(a_4-k)^2>4b$, then we have two different real eigenvalues λ_1 and λ_2.
(2) If $(a_4-k)^2=4b$, then we have two real and multiple eigenvalues λ_1 and λ_2.
(3) If $(a_4-k)^2<4b$, we have two complex eigenvalues λ_1 and λ_2.

3.1. Constructing the Cauchy Matrix in the Case 1

Using Maple, we obtain the eigenvectors of the matrix (8):

$$\begin{gathered}\overrightarrow{v}_1=\begin{pmatrix}0\\0\\-\frac{2b}{a_4-k+\sqrt{(a_4-k)^2-4b}}\\0\\1\end{pmatrix},\ \ \overrightarrow{v}_2=\begin{pmatrix}0\\0\\-\frac{2b}{a_4-k-\sqrt{(a_4-k)^2-4b}}\\0\\1\end{pmatrix},\ \ \overrightarrow{v}_3=\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix},\\ \overrightarrow{v}_4=\begin{pmatrix}0\\-\frac{a_4a_5-a_4k-a_5^2+a_5k-b}{a_4}\\-a_5+k\\0\\1\end{pmatrix},\ \ \overrightarrow{v}_5=\begin{pmatrix}-c\,(a_5+a_1-a_2)\\-ca_3\\a_1-a_2+k\\-\frac{(a_5+a_1-a_2)a_6c}{a_1-a_2+a_7}\\1\end{pmatrix},\end{gathered}\tag{10}$$

where $c=\frac{a_1^2-2a_1a_2+a_1a_4+a_1k+a_2^2-a_2a_4-a_2k+a_4k+b}{a_1a_8-a_2a_8-a_3a_4+a_5a_8}$.

Let us denote $\alpha_{31} = -\frac{2b}{a_4-k+\sqrt{(a_4-k)^2-4b}}, \alpha_{32} = -\frac{2b}{a_4-k-\sqrt{(a_4-k)^2-4b}}, \alpha_{24} = -\frac{a_4a_5-a_4k-a_5^2+a_5k-b}{a_4}, \alpha_{34} = -a_5+k, \alpha_{15} = -c\left(a_5+a_1-a_2\right), \alpha_{25} = -ca_3, \alpha_{35} = a_1-a_2+k, \alpha_{45} = -\frac{(a_5+a_1-a_2)a_6c}{a_1-a_2+a_7}$, and define the matrix

$$B = [\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3, \overrightarrow{v}_4, \overrightarrow{v}_5] = \begin{pmatrix} 0 & 0 & 0 & 0 & \alpha_{15} \\ 0 & 0 & 0 & \alpha_{24} & \alpha_{25} \\ \alpha_{31} & \alpha_{32} & 0 & \alpha_{34} & \alpha_{35} \\ 0 & 0 & 1 & 0 & \alpha_{45} \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

containing eigenvectors and its inverse matrix

$$B^{-1} = \begin{pmatrix} \frac{\alpha_{24}(\alpha_{32}-\alpha_{35})-\alpha_{25}(\alpha_{32}-\alpha_{34})}{\alpha_{15}\alpha_{24}(\alpha_{31}-\alpha_{32})} & \frac{\alpha_{32}-\alpha_{34}}{\alpha_{24}(\alpha_{31}-\alpha_{32})} & \frac{1}{\alpha_{31}-\alpha_{32}} & 0 & -\frac{\alpha_{32}}{\alpha_{31}-\alpha_{32}} \\ -\frac{\alpha_{24}(\alpha_{31}-\alpha_{35})-\alpha_{25}(\alpha_{31}-\alpha_{34})}{\alpha_{15}\alpha_{24}(\alpha_{31}-\alpha_{32})} & -\frac{\alpha_{31}-\alpha_{34}}{\alpha_{24}(\alpha_{31}-\alpha_{32})} & -\frac{1}{\alpha_{31}-\alpha_{32}} & 0 & \frac{\alpha_{31}}{\alpha_{31}-\alpha_{32}} \\ -\frac{\alpha_{45}}{\alpha_{15}} & 0 & 0 & 1 & 0 \\ -\frac{\alpha_{25}}{\alpha_{15}\alpha_{24}} & \frac{1}{\alpha_{24}} & 0 & 0 & 0 \\ \frac{1}{\alpha_{15}} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us write now the Cauchy matrix $C(t,s)$ of the the system (7). The Cauchy matrix can be written as $C(t,s) = e^{A(t-s)}$. In our case A is diagonalized: $A = BDB^{-1}$, we have $e^{A(t-s)} = Be^{D(t-s)}B^{-1}$, where the matrix D is diagonal, containing the eigenvalues of the matrix A. The columns $\overrightarrow{C}_i\,(t,s)_{1\le i\le 5}$ of the Cauchy matrix $C(t,s)$ of system (7) are the following ones:

$$\begin{aligned} \overrightarrow{C}_1(t,s) = {} & \frac{\alpha_{24}\left(\alpha_{32}-\alpha_{35}\right)-\alpha_{25}\left(\alpha_{32}-\alpha_{34}\right)}{\alpha_{15}\alpha_{24}\left(\alpha_{31}-\alpha_{32}\right)} \begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k+\sqrt{(a_4-k)^2-4b}} \\ 0 \\ 1 \end{pmatrix} e^{\left(\frac{-a_4-k+\sqrt{(a_4-k)^2-4b}}{2}\right)(t-s)} - \\ & \frac{\alpha_{24}\left(\alpha_{31}-\alpha_{35}\right)-\alpha_{25}\left(\alpha_{31}-\alpha_{34}\right)}{\alpha_{15}\alpha_{24}\left(\alpha_{31}-\alpha_{32}\right)} \begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k-\sqrt{(a_4-k)^2-4b}} \\ 0 \\ 1 \end{pmatrix} e^{\left(\frac{-a_4-k-\sqrt{(a_4-k)^2-4b}}{2}\right)(t-s)} - \\ & \frac{\alpha_{45}}{\alpha_{15}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-a_7(t-s)} - \frac{\alpha_{25}}{\alpha_{15}\alpha_{24}} \begin{pmatrix} 0 \\ -\frac{a_4a_5-a_4k-a_5^2+a_5k-b}{a_4} \\ -a_5+k \\ 0 \\ 1 \end{pmatrix} e^{-a_5(t-s)} + \\ & \frac{1}{\alpha_{15}} \begin{pmatrix} -c\left(a_5+a_1-a_2\right) \\ -ca_3 \\ a_1-a_2+k \\ -\frac{(a_5+a_1-a_2)a_6c}{a_1-a_2+a_7} \\ 1 \end{pmatrix} e^{(a_1-a_2)(t-s)} \end{aligned}$$

$$\begin{aligned}
\overrightarrow{C}_2(t,s) &= \frac{\alpha_{32}-\alpha_{34}}{\alpha_{24}(\alpha_{31}-\alpha_{32})}\begin{pmatrix} 0\\ 0\\ -\frac{2b}{a_4-k+\sqrt{(a_4-k)^2-4b}}\\ 0\\ 1\end{pmatrix} e^{\left(\frac{-a_4-k+\sqrt{(a_4-k)^2-4b}}{2}\right)(t-s)} - \\
&\quad \frac{\alpha_{31}-\alpha_{34}}{\alpha_{24}(\alpha_{31}-\alpha_{32})}\begin{pmatrix} 0\\ 0\\ -\frac{2b}{a_4-k-\sqrt{(a_4-k)^2-4b}}\\ 0\\ 1\end{pmatrix} e^{\left(\frac{-a_4-k-\sqrt{(a_4-k)^2-4b}}{2}\right)(t-s)} + \\
&\quad \frac{1}{\alpha_{24}}\begin{pmatrix} 0\\ -\frac{a_4a_5-a_4k-a_5^2+a_5k-b}{a_4}\\ -a_5+k\\ 0\\ 1\end{pmatrix} e^{-a_5(t-s)}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{C}_3(t,s) &= \frac{1}{\alpha_{31}-\alpha_{32}}\begin{pmatrix} 0\\ 0\\ -\frac{2b}{a_4-k+\sqrt{(a_4-k)^2-4b}}\\ 0\\ 1\end{pmatrix} e^{\left(\frac{-a_4-k+\sqrt{(a_4-k)^2-4b}}{2}\right)(t-s)} - \\
&\quad \frac{1}{\alpha_{31}-\alpha_{32}}\begin{pmatrix} 0\\ 0\\ -\frac{2b}{a_4-k-\sqrt{(a_4-k)^2-4b}}\\ 0\\ 1\end{pmatrix} e^{\left(\frac{-a_4-k-\sqrt{(a_4-k)^2-4b}}{2}\right)(t-s)}
\end{aligned}$$

$$\overrightarrow{C}_4(t,s) = \begin{pmatrix} 0\\ 0\\ 0\\ 1\\ 0\end{pmatrix} e^{-a_7(t-s)}$$

$$\begin{aligned}
\overrightarrow{C}_5(t,s) &= -\frac{\alpha_{32}}{\alpha_{31}-\alpha_{32}}\begin{pmatrix} 0\\ 0\\ -\frac{2b}{a_4-k+\sqrt{(a_4-k)^2-4b}}\\ 0\\ 1\end{pmatrix} e^{\left(\frac{-a_4-k+\sqrt{(a_4-k)^2-4b}}{2}\right)(t-s)} + \\
&\quad \frac{\alpha_{31}}{\alpha_{31}-\alpha_{32}}\begin{pmatrix} 0\\ 0\\ -\frac{2b}{a_4-k-\sqrt{(a_4-k)^2-4b}}\\ 0\\ 1\end{pmatrix} e^{\left(\frac{-a_4-k-\sqrt{(a_4-k)^2-4b}}{2}\right)(t-s)}
\end{aligned}$$

3.2. Constructing the Cauchy Matrix in the Case 2

We have the eigenvalues

$$\lambda_1 = \lambda_2 = -\frac{a_4 + k}{2}, \;\; \lambda_3 = -a_7, \;\; \lambda_4 = -a_5, \;\; \lambda_5 = a_1 - a_2, \tag{11}$$

Consider the following set of vectors

$$\begin{gathered} \overrightarrow{v}_1 = \begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k} \\ 0 \\ 1 \end{pmatrix}, \;\; \overrightarrow{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{2}{a_4-k} \end{pmatrix}, \;\; \overrightarrow{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \\ \overrightarrow{v}_4 = \begin{pmatrix} 0 \\ -\frac{a_4a_5 - a_4k - a_5^2 + a_5k - b}{a_4} \\ -a_5 + k \\ 0 \\ 1 \end{pmatrix}, \;\; \overrightarrow{v}_5 = \begin{pmatrix} -c\left(a_5 + a_1 - a_2\right) \\ -ca_3 \\ a_1 - a_2 + k \\ -\frac{(a_5 + a_1 - a_2)a_6c}{a_1 - a_2 + a_7} \\ 1 \end{pmatrix}, \end{gathered} \tag{12}$$

here $\overrightarrow{v}_1, \overrightarrow{v}_3, \overrightarrow{v}_4, \overrightarrow{v}_5$ are the eigenvectors of matrix (8) and $\overrightarrow{v}_2$ is a root vector for $\overrightarrow{v}_1$.

Let us denote $\beta_{31} = -\frac{2b}{a_4-k}, \beta_{52} = \frac{2}{a_4-k}, \beta_{24} = -\frac{a_4a_5-a_4k-a_5^2+a_5k-b}{a_4}, \beta_{34} = -a_5 + k, \beta_{15} = -c\left(a_5 + a_1 - a_2\right), \beta_{25} = -ca_3, \beta_{35} = a_1 - a_2 + k, \beta_{45} = -\frac{(a_5+a_1-a_2)a_6c}{a_1-a_2+a_7}$, and define the matrix

$$B = [\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3, \overrightarrow{v}_4, \overrightarrow{v}_5] = \begin{pmatrix} 0 & 0 & 0 & 0 & \beta_{15} \\ 0 & 0 & 0 & \beta_{24} & \beta_{25} \\ \beta_{31} & 0 & 0 & \beta_{34} & \beta_{35} \\ 0 & 0 & 1 & 0 & \beta_{45} \\ 1 & \beta_{52} & 0 & 1 & 1 \end{pmatrix}$$

and its inverse matrix

$$B^{-1} = \begin{pmatrix} -\frac{\beta_{24}\beta_{35} - \beta_{25}\beta_{34}}{\beta_{31}\beta_{15}\beta_{24}} & -\frac{\beta_{34}}{\beta_{24}\beta_{31}} & \frac{1}{\beta_{31}} & 0 & 0 \\ -\frac{\beta_{24}(\beta_{31} - \beta_{35}) - \beta_{25}(\beta_{31} - \beta_{34})}{\beta_{31}\beta_{52}\beta_{24}\beta_{15}} & -\frac{\beta_{31} - \beta_{34}}{\beta_{31}\beta_{24}\beta_{52}} & -\frac{1}{\beta_{31}\beta_{52}} & 0 & \frac{1}{\beta_{52}} \\ -\frac{\beta_{45}}{\beta_{15}} & 0 & 0 & 1 & 0 \\ -\frac{\beta_{25}}{\beta_{15}\beta_{24}} & \frac{1}{\beta_{24}} & 0 & 0 & 0 \\ \frac{1}{\beta_{15}} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us denote: $\overrightarrow{u}_1(t) = \overrightarrow{v}_1 e^{\lambda_1 t}$, $\overrightarrow{u}_2(t) = (\overrightarrow{v}_2 + t\overrightarrow{v}_1)e^{\lambda_1 t}$, $\overrightarrow{u}_i(t) = \overrightarrow{v}_i e^{\lambda_i t}$, $3 \leq i \leq 5$, $\overrightarrow{w}_j(t,s) = \overrightarrow{u}_j(t-s)$, $1 \leq j \leq 5$.

Let us build the Cauchy matrix $C(t,s) = \left\{\overrightarrow{C}_i(t,s)\right\}_{1 \leq i \leq 5}$, where $\overrightarrow{C}_i(t,s) = \sum\limits_{j=1}^{5} b_{ji}\, \overrightarrow{w}_j(t,s)$, $1 \leq i \leq 5$.

We have to find b_{ji}, $1 \leq i, j \leq 5$ in this representation. Taking into account that $C(s,s) = I$, where I is the identity (5×5)-matrix, we can write: $\overrightarrow{C}_i(s,s) = \sum\limits_{j=1}^{5} b_{ji}\, \overrightarrow{v}_j$, $1 \leq i \leq 5$.

Setting $i = 1, 2, 3, 4, 5$, we obtain

$$\overrightarrow{C}_1(s,s) = \sum_{j=1}^{5} b_{j1}\,\overrightarrow{v}_j = B\begin{pmatrix} b_{11}\\ b_{21}\\ b_{31}\\ b_{41}\\ b_{51}\end{pmatrix} = \begin{pmatrix} 1\\ 0\\ 0\\ 0\\ 0\end{pmatrix} \quad\Rightarrow\quad \begin{pmatrix} b_{11}\\ b_{21}\\ b_{31}\\ b_{41}\\ b_{51}\end{pmatrix} = \begin{pmatrix} -\frac{\beta_{24}\beta_{35}-\beta_{25}\beta_{34}}{\beta_{31}\beta_{15}\beta_{24}}\\ -\frac{\beta_{24}(\beta_{31}-\beta_{35})-\beta_{25}(\beta_{31}-\beta_{34})}{\beta_{31}\beta_{52}\beta_{24}\beta_{15}}\\ -\frac{\beta_{45}}{\beta_{15}}\\ -\frac{\beta_{25}}{\beta_{15}\beta_{24}}\\ \frac{1}{\beta_{15}}\end{pmatrix}$$

$$\overrightarrow{C}_2(s,s) = \sum_{j=1}^{5} b_{j2}\,\overrightarrow{v}_j = B\begin{pmatrix} b_{12}\\ b_{22}\\ b_{32}\\ b_{42}\\ b_{52}\end{pmatrix} = \begin{pmatrix} 0\\ 1\\ 0\\ 0\\ 0\end{pmatrix} \quad\Rightarrow\quad \begin{pmatrix} b_{12}\\ b_{22}\\ b_{32}\\ b_{42}\\ b_{52}\end{pmatrix} = \begin{pmatrix} -\frac{\beta_{34}}{\beta_{24}\beta_{31}}\\ -\frac{\beta_{31}-\beta_{34}}{\beta_{31}\beta_{24}\beta_{52}}\\ 0\\ \frac{1}{\beta_{24}}\\ 0\end{pmatrix}$$

$$\overrightarrow{C}_3(s,s) = \sum_{j=1}^{5} b_{j3}\,\overrightarrow{v}_j = B\begin{pmatrix} b_{13}\\ b_{23}\\ b_{33}\\ b_{43}\\ b_{53}\end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 1\\ 0\\ 0\end{pmatrix} \quad\Rightarrow\quad \begin{pmatrix} b_{13}\\ b_{23}\\ b_{33}\\ b_{43}\\ b_{53}\end{pmatrix} = \begin{pmatrix} \frac{1}{\beta_{31}}\\ -\frac{1}{\beta_{31}\beta_{52}}\\ 0\\ 0\\ 0\end{pmatrix}$$

$$\overrightarrow{C}_4(s,s) = \sum_{j=1}^{5} b_{j4}\,\overrightarrow{v}_j = B\begin{pmatrix} b_{14}\\ b_{24}\\ b_{34}\\ b_{44}\\ b_{54}\end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\\ 1\\ 0\end{pmatrix} \quad\Rightarrow\quad \begin{pmatrix} b_{14}\\ b_{24}\\ b_{34}\\ b_{44}\\ b_{54}\end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 1\\ 0\\ 0\end{pmatrix}$$

$$\overrightarrow{C}_5(s,s) = \sum_{j=1}^{5} b_{j5}\,\overrightarrow{v}_j = B\begin{pmatrix} b_{15}\\ b_{25}\\ b_{35}\\ b_{45}\\ b_{55}\end{pmatrix} = \begin{pmatrix} 0\\ 0\\ 0\\ 0\\ 1\end{pmatrix} \quad\Rightarrow\quad \begin{pmatrix} b_{15}\\ b_{25}\\ b_{35}\\ b_{45}\\ b_{55}\end{pmatrix} = \begin{pmatrix} 0\\ \frac{1}{\beta_{52}}\\ 0\\ 0\\ 0\end{pmatrix}$$

Substituting the coefficients b_{ji}, $1 \le i, j \le 5$ into the equality $\overrightarrow{C}_i(t,s) = \sum_{j=1}^{5} b_{ji} \overrightarrow{w}_j(t,s)$, $1 \le i \le 5$ we obtain

$$\begin{aligned}
\overrightarrow{C}_1(t,s) \quad = \quad & -\frac{\beta_{24}\beta_{35}-\beta_{25}\beta_{34}}{\beta_{31}\beta_{15}\beta_{24}} \begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k} \\ 0 \\ 1 \end{pmatrix} e^{\left(-\frac{a_4+k}{2}\right)(t-s)} - \\
& \frac{\beta_{24}(\beta_{31}-\beta_{35})-\beta_{25}(\beta_{31}-\beta_{34})}{\beta_{31}\beta_{52}\beta_{24}\beta_{15}} \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{2}{a_4-k} \end{pmatrix} + (t-s) \begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k} \\ 0 \\ 1 \end{pmatrix} \right] e^{\left(-\frac{a_4+k}{2}\right)(t-s)} - \\
& \frac{\beta_{45}}{\beta_{15}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-a_7(t-s)} - \frac{\beta_{25}}{\beta_{15}\beta_{24}} \begin{pmatrix} 0 \\ -\frac{a_4a_5-a_4k-a_5^2+a_5k-b}{a_4} \\ -a_5+k \\ 0 \\ 1 \end{pmatrix} e^{-a_5(t-s)} + \\
& \frac{1}{\beta_{15}} \begin{pmatrix} -c(a_5+a_1-a_2) \\ -ca_3 \\ a_1-a_2+k \\ -\frac{(a_5+a_1-a_2)a_6c}{a_1-a_2+a_7} \\ 1 \end{pmatrix} e^{(a_1-a_2)(t-s)}
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{C}_2(t,s) = & \left[-\frac{\beta_{34}}{\beta_{24}\beta_{31}} \begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k} \\ 0 \\ 1 \end{pmatrix} - \frac{\beta_{31}-\beta_{34}}{\beta_{31}\beta_{24}\beta_{52}} \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{2}{a_4-k} \end{pmatrix} + (t-s) \begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k} \\ 0 \\ 1 \end{pmatrix} \right] \right] e^{\left(-\frac{a_4+k}{2}\right)(t-s)} + \\
& \frac{1}{\beta_{24}} \begin{pmatrix} 0 \\ -\frac{a_4a_5-a_4k-a_5^2+a_5k-b}{a_4} \\ -a_5+k \\ 0 \\ 1 \end{pmatrix} e^{-a_5(t-s)}
\end{aligned}$$

$$\overrightarrow{C}_3(t,s) = \left[\frac{1}{\beta_{31}} \begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k} \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\beta_{31}\beta_{52}} \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{2}{a_4-k} \end{pmatrix} + (t-s) \begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k} \\ 0 \\ 1 \end{pmatrix} \right] \right] e^{\left(-\frac{a_4+k}{2}\right)(t-s)}$$

$$\overrightarrow{C}_4(t,s) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-a_7(t-s)}$$

$$\overrightarrow{C}_5(t,s) = \frac{1}{\beta_{52}}\left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{2}{a_4-k} \end{pmatrix} + (t-s)\begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k} \\ 0 \\ 1 \end{pmatrix}\right] e^{\left(-\frac{a_4+k}{2}\right)(t-s)}$$

3.3. Constructing the Cauchy Matrix in the Case 3

We have the eigenvalues

$$\lambda_1 = \frac{-a_4 - k + i\sqrt{4b-(a_4-k)^2}}{2}, \ \lambda_2 = \frac{-a_4 - k - i\sqrt{4b-(a_4-k)^2}}{2}, \quad (13)$$
$$\lambda_3 = -a_7, \quad \lambda_4 = -a_5, \quad \lambda_5 = a_1 - a_2,$$

$$\overrightarrow{v}_1 = \begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k+i\sqrt{4b-(a_4-k)^2}} \\ 0 \\ 1 \end{pmatrix}, \ \overrightarrow{v}_2 = \begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k-i\sqrt{4b-(a_4-k)^2}} \\ 0 \\ 1 \end{pmatrix}, \ \overrightarrow{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad (14)$$
$$\overrightarrow{v}_4 = \begin{pmatrix} 0 \\ -\frac{a_4a_5 - a_4k - a_5^2 + a_5k - b}{a_4} \\ k - a_5 \\ 0 \\ 1 \end{pmatrix}, \ \overrightarrow{v}_5 = \begin{pmatrix} -c(a_5 + a_1 - a_2) \\ -ca_3 \\ a_1 - a_2 + k \\ -\frac{(a_5+a_1-a_2)a_6c}{a_1-a_2+a_7} \\ 1 \end{pmatrix},$$

We can write first two vector-solutions as follows:

$$\overrightarrow{u_1}(t) = \begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k+i\sqrt{4b-(a_4-k)^2}} \\ 0 \\ 1 \end{pmatrix} \cdot e^{\left(\frac{-a_4-k+i\sqrt{4b-(a_4-k)^2}}{2}\right)t} =$$
$$\begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k+i\sqrt{4b-(a_4-k)^2}} \\ 0 \\ 1 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}t} \cdot \left(\cos\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}t\right) + i\sin\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}t\right)\right)$$
$$\overrightarrow{u_2}(t) = \begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k-i\sqrt{4b-(a_4-k)^2}} \\ 0 \\ 1 \end{pmatrix} \cdot e^{\left(\frac{-a_4-k-i\sqrt{4b-(a_4-k)^2}}{2}\right)t} =$$
$$\begin{pmatrix} 0 \\ 0 \\ -\frac{2b}{a_4-k-i\sqrt{4b-(a_4-k)^2}} \\ 0 \\ 1 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}t} \cdot \left(\cos\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}t\right) - i\sin\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}t\right)\right)$$

Passing to real solutions:

$$\overrightarrow{w_1}(t) = \frac{\overrightarrow{u_1}(t) + \overrightarrow{u_2}(t)}{2} = \begin{pmatrix} 0 \\ 0 \\ \frac{k-a_4}{2} \\ 0 \\ 1 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}t} \cdot \cos\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}t\right) + \begin{pmatrix} 0 \\ 0 \\ -\frac{\sqrt{4b-(a_4-k)^2}}{2} \\ 0 \\ 0 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}t} \cdot \sin\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}t\right)$$

$$\overrightarrow{w_2}(t) = \frac{\overrightarrow{u_1}(t) - \overrightarrow{u_2}(t)}{2i} = \begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{4b-(a_4-k)^2}}{2} \\ 0 \\ 0 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}t} \cdot \cos\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}t\right) + \begin{pmatrix} 0 \\ 0 \\ \frac{a_4-k}{2} \\ 0 \\ 1 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}t} \cdot \sin\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}t\right)$$

$$\overrightarrow{w_3}(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot e^{-a_7 t}$$

$$\overrightarrow{w_4}(t) = \begin{pmatrix} 0 \\ -\frac{a_4a_5-a_4k-a_5^2+a_5k-b}{a_4} \\ k-a_5 \\ 0 \\ 1 \end{pmatrix} \cdot e^{-a_5 t}$$

$$\overrightarrow{w_5}(t) = \begin{pmatrix} -c(a_5+a_1-a_2) \\ -ca_3 \\ a_1-a_2+k \\ -\frac{(a_5+a_1-a_2)a_6c}{a_1-a_2+a_7} \\ 1 \end{pmatrix} \cdot e^{(a_1-a_2)t}$$

Let us construct now the Cauchy matrix $C(t,s) = \left\{\overrightarrow{C_i}(t,s)\right\}_{i=1,\dots,5}$ of the system. Let us define $\overrightarrow{w_i}(t,s) = \overrightarrow{w_i}(t-s)$, then

$$\overrightarrow{C_i}(t,s) = b_{1i}\overrightarrow{w_1}(t,s) + b_{2i}\overrightarrow{w_2}(t,s) + b_{3i}\overrightarrow{w_3}(t,s) + b_{4i}\overrightarrow{w_4}(t,s) + b_{5i}\overrightarrow{w_5}(t,s) =$$

$$b_{1i}\cdot\left[\begin{pmatrix} 0 \\ 0 \\ \frac{k-a_4}{2} \\ 0 \\ 1 \end{pmatrix}\cdot e^{-\frac{a_4+k}{2}(t-s)}\cdot\cos\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}(t-s)\right) + \begin{pmatrix} 0 \\ 0 \\ -\frac{\sqrt{4b-(a_4-k)^2}}{2} \\ 0 \\ 0 \end{pmatrix}\cdot e^{-\frac{a_4+k}{2}(t-s)}\cdot\sin\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}(t-s)\right)\right] +$$

$$b_{2i}\cdot\left[\begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{4b-(a_4-k)^2}}{2} \\ 0 \\ 0 \end{pmatrix}\cdot e^{-\frac{a_4+k}{2}(t-s)}\cdot\cos\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}(t-s)\right) + \begin{pmatrix} 0 \\ 0 \\ \frac{a_4-k}{2} \\ 0 \\ 1 \end{pmatrix}\cdot e^{-\frac{a_4+k}{2}(t-s)}\cdot\sin\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}(t-s)\right)\right] +$$

$$b_{3i}\cdot\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\cdot e^{-a_7(t-s)} + b_{4i}\cdot\begin{pmatrix} 0 \\ -\frac{a_4a_5-a_4k-a_5^2+a_5k-b}{a_4} \\ k-a_5 \\ 0 \\ 1 \end{pmatrix}\cdot e^{-a_5(t-s)} +$$

$$b_{5i}\cdot\begin{pmatrix} -c(a_5+a_1-a_2) \\ -ca_3 \\ a_1-a_2+k \\ -\frac{(a_5+a_1-a_2)a_6c}{a_1-a_2+a_7} \\ 1 \end{pmatrix}\cdot e^{(a_1-a_2)(t-s)}$$

We have to find $b_{1i}, b_{2i}, b_{3i}, b_{4i}, b_{5i}$ in this representation. Taking into account that $C(s,s) = I$, where I is the identity (5×5) matrix, we can write:

$$\begin{aligned}\overrightarrow{C_i}(s,s) &= b_{1i}\cdot\begin{pmatrix} 0 \\ 0 \\ \frac{k-a_4}{2} \\ 0 \\ 1 \end{pmatrix} + b_{2i}\cdot\begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{4b-(a_4-k)^2}}{2} \\ 0 \\ 0 \end{pmatrix} + b_{3i}\cdot\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \\ &\quad b_{4i}\cdot\begin{pmatrix} 0 \\ -\frac{a_4a_5-a_4k-a_5^2+a_5k-b}{a_4} \\ k-a_5 \\ 0 \\ 1 \end{pmatrix} + b_{5i}\cdot\begin{pmatrix} -c(a_5+a_1-a_2) \\ -ca_3 \\ a_1-a_2+k \\ -\frac{(a_5+a_1-a_2)a_6c}{a_1-a_2+a_7} \\ 1 \end{pmatrix}\end{aligned}$$

Let us denote $\gamma_{32} = \frac{\sqrt{4b-(a_4-k)^2}}{2}, \gamma_{24} = -\frac{a_4a_5-a_4k-a_5^2+a_5k-b}{a_4}, \gamma_{15} = -c\left(a_5+a_1-a_2\right), \gamma_{25} = -ca_3, \gamma_{35} = a_1-a_2+k, \gamma_{45} = -\frac{(a_5+a_1-a_2)a_6c}{a_1-a_2+a_7}$, and define the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & \gamma_{15} \\ 0 & 0 & 0 & \gamma_{24} & \gamma_{25} \\ \frac{k-a_4}{2} & \gamma_{32} & 0 & k-a_5 & \gamma_{35} \\ 0 & 0 & 1 & 0 & \gamma_{45} \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and its inverse matrix

$$B^{-1} = \begin{pmatrix} -\frac{\gamma_{24}-\gamma_{25}}{\gamma_{15}\gamma_{24}} & -\frac{1}{\gamma_{24}} & 0 & 0 & 1 \\ -\frac{1}{2}\frac{\gamma_{24}(2\gamma_{35}-a_4+k)+\gamma_{25}(a_4-2a_5+3k)}{\gamma_{32}\gamma_{15}\gamma_{24}} & \frac{1}{2}\frac{a_4-3k+2a_5}{\gamma_{24}\gamma_{32}} & \frac{1}{\gamma_{32}} & 0 & \frac{1}{2}\frac{a_4-k}{\gamma_{32}} \\ -\frac{\gamma_{45}}{\gamma_{15}} & 0 & 0 & 1 & 0 \\ -\frac{\gamma_{25}}{\gamma_{15}\gamma_{24}} & \frac{1}{\gamma_{24}} & 0 & 0 & 0 \\ \frac{1}{\gamma_{15}} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us build the Cauchy matrix $C(t,s) = \left\{\overrightarrow{C}_i(t,s)\right\}_{1\le i\le 5}$, where $\overrightarrow{C}_i(t,s) = \sum_{j=1}^{5} b_{ji}\overrightarrow{w}_j(t,s)$, $1 \le i \le 5$.

We have to find b_{ji}, $1 \le i,j \le 5$ in this representation. Taking into account that $C(s,s) = I$, where I is the identity (5×5)-matrix, we can write: $\overrightarrow{C}_i(s,s) = \sum_{j=1}^{5} b_{ji}\overrightarrow{v}_j$, $1 \le i \le 5$.

Setting $i = 1,2,3,4,5$, we obtain

$$\overrightarrow{C}_1(s,s) = \sum_{j=1}^{5} b_{j1}\overrightarrow{v}_j = B\begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \\ b_{51} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \\ b_{51} \end{pmatrix} = \begin{pmatrix} -\frac{\gamma_{24}-\gamma_{25}}{\gamma_{15}\gamma_{24}} \\ -\frac{1}{2}\frac{\gamma_{24}(2\gamma_{35}-a_4+k)+\gamma_{25}(a_4-2a_5+3k)}{\gamma_{32}\gamma_{15}\gamma_{24}} \\ -\frac{\gamma_{45}}{\gamma_{15}} \\ -\frac{\gamma_{25}}{\gamma_{15}\gamma_{24}} \\ \frac{1}{\gamma_{15}} \end{pmatrix}$$

$$\overrightarrow{C}_2(s,s) = \sum_{j=1}^{5} b_{j2}\overrightarrow{v}_j = B\begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \\ b_{42} \\ b_{52} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \\ b_{42} \\ b_{52} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\gamma_{24}} \\ \frac{1}{2}\frac{a_4-3k+2a_5}{\gamma_{24}\gamma_{32}} \\ 0 \\ \frac{1}{\gamma_{24}} \\ 0 \end{pmatrix}$$

$$\overrightarrow{C}_3(s,s) = \sum_{j=1}^{5} b_{j3}\overrightarrow{v}_j = B\begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \\ b_{43} \\ b_{53} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \\ b_{43} \\ b_{53} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\gamma_{32}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\overrightarrow{C}_4(s,s) = \sum_{j=1}^{5} b_{j4}\overrightarrow{v}_j = B\begin{pmatrix} b_{14} \\ b_{24} \\ b_{34} \\ b_{44} \\ b_{54} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} b_{14} \\ b_{24} \\ b_{34} \\ b_{44} \\ b_{54} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\overrightarrow{C}_5(s,s) = \sum_{j=1}^{5} b_{j5} \overrightarrow{v}_j = B \begin{pmatrix} b_{15} \\ b_{25} \\ b_{35} \\ b_{45} \\ b_{55} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} b_{15} \\ b_{25} \\ b_{35} \\ b_{45} \\ b_{55} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2}\frac{a_4-k}{\gamma_{32}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Substituting the coefficients $b_{ji},\ 1 \le i,j \le 5$ into equality $\overrightarrow{C}_i(t,s) = \sum_{j=1}^{5} b_{ji} \overrightarrow{w}_j(t,s),\ \ 1 \le i \le 5$ we obtain

$$\overrightarrow{C}_1(t,s) = -\frac{\gamma_{24}-\gamma_{25}}{\gamma_{15}\gamma_{24}} \cdot \left[\begin{pmatrix} 0 \\ 0 \\ \frac{k-a_4}{2} \\ 0 \\ 1 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}(t-s)} \cdot \cos\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}(t-s)\right) + \begin{pmatrix} 0 \\ 0 \\ -\frac{\sqrt{4b-(a_4-k)^2}}{2} \\ 0 \\ 0 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}(t-s)} \cdot \sin\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}(t-s)\right) \right] -$$

$$\frac{1}{2}\frac{\gamma_{24}(2\gamma_{35}-a_4+k)+\gamma_{25}(a_4-2a_5+3k)}{\gamma_{32}\gamma_{15}\gamma_{24}} \cdot \left[\begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{4b-(a_4-k)^2}}{2} \\ 0 \\ 0 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}(t-s)} \cdot \cos\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}(t-s)\right) + \begin{pmatrix} 0 \\ 0 \\ \frac{a_4-k}{2} \\ 0 \\ 1 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}(t-s)} \cdot \sin\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}(t-s)\right) \right] -$$

$$\frac{\gamma_{45}}{\gamma_{15}} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot e^{-a_7(t-s)} - \frac{\gamma_{25}}{\gamma_{15}\gamma_{24}} \cdot \begin{pmatrix} 0 \\ -\frac{a_4a_5-a_4k-a_5^2+a_5k-b}{a_4} \\ a_5-k \\ 0 \\ 1 \end{pmatrix} \cdot e^{-a_5(t-s)} +$$

$$\frac{1}{\gamma_{15}} \cdot \begin{pmatrix} -c(a_5+a_1-a_2) \\ -ca_3 \\ a_1-a_2+k \\ -\frac{(a_5+a_1-a_2)a_6c}{a_1-a_2+a_7} \\ 1 \end{pmatrix} \cdot e^{(a_1-a_2)(t-s)}$$

$$\overrightarrow{C}_2(t,s) = -\frac{1}{\gamma_{24}} \cdot \left[\begin{pmatrix} 0 \\ 0 \\ \frac{k-a_4}{2} \\ 0 \\ 1 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}(t-s)} \cdot \cos\left(\frac{\sqrt{4b-(a_4-k)^2}}{2} (t-s) \right) + \begin{pmatrix} 0 \\ 0 \\ -\frac{\sqrt{4b-(a_4-k)^2}}{2} \\ 0 \\ 0 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}(t-s)} \cdot \sin\left(\frac{\sqrt{4b-(a_4-k)^2}}{2} (t-s) \right) \right] +$$

$$\frac{1}{2} \frac{a_4-3k+2a_5}{\gamma_{24}\gamma_{32}} \cdot \left[\begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{4b-(a_4-k)^2}}{2} \\ 0 \\ 0 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}(t-s)} \cdot \cos\left(\frac{\sqrt{4b-(a_4-k)^2}}{2} (t-s) \right) + \begin{pmatrix} 0 \\ 0 \\ \frac{a_4-k}{2} \\ 0 \\ 1 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}(t-s)} \cdot \sin\left(\frac{\sqrt{4b-(a_4-k)^2}}{2} (t-s) \right) \right] +$$

$$\frac{1}{\gamma_{24}} \cdot \begin{pmatrix} 0 \\ -\frac{a_4a_5-a_4k-a_5^2+a_5k-b}{a_4} \\ a_5-k \\ 0 \\ 1 \end{pmatrix} \cdot e^{-a_5(t-s)}$$

$$\overrightarrow{C}_3(t,s) = \frac{1}{\gamma_{32}} \cdot \left[\begin{pmatrix} 0 \\ 0 \\ \frac{\sqrt{4b-(a_4-k)^2}}{2} \\ 0 \\ 0 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}(t-s)} \cdot \cos\left(\frac{\sqrt{4b-(a_4-k)^2}}{2} (t-s) \right) + \begin{pmatrix} 0 \\ 0 \\ \frac{a_4-k}{2} \\ 0 \\ 1 \end{pmatrix} \cdot e^{-\frac{a_4+k}{2}(t-s)} \cdot \sin\left(\frac{\sqrt{4b-(a_4-k)^2}}{2} (t-s) \right) \right]$$

$$\overrightarrow{C}_4(t,s) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot e^{-a_7(t-s)}$$

$$\overrightarrow{C}_5\left(t,s\right)=\begin{pmatrix}0\\0\\\frac{k-a_4}{2}\\0\\1\end{pmatrix}\cdot e^{-\frac{a_4+k}{2}(t-s)}\cdot\cos\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}(t-s)\right)+$$
$$\begin{pmatrix}0\\0\\-\frac{\sqrt{4b-(a_4-k)^2}}{2}\\0\\0\end{pmatrix}\cdot e^{-\frac{a_4+k}{2}(t-s)}\cdot\sin\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}(t-s)\right)+$$
$$\frac{1}{2}\frac{a_4-k}{\gamma_{32}}\left[\begin{array}{c}\begin{pmatrix}0\\0\\\frac{\sqrt{4b-(a_4-k)^2}}{2}\\0\\0\end{pmatrix}\cdot e^{-\frac{a_4+k}{2}(t-s)}\cdot\cos\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}\left(t-s\right)\right)+\\\begin{pmatrix}0\\0\\\frac{a_4-k}{2}\\0\\1\end{pmatrix}\cdot e^{-\frac{a_4+k}{2}(t-s)}\cdot\sin\left(\frac{\sqrt{4b-(a_4-k)^2}}{2}\left(t-s\right)\right)\end{array}\right]$$

4. System with Uncertain Coefficient in the Distributed Control

Consider the following system of equations

$$\left\{\begin{array}{c}\frac{dV}{dt}=\beta V\left(t\right)-\gamma F\left(t\right)V\left(t\right)\\\frac{dC}{dt}=\zeta\left(m(t)\right)\alpha F\left(t\right)V\left(t\right)-\mu_c\left(C\left(t\right)-C^*\right)\\\frac{dF}{dt}=\rho C-\eta\gamma F\left(t\right)V\left(t\right)-\mu_f F\left(t\right)-\left(b+\triangle b(t)\right)u\left(t\right)\\\frac{dm}{dt}=\sigma V\left(t\right)-\mu_m m\left(t\right)\\\frac{du}{dt}=F\left(t\right)-F^*-ku\left(t\right)\end{array}\right. . \tag{15}$$

Appearing $\triangle b(t)$ in the third equation can be explained by the individual reaction of the human body on the drug. Of course sensitivity of different patients' reactions can be different and it can be variable in time. We assume below that $\triangle b(t)$ is essentially a bounded function.

This system can be rewritten in the form

$$\left\{\begin{array}{c}x_1'=\left(a_1-a_2\right)x_1+g_1\left(x_1(t),x_3(t)\right)\\x_2'=a_3x_1-a_5x_2+g_2\left(x_1(t),x_3(t)\right)\\x_3'=-a_8\,x_1+a_4x_2-a_4x_3-\left(b+\triangle b(t)\right)x_5+g_3\left(x_1(t),x_3(t)\right)\\x_4'=a_6x_1-a_7x_4\\x_5'=x_3-kx_5\end{array}\right. , \tag{16}$$

where $g_i(x_1(t),x_3(t))\left(t\right),\ 1\le i\le 3$ results of "mistakes" we made in the process of the linearization.

It is clear that the model described by systems (15) and (16) were obtained under the assumption that various factors $\triangle g_i(t)$ acting on the antigen, plasma cell and antibody concentrations, were neglected. In reality these factors act although they are "small". Denote the so-called right-hand sides $G_i(t)=g_i(x_1(t),x_3(t))+\triangle g_i(t)$ for $i=1,2,3$ and $G_i(t)=\triangle g_i(t)$ for $i=4,5$. Denote $F(t)=col\{G_1(t),...,G_5(t)\}$, assume that $F(t)\in L_\infty^5$.

Consider the system

$$X'=AX+\Delta B\left(t\right)X+F\left(t\right), \tag{17}$$

where

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{pmatrix}, \quad \Delta B\,(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\triangle\, b(t) \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The natural problem is to estimate an influence of the right-hand side $F(t)$ on the solution $X(t)$.

The general solution of the system

$$X' - AX = Z \tag{18}$$

can be represented in the following form (see, for example, [11,12])

$$X(t) = \int_0^t C\,(t,s)\,Z\,(s)\,ds + C\,(t,0)\,X\,(0)\,. \tag{19}$$

Without loss of generality, $X\,(0) = col\{0,0,0,0,0\}$. Substituting (19) into (17) we obtain

$$Z(t) - \Delta B\,(t)\int_0^t C\,(t,s)\,Z\,(s)\,ds = F(t), \tag{20}$$

which can be written in the operator form as

$$Z\,(t) = (\Omega Z)\,(t) + F\,(t)\,, \tag{21}$$

where the operator $\Omega : L_\infty^5 \to L_\infty^5$ (L_∞^5 is the space of five vector-functions with essentially bounded components) is defined by the equality

$$(\Omega Z)\,(t) = \Delta B\,(t)\int_0^t C\,(t,s)\,Z\,(s)\,ds.$$

Denote $||\Omega||$ the norm of the operator Ω.

Estimating $\|\Omega\|$ for $(a_4 - k)^2 - 4b > 0$, we obtain

$$\|\Omega\| \le \max_{1\le j\le 5}\left(ess\sup_{t\ge 0}\int_0^t \sum_{i=1}^{5}\left|(\Delta B\,(t)\,C(t,s))_{ij}\right| ds\right).$$

Denoting $Q_j = ess\sup_{t\ge 0}\int\limits_0^t \sum\limits_{i=1}^{5}\left|(\Delta B\,(t)\,C(t,s))_{ij}\right| ds$ and $\triangle b^* = ess\sup_{t\ge 0}|\triangle b(t)|$, we obtain

$$
\begin{aligned}
Q_1 &= \triangle b^* \left[\begin{array}{c} \left|\frac{\alpha_{24}(\alpha_{32}-\alpha_{35})-\alpha_{25}(\alpha_{32}-\alpha_{34})}{\alpha_{15}\alpha_{24}(\alpha_{31}-\alpha_{32})}\right| \frac{1}{|\lambda_1|}+ \\ \left|\frac{\alpha_{24}(\alpha_{31}-\alpha_{35})-\alpha_{25}(\alpha_{31}-\alpha_{34})}{\alpha_{15}\alpha_{24}(\alpha_{31}-\alpha_{32})}\right| \frac{1}{|\lambda_2|} + \left|\frac{\alpha_{25}}{a_5\alpha_{15}\alpha_{24}}\right| \end{array} \right], \\
Q_2 &= \triangle b^* \left[\left|\frac{\alpha_{32}-\alpha_{34}}{\alpha_{24}(\alpha_{31}-\alpha_{32})}\right| \frac{1}{|\lambda_1|} + \left|\frac{\alpha_{31}-\alpha_{34}}{\alpha_{24}(\alpha_{31}-\alpha_{32})}\right| \frac{1}{|\lambda_2|} + \frac{1}{|a_5\alpha_{24}|} \right], \\
Q_3 &= \triangle b^* \left[\frac{1}{|\alpha_{31}-\alpha_{32}|}\frac{1}{|\lambda_1|} + \frac{1}{|\alpha_{31}-\alpha_{32}|}\frac{1}{|\lambda_2|} \right], \\
Q_4 &= 0, \\
Q_5 &= \triangle b^* \left[\left|\frac{\alpha_{32}}{\alpha_{31}-\alpha_{32}}\right| \frac{1}{|\lambda_1|} + \left|\frac{\alpha_{31}}{\alpha_{31}-\alpha_{32}}\right| \frac{1}{|\lambda_2|} \right].
\end{aligned} \tag{22}
$$

Theorem 2. *Let the assumption of Theorem 1 be fulfilled,* $(a_4 - k)^2 > 4b$ *and the inequality* $\max_{1\le j\le 5}\{|Q_j|\} < 1$ *be true. Then system* (16) *is exponential stable.*

Proof. The inequality in the condition of Theorem 2 implies that the norm $\|\Omega\|$ of the operator Ω is less than one. In this case there exists the inverse operator $(I-\Omega)^{-1} : L^5_\infty \longrightarrow L^5_\infty$ and $Z = (I-\Omega)^{-1}F = (I+\Omega+\Omega^2+\ldots)F$. It is clear that $||Z||_{L^5_\infty} \le \frac{1}{1-||\Omega||}||F||_{L^5_\infty}$. It means that all components of the solution-vector Z of system (21) are bounded. The Cauchy matrix of system (16) satisfies the exponential estimate i.e., there exist such positive N, M that

$$|C_{ij}(t,s)| \le Ne^{-M(t-s)}, 0 \le s \le t < \infty.$$

Then all components of the solution-vector $X(t)$ of system (17) are bounded, according to representation (19). The exponential stability of the homogeneous system

$$X'(t) = AX(t) + \triangle B(t)X(t)$$

follows now from Bohl-Perron theorem (see, for example, [11] p. 500, [12] p. 93). □

Example 1. *Substituting the values from Remark 1 and setting* $k = 4, b = 1$ *we obtain*

$$Q_1 \le 327.0253788,\ \ Q_2 \le 0.000001437277837,\ \ Q_3 \le 1.154699764,\ \ Q_4 = 0,\ \ Q_5 \le 0.5773500802.$$

The inequality $327.0253788 \cdot \triangle b^* < 1$ *implies the inequality* $\max_{1\le j\le 5}\{|Q_j|\} < 1$.
Thus if, according to Theorem 2 $\triangle b^* < 0.003057866651$, *then the system* (16) *is exponentially stable.*

Let us estimate $\|\Omega\|$ for $(a_4 - k)^2 = 4b$. Denoting $P_j = ess\sup_{t\ge 0}\int\limits_0^t \sum\limits_{i=1}^5 \left|(\Delta B(t)\, C(t,s))_{ij}\right| ds$, we obtain

$$
\begin{aligned}
P_1 &= \triangle b^* \left[\begin{array}{c} \left|\frac{\beta_{24}\beta_{35}-\beta_{25}\beta_{34}}{\beta_{31}\beta_{15}\beta_{24}}\right| \frac{2}{|a_4+k|} + \left|\frac{\beta_{24}(\beta_{31}-\beta_{35})-\beta_{25}(\beta_{31}-\beta_{34})}{\beta_{31}\beta_{52}\beta_{24}\beta_{15}}\right| \left[\frac{4}{|a_4^2-k^2|} + \frac{2}{|a_4+k|}\right] \\ + \left|\frac{\beta_{25}}{\beta_{15}\beta_{24}}\right| \frac{1}{|a_5|} + \frac{1}{|\beta_{15}|}\frac{1}{|a_1-a_2|} \end{array} \right], \\
P_2 &= \triangle b^* \left[\left|\frac{\beta_{34}}{\beta_{24}\beta_{31}}\right| \frac{2}{|a_4+k|} + \left|\frac{\beta_{31}-\beta_{34}}{\beta_{31}\beta_{24}\beta_{52}}\right| \left[\frac{4}{|a_4^2-k^2|} + \frac{2}{|a_4+k|}\right] + \frac{1}{|\beta_{24}|}\frac{1}{|a_5|} \right], \\
P_3 &= \triangle b^* \left[\frac{1}{|\beta_{31}|}\frac{2}{|a_4+k|} + \frac{1}{|\beta_{31}\beta_{52}|}\left[\frac{4}{|a_4^2-k^2|} + \frac{2}{|a_4+k|}\right]\right], \\
P_4 &= 0, \\
P_5 &= \triangle b^* \frac{1}{|\beta_{52}|}\left[\frac{4}{|a_4^2-k^2|} + \frac{2}{|a_4+k|}\right].
\end{aligned} \tag{23}
$$

Theorem 3. *Let the assumption of Theorem 1 be fulfilled, $(a_4-k)^2 = 4b$ and the inequality $\max_{1\le j\le 5}\{|P_j|\} < 1$ be true. Then system (16) is exponential stable.*

The proof of Theorem 3 repeats the proof of Theorem 2.

Example 2. *Substituting the values from Remark 1 and setting $k = 1, b = 0.249999902$, we obtain the inequalities*

$$P_1 \le 4.735918812 \cdot 10^{13}, \;\; P_2 \le 2.047987177 \cdot 10^5, \;\; P_3 \le 9.999999608, \;\; P_4 = 0, \;\; P_5 \le 2.999999216.$$

The inequality $4.735918812 \cdot 10^{13} \cdot \triangle b^* < 1$ implies the inequality $\max_{1\le j\le 5}\{|P_j|\} < 1$.

Thus if $\triangle b^* < 2.111522684 \cdot 10^{-14}$, then the system (16) is exponentially stable, according to Theorem 3.

Let us estimate $\|\Omega\|$ for $(a_4-k)^2 - 4b < 0$. Denoting $R_j = ess\sup_{t\ge 0} \int_0^t \sum_{i=1}^5 \left|(\Delta B(t)\, C(t,s))_{ij}\right| ds$ we obtain

$$
\begin{aligned}
R_1 &= \triangle b^* \left[\begin{array}{c} \left|\frac{\gamma_{24}-\gamma_{25}}{\gamma_{15}\gamma_{24}}\right| \frac{2}{|a_4+k|} + \left|\frac{\gamma_{24}(2\gamma_{35}-a_4+k)+\gamma_{25}(a_4-2a_5+3k)}{\gamma_{32}\gamma_{15}\gamma_{24}}\right| \frac{1}{|a_4+k|} \\ + \left|\frac{\gamma_{25}}{\gamma_{15}\gamma_{24}}\right| \frac{1}{|a_5|} + \left|\frac{1}{\gamma_{15}}\right| \frac{1}{|a_1-a_2|} \end{array} \right], \\
R_2 &= \triangle b^* \left[\frac{1}{|\gamma_{24}|}\frac{2}{|a_4+k|} + \left|\frac{a_4-3k+2a_5}{\gamma_{24}\gamma_{32}}\right| \frac{1}{|a_4+k|} + \frac{1}{|\gamma_{24}|}\frac{1}{|a_5|}\right], \\
R_3 &= \triangle b^* \frac{1}{|\gamma_{32}|}\frac{2}{|a_4+k|}, \\
R_4 &= 0, \\
R_5 &= \triangle b^* \left[\frac{2}{|a_4+k|} + \left|\frac{a_4-k}{\gamma_{32}}\right| \frac{1}{|a_4+k|}\right].
\end{aligned} \tag{24}
$$

Theorem 4. *Let the assumption of Theorem 1 be fulfilled, $(a_4-k)^2 < 4b$ and the inequality $\max_{1\le j\le 5}\{|R_j|\} < 1$ be true. Then system (16) is exponential stable.*

The proof of Theorem 3 repeats the proof of Theorem 2.

Example 3. *Substituting the values from Remark 1 and setting $k = 1, b = 2$ we obtain*

$$R_1 \leq 133.8894553, \;\; R_2 \leq 6.173038374 \cdot 10^{-7}, \;\; R_3 \leq 1.511857554, \;\; R_4 = 0, \;\; R_5 \leq 0.7559286288.$$

The inequality $133.8894553 \cdot \triangle b^* < 1$ *implies the inequality* $\max_{1\leq j\leq 5}\left\{\left|R_j\right|\right\} < 1$.
Thus if $\triangle b^* < 0.7468848071$, *then the system* (16) *is exponentially stable, according to Theorem 4.*

5. Influence of Changes in the Right-Hand Side on Behavior of Solutions

Constructing system we neglect the influence of different factors that seem us nonessential. The Cauchy matrix $C(t,s)$ allows us to estimate the influences of all these factors on the solution.

Consider the system

$$Y'(t) - AY(t) = G(t) + \triangle G(t), \tag{25}$$

where the matrix A is defined by (8) is the matrix of the coefficients of system (7) and $\triangle G(t) \in L^5_\infty$ describes a change of the right-hand side. In the following assertion we estimate the difference between the solution-vector $Y(t) = col\{y_1(t), ..., y_5(t)\}$ of the system (25) and the solution $X(t) = col\{x_1(t), ..., x_5(t)\}$ of the system (7).

Theorem 5. *Under the assumption of Theorem 1 the system (7) is exponentially stable and the following inequality*

$$\|Y(t) - X(t)\| \leq \|C\| \, \|\triangle G(t)\|,$$

is true, where

$$\|C\| = \max_{1\leq i\leq 5}\left(\sup_{t\geq 0}\int_0^t \sum_{j=0}^{5}\left|c_{ij}(t,s)\right|\right)ds, \quad \|\triangle G(t)\| = \max_{1\leq i\leq 5} ess\sup_{t\geq 0}\left|\triangle G_i(t)\right|,$$

$$||Y(t) - X(t)|| = max_{1\leq i\leq 5} sup_{t\geq 0}|y_i(t) - x_i(t)|.$$

The proof follows from the representation of solution of system (7).

The estimates of $\|C\|$ can be obtained through the estimates of the elements of the Cauchy matrix obtained in Section 3.

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