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Two Types of Trilocality of Probability and Correlation Tensors

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Abstract: In this work, we discuss two types of trilocality of probability tensors (PTs) $\mathbf{P} = [\![P(a_1a_2a_3)]\!]$ over an outcome set Ω_3 and correlation tensors (CTs) $\mathbf{P} = [\![P(a_1a_2a_3|x_1x_2x_3)]\!]$ over an outcome-input set Δ_3 based on a triangle network and described by continuous (integral) and discrete (sum) trilocal hidden variable models (C-triLHVMs and D-triLHVMs). We say that a PT (or CT) $\mathbf{P}$ is C-trilocal (resp. D-trilocal) if it can be described by a C-triLHVM (resp. D-triLHVM). It is proved that a PT (resp. CT) is D-trilocal if and only if it can be realized in a triangle network by three shared separable states and a local POVM (resp. a set of local POVMs) performed at each node; a CT is C-trilocal (resp. D-trilocal) if and only if it can be written as a convex combination of the product deterministic CTs with a C-trilocal (resp. D-trilocal) PT as a coefficient tensor. Some properties of the sets consisting of C-trilocal and D-trilocal PTs (resp. C-trilocal and D-trilocal CTs) are proved, including their path-connectedness and partial star-convexity.

Keywords: C-trilocality; D-trilocality; bell locality; probability tensor; correlation tensor

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1. Introduction

Quantum networks [1–4] have recently attracted much interest as they have been identified as a promising platform for quantum information processing, such as long-distance quantum communication [5,6]. In an abstract sense, a quantum network consists of several sources, which distribute entangled quantum states to spatially separated nodes; then, the quantum information is processed locally in these nodes. This may be seen as a generalization of a classical causal model [7,8], where the shared classical information between the nodes is replaced by quantum states. Clearly, it is important to understand the quantum correlations that arise in such a quantum network. Recent developments have shown that the network structure and topology lead to novel notions of nonlocality [9,10], as well as new concepts of entanglement and separability [11–13], which differ from the traditional concepts and definitions [14,15]. Dealing with these new concepts requires theoretical tools for their analysis. Thus far, examples of entanglement criteria for the network scenario have been derived using the mutual information [11,12], the fidelity with pure states [12,13], or covariance matrices build from measurement probabilities [16,17]. According to Bell's local causality assumption [18,19], the different systems measured in the experiment are considered to be all in an initial joint "hidden" state λ, where λ is arbitrary and could even describe the state of the entire universe prior to the measurement choices. The probability $P(o|m, \lambda)$ of obtaining measurement outcome o of any particular system can depend arbitrarily on the global state λ and on the type m of measurement performed on that system, but not on the measurements performed on distant systems.

Focusing on quantum networks, a completely different approach to multipartite nonlocality was proposed [20–22]. For the case where distant observers share entanglement distributed by independent several sources, the observers may correlate distant quantum systems and establish strong correlations across the entire network by performing joint entangled measurements, such as the well-known Bell state measurement used in quantum teleportation [23]. It turns out that this situation is fundamentally different from standard multipartite nonlocality, and allows for radically novel phenomena. As regards correlations,

it is now possible to witness quantum nonlocality in experiments where all the observers perform a fixed measurement; i.e., they receive no input [24–27]. This effect of quantum nonlocality without inputs is remarkable, and radically departs from previous forms of quantum nonlocality [9].

Recently, Kraft et al. [28] demonstrated that the theory of quantum coherence provides powerful tools for analyzing correlations in quantum networks and provided a direct link between the theory of multisubspace coherence [29,30] and the approach to quantum networks using covariance matrices [16,17]. Patricia et al. [31] derived sufficient conditions for entanglement to give rise to genuine multipartite nonlocality in networks and found that any network where the parties are connected by bipartite pure entangled states is genuine multipartite nonlocal, independently of the amount of entanglement in the shared states and of the topology of the network. Šupić et al. [32] introduced a notion of genuine network quantum nonlocality and showed several examples of correlations that are genuine network nonlocal, considering the so-called bilocality network of entanglement swapping. Recently, Tavakoli et al. [33] contributed a review paper by discussing the main concepts, methods, results, and future challenges in the emerging topic of Bell nonlocality in networks. Some open problems were listed at the end of their paper. In particular, the authors said that, "in the triangle network with no inputs and binary outputs, the conjecture that the local and quantum sets are identical remains open".

When a triangle network consisting of three quantum systems S_1, S_2 and S_3 (refer to Figure 1 below) is locally measured one time, the probabilities $P(a_1, a_2, a_3)$ of obtaining outcomes a_1, a_2, a_3 at nodes S_1, S_2 and S_3 form a nonnegative tensor $\mathbf{P} = [\![P(a_1, a_2, a_3)]\!]$ over $\Omega_3 = [o_1] \times [o_2] \times [o_3]$ with

$$\sum_{a_1, a_2, a_3} P(a_1, a_2, a_3) = 1,$$

$[o_i]$ denotes the set consisting of outcomes $1, 2, \ldots, o_i$ at node S_i. We call it a probability tensor (PT) over Ω_3. When a triangle network is locally measured many times, the conditional probabilities $P(a_1 a_2 a_3 | x_1 x_2 x_3)$ of obtaining outcomes a_1, a_2, a_3 at nodes S_1, S_2 and S_3 form a nonnegative tensor $\mathbf{P} = [\![P(a_1, a_2, a_3 | x_1, x_2, x_3)]\!]$ over $\Delta_3 = \Omega_3 \times [m_1] \times [m_2] \times [m_3]$ with

$$\sum_{a_1, a_2, a_3} P(a_1, a_2, a_3 | x_1, x_2, x_3) = 1$$

for all $(x_1, x_2, x_3) \in [m_1] \times [m_2] \times [m_3]$, $[m_i]$ denotes the set consisting of inputs $1, 2, \ldots, m_i$ at node S_i. We call it a correlation tensor (CT) over Δ_3.

In this work, we aim to introduce and discuss two types of trilocality of PTs and CTs, called C-trilocality and D-trilocality, according to their descriptions of continuous (integral) and discrete (sum) the types of trilocal hidden variable models. In Section 2, we will define and discuss the C-trilocality and D-trilocality of a PT. Section 3 is devoted to introduce and discuss the C-trilocality and D-trilocality of a CT. In Section 4, we will give a summary and list some open questions.
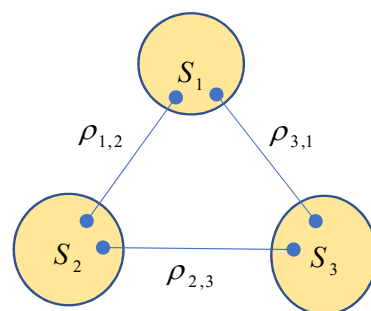

Figure 1. A triangle quantum network where the Hilbert spaces of systems S_1, S_2 and S_3 are $\mathcal{H}^{(1)} = \mathcal{H}_2^{(1)} \otimes \mathcal{H}_1^{(1)}, \mathcal{H}^{(2)} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)}$, and $\mathcal{H}^{(3)} = \mathcal{H}_1^{(3)} \otimes \mathcal{H}_2^{(3)}$, respectively.

2. Trilocality of Probability Tensors

In what follows, we use $\mathcal{H}_A$ and $\mathcal{H}_B$ to denote the finite-dimensional complex Hilbert spaces describing quantum systems A and B, respectively. The composite system of A and B is then described by the Hilbert space $\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$. We also use I_X and $\mathcal{D}_X$ to denote the identity operator on a Hilbert space $\mathcal{H}_X$ and the set of all quantum states of the system X described by $\mathcal{H}_X$, respectively, where $X = A, B$ and AB. We also use the notation $[m] = \{1, 2, \ldots, m\}$ for every positive integer m.

2.1. Triangle Quantum Networks

Considering a system-based network $\mathcal{N}$ with N nodes S_n (quantum systems), the topological structure of the network can be described by a directed graph $G(\mathcal{N}) = (V(\mathcal{N}), E(\mathcal{N}))$ with the set $V(\mathcal{N}) = \{S_1, S_2, \ldots, S_N\}$ of vertices and the set $E(\mathcal{N})$ of edges where $S_i S_j \in E(\mathcal{N})$ if and only if S_i and S_j share a resource $\rho_{i,j}$ (a quantum state of a system $\mathcal{H}_i \otimes \mathcal{H}_j$). Put $n(S_i) = \{S_j : S_i S_j \in E(\mathcal{N})\}$ and assume that each node shares a resource with at least one node, i.e., $n(S_i) \neq \varnothing$ for all $i = 1, 2, \ldots, N$. The state $\rho_{\mathcal{N}}$ of the network $\mathcal{N}$, called the network state, is the tensor product of all shared states $\rho_{i,j}$ in a certain order that you chose. Clearly, the feature of a network $\mathcal{N}$ is determined by its topology $G(\mathcal{N})$ together with its network state $\rho_{\mathcal{N}}$.

For example, for a triangle network $\mathcal{TN}$ given by Figure 1, we have

$$V(\mathcal{TN}) = \{S_1, S_2, S_3\}, E(\mathcal{TN}) = \{S_1 S_2, S_2 S_3, S_3 S_1\},$$

and the network state $\rho_{\mathcal{TN}}$ of $\mathcal{TN}$ reads

$$\rho_{\mathcal{TN}} = \rho_{1,2} \otimes \rho_{2,3} \otimes \rho_{3,1} \in \mathcal{D}(\mathcal{H}_1^{(1)} \otimes \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)} \otimes \mathcal{H}_1^{(3)} \otimes \mathcal{H}_2^{(3)} \otimes \mathcal{H}_2^{(1)}), \tag{1}$$

where

$$\rho_{1,2} \in \mathcal{D}(\mathcal{H}_1^{(1)} \otimes \mathcal{H}_1^{(2)}), \rho_{2,3} \in \mathcal{D}(\mathcal{H}_2^{(2)} \otimes \mathcal{H}_1^{(3)}), \rho_{3,1} \in \mathcal{D}(\mathcal{H}_2^{(3)} \otimes \mathcal{H}_2^{(1)}). \tag{2}$$

To explore the property of the network, a POVM measurement $M^{(n)} = \{M_{a_n}^{(n)}\}_{a_n=1}^{d_n}$ is performed at each node S_n. Put $\mathcal{M} = \{M^{(n)}\}_{n=1}^N$. The observed probability distribution over the outcomes reads

$$P_{\mathcal{N}}^{\mathcal{M}}(a_1, \ldots, a_N) = \text{tr}[(\otimes_{n=1}^N M_{a_n}^{(n)}) \widetilde{\rho_{\mathcal{N}}}] \tag{3}$$

where $\otimes_{n=1}^N M_{a_n}^{(n)}$ are positive operators on the Hilbert space $\mathcal{H}_{\text{net}} := \otimes_{i=1}^N \mathcal{H}^{(i)}$, $\widetilde{\rho_{\mathcal{N}}}$ denotes the state of $\mathcal{H}_{\text{net}}$ obtained from the network state $\rho_{\mathcal{N}}$ after performing the canonical unitary transformation $\mathcal{T}$ from the space $\mathcal{H}_{\text{state}}$ of $\rho_{\mathcal{N}}$ onto $\mathcal{H}_{\text{net}}$, i.e., $\widetilde{\rho_{\mathcal{N}}} = \mathcal{T} \rho_{\mathcal{N}} \mathcal{T}^\dagger$. We call $\widetilde{\rho_{\mathcal{N}}}$ the measurement state.

Let us consider the triangle network given by Figure 1. To find out the state $\widetilde{\rho_{\mathcal{TN}}}$, we write

$$\rho_{1,2} = \sum_{i=1}^r \alpha(i) X_i^{(1)} \otimes X_i^{(2)} \in \mathcal{D}(\mathcal{H}_1^{(1)} \otimes \mathcal{H}_1^{(2)}),$$

$$\rho_{2,3} = \sum_{j=1}^s \beta(j) Y_j^{(2)} \otimes Y_j^{(3)} \in \mathcal{D}(\mathcal{H}_2^{(2)} \otimes \mathcal{H}_1^{(3)}),$$

$$\rho_{3,1} = \sum_{k=1}^t \gamma(k) Z_k^{(3)} \otimes Z_k^{(1)} \in \mathcal{D}(\mathcal{H}_2^{(3)} \otimes \mathcal{H}_2^{(1)}).$$

Thus, the network state reads

$$\rho_{\mathcal{TN}} = \sum_{i,j,k} \alpha(i) \beta(j) \gamma(k) (X_i^{(1)} \otimes X_i^{(2)}) \otimes (Y_j^{(2)} \otimes Y_j^{(3)}) \otimes (Z_k^{(3)} \otimes Z_k^{(1)}),$$

resulting in the measurement state

$$\widetilde{\rho_{\mathcal{TN}}} = \sum_{i,j,k} \alpha(i)\beta(j)\gamma(k)(Z_k^{(1)} \otimes X_i^{(1)}) \otimes (X_i^{(2)} \otimes Y_j^{(2)}) \otimes (Y_j^{(3)} \otimes Z_k^{(3)}),$$

a state of

$$\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \mathcal{H}^{(3)} = (\mathcal{H}_2^{(1)} \otimes \mathcal{H}_1^{(1)}) \otimes (\mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)}) \otimes (\mathcal{H}_1^{(3)} \otimes \mathcal{H}_2^{(3)}).$$

Here, the action of $\mathcal{T}$ is

$$|x_1^{(1)} x_1^{(2)} x_2^{(2)} x_1^{(3)} x_2^{(3)} x_2^{(1)}\rangle \mapsto |x_2^{(1)} x_1^{(1)}\rangle \otimes |x_1^{(2)} x_2^{(2)}\rangle \otimes |x_1^{(3)} x_2^{(3)}\rangle$$

for all $|x_j^{(i)}\rangle \in \mathcal{H}_j^{(i)}$. The joint probability is given by

$$
\begin{aligned}
P_{\mathcal{TN}}^{\mathcal{M}}(a_1, a_2, a_3) &= \mathrm{tr}[(\otimes_{n=1}^3 M_{a_n}^{(n)})\widetilde{\rho_{\mathcal{TN}}}] \\
&= \sum_{i,j,k} \alpha(i)\beta(j)\gamma(k)\mathrm{tr}[M_{a_1}^{(1)}(Z_k^{(1)} \otimes X_i^{(1)})] \\
&\quad \times \mathrm{tr}[M_{a_2}^{(2)}(X_i^{(2)} \otimes Y_j^{(2)})] \times \mathrm{tr}[M_{a_3}^{(3)}(Y_j^{(3)} \otimes Z_k^{(3)})].
\end{aligned}
\tag{4}
$$

In particular, when the shared states $\rho_{i,j}$ are separable, they can be written as convex combinations of product states. Then, we can assume that the coefficients $\alpha(i), \beta(j), \gamma(k)$ are probability distributions (PDs) of i, j, k and that the operators $X_i^{(t)}, Y_j^{(t)}$ and $Z_K^{(t)}$ are all states. Put

$$P_1(a_1|k,i) = \mathrm{tr}[M_{a_1}^{(1)}(Z_k^{(1)} \otimes X_i^{(1)})],$$

$$P_2(a_2|i,j) = \mathrm{tr}[M_{a_2}^{(2)}(X_i^{(2)} \otimes Y_j^{(2)})],$$

$$P_3(a_3|j,k) = \mathrm{tr}[M_{a_3}^{(3)}(Y_j^{(3)} \otimes Z_k^{(3)})],$$

which are PDs of outcomes a_1, a_2, a_3, respectively. Thus, in this case, Equation (4) becomes

$$P_{\mathcal{TN}}^{\mathcal{M}}(a_1, a_2, a_3) = \sum_{i,j,k} \alpha(i)\beta(j)\gamma(k)P_1(a_1|k,i)P_2(a_2|i,j)P_3(a_3|j,k) \tag{5}$$

for all possible a_1, a_2, a_3. This is just the motivation for introducing the concept of D-trilocality; see Section 2.2.

2.2. Trilocality of Probability Tensors

The central question is whether a given probability distribution may originate from a network with a given topology [28]. The usual Bell nonlocality of a quantum state or a quantum network is the property that is exhibited by performing a set of non-compatible local POVM measurement.

Renou et al. [9] pointed out that quantum nonlocality can be demonstrated without the need of having various input settings, but only by considering the joint statistics of fixed local measurement outputs. They call this property quantum nonlocality without inputs. For example, when a triangle network is measured by just one local POVM $\mathcal{M}$, joint probabilities $P_{\mathcal{TN}}^{\mathcal{M}}(a_1, a_2, a_3)$ are obtained, which form a nonnegative tensor $\mathbf{P}_{\mathcal{TN}}^{\mathcal{M}} = [\![P_{\mathcal{TN}}^{\mathcal{M}}(a_1, a_2, a_3)]\!]$ over the index set $\Omega_3 = [o_1] \times [o_2] \times [o_3]$. Generally, when a function $P : \Omega_3 \to [0,1]$ satisfies the completeness condition:

$$\sum_{a_1, a_2, a_3} P(a_1, a_2, a_3) = 1,$$

we call it a probability tensor (PT) over Ω_3, denoted by $\mathbf{P} = [\![P(a_1, a_2, a_3)]\!]$.

Fritz in ([22] Definition 2.12) called a probability tensor $\mathbf{P} = [\![P(a_1, a_2, a_3)]\!]$ over Ω_3 *classical in* C_3 if it can be written as

$$
\begin{aligned}
P(a_1, a_2, a_3) &= \iiint_\Lambda q_1(\lambda_1)q_2(\lambda_2)q_3(\lambda_3)P_1(a_1|\lambda_3\lambda_1)P_2(a_2|\lambda_1\lambda_2) \\
&\times P_3(a_3|\lambda_2\lambda_3)\mathrm{d}\lambda_1\mathrm{d}\lambda_2\mathrm{d}\lambda_3
\end{aligned} \tag{6}
$$

for appropriate (conditional) distributions $q_1(\lambda_1), q_2(\lambda_2), q_3(\lambda_3), P_1(a_1|\lambda_3\lambda_1), P_2(a_2|\lambda_1\lambda_2)$, and $P_3(a_3|\lambda_2\lambda_3)$. It was proved ([22] Proposition 2.13) that classical correlations in C_3 are monogamous in the sense that a_1 is independent of λ_1 (i.e., $I(a_1 : \lambda_1) = 0$) and a_3 is independent of λ_2 (i.e., $I(a_3 : \lambda_2) = 0$) whenever $P(a_1 = a_3) = 1$. Since the representation (6) is given by the integral of hidden variables, we call it a *continuous trilocal hidden variable model* (C-triLHVM) for $\mathbf{P}$.

Motivated by this work, we introduce the following concepts of trilocality of tripartite PTs.

Definition 1. *Let* $\mathbf{P} = [\![P(a_1, a_2, a_3)]\!]$ *be a PT over* Ω_3.

(1) $\mathbf{P}$ is said to be *C-trilocal* if it has a C-triLHVM:

$$
\begin{aligned}
P(a_1, a_2, a_3) &= \iiint_{\Lambda_1 \times \Lambda_2 \times \Lambda_3} q_1(\lambda_1)q_2(\lambda_2)q_3(\lambda_3)P_1(a_1|\lambda_3\lambda_1)P_2(a_2|\lambda_1\lambda_2) \\
&\times P_3(a_3|\lambda_2\lambda_3)\mathrm{d}\mu_1(\lambda_1)\mathrm{d}\mu_2(\lambda_2)\mathrm{d}\mu_3(\lambda_3)
\end{aligned} \tag{7}
$$

for some product measure space

$$(\Lambda, \Sigma, \mu) = (\Lambda_1 \times \Lambda_2 \times \Lambda_3, \Sigma_1 \times \Sigma_2 \times \Sigma_3, \mu_1 \times \mu_2 \times \mu_3),$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\mathrm{d}\mu(\lambda) = \mathrm{d}\mu_1(\lambda_1)\mathrm{d}\mu_2(\lambda_2)\mathrm{d}\mu_3(\lambda_3)$, and

(a) $q_j(\lambda_j)$ is a density function (DF) of λ_j, i.e., $q_j(\lambda_j) \geq 0$ for all λ_j in Λ_j such that $\int_{\Lambda_j} q_j(\lambda_j)\mathrm{d}\mu_j(\lambda_j) = 1$;

(b) $P_1(a_1|\lambda_3\lambda_1), P_2(a_2|\lambda_1\lambda_2)$ and $P_3(a_3|\lambda_2\lambda_3)$, called response functions (RSs) at nodes 1, 2 and 3, are PDs of a_1, a_2 and a_3, respectively, for each $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ in Λ and are Ω-measurable on Λ w.r.t. $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ for each $a = (a_1, a_2, a_3)$ in Ω_3.

(2) $\mathbf{P}$ is said to be *D-trilocal* if it has a D-triLHVM:

$$
P(a_1, a_2, a_3) = \sum_{\lambda_1=1}^{n_1} \sum_{\lambda_2=1}^{n_2} \sum_{\lambda_3=1}^{n_3} q_1(\lambda_1)q_2(\lambda_2)q_3(\lambda_3)P_1(a_1|\lambda_3\lambda_1)P_2(a_2|\lambda_1\lambda_2)P_3(a_3|\lambda_2\lambda_3) \tag{8}
$$

for all $a_k \in [o_k] (k = 1, 2, 3)$, where $q_k(\lambda_k), P_1(a_1|\lambda_3\lambda_1), P_2(a_2|\lambda_1\lambda_2)$ and $P_3(a_3|\lambda_2\lambda_3)$ are PDs of λ_k, a_1, a_2 and a_3, respectively.

(3) $\mathbf{P}$ is said to be *C-nontrilocal* (resp. *D-nontrilocal*) if it is not C-trilocal (resp. not D-trilocal).
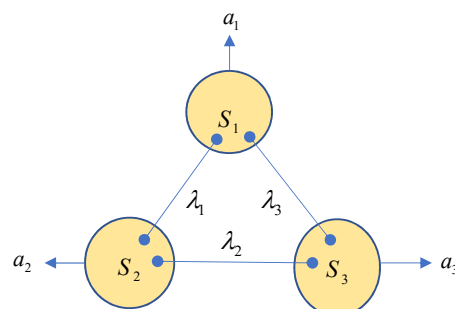
Please refer to Figure 2.

Figure 2. A trilocal scenario.

We use $\mathcal{PT}^{\text{C-trilocal}}(\Omega_3)$ and $\mathcal{PT}^{\text{D-trilocal}}(\Omega_3)$ to denote the sets of all C-trilocal and D-trilocal PTs over Ω_3, respectively. Obviously, $\mathcal{PT}^{\text{C-trilocal}}(\Omega_3) \supset \mathcal{PT}^{\text{D-trilocal}}(\Omega_3)$.

When **P** has a C-triLHVM (7), by letting

$$\mathrm{d}\gamma_k(\lambda_k) = q_k(\lambda_k)\mathrm{d}\mu_k(\lambda_k)(k = 1, 2, 3),$$

equivalently, defining measures γ_k on Σ_k as

$$\gamma_k(E_k) = \int_{\Lambda_k} \chi_{E_k}(\lambda_k) q_k(\lambda_k) \mathrm{d}\mu_k(\lambda_k), \ \forall E_k \in \Sigma_k,$$

where $\chi_{E_k}(\lambda_k)$ is the characteristic function of E_k, we obtain a product probability space

$$(\Lambda, \Sigma, \gamma) = (\Lambda_1 \times \Lambda_2 \times \Lambda_3, \Sigma_1 \times \Sigma_2 \times \Sigma_3, \gamma_1 \times \gamma_2 \times \gamma_3).$$

In this setting, the C-triLHVM (7) becomes

$$P(a_1, a_2, a_3) = \int_{\Lambda} P_1(a_1|\lambda_3\lambda_1) P_2(a_2|\lambda_1\lambda_2) P_3(a_3|\lambda_2\lambda_3) \mathrm{d}\gamma(\lambda), \tag{9}$$

where $\mathrm{d}\gamma(\lambda) = \mathrm{d}\gamma_1(\lambda_1)\mathrm{d}\gamma_2(\lambda_2)\mathrm{d}\gamma_3(\lambda_3)$.

Conversely, every C-triLHVM (9) can be written as a C-triLHVM (7) by letting $q_k(\lambda_k) \equiv 1$. This leads to the following conclusion.

Proposition 1. *A tripartite PT* $\mathbf{P} = [\![P(a_1, a_2, a_3)]\!]$ *over* Ω_3 *is C-trilocal if and only if it admits a C-triLHVM (9) for a product probability space*

$$(\Lambda, \Sigma, \gamma) = (\Lambda_1 \times \Lambda_2 \times \Lambda_3, \Sigma_1 \times \Sigma_2 \times \Sigma_3, \gamma_1 \times \gamma_2 \times \gamma_3).$$

Example 1. *Consider the PT* $\mathbf{P}_{\text{cube}} = [\![P(a_1, a_2, a_3)]\!]$ *over* Ω_3 *defined by Riemann integral*

$$P(a_1, a_2, a_3) = \iiint_{[0,1]^3} P_1(a_1|\lambda_3\lambda_1) P_2(a_2|\lambda_1\lambda_2) P_3(a_3|\lambda_2\lambda_3) \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \mathrm{d}\lambda_3, \tag{10}$$

where

$$P_1(a_1|\lambda_3\lambda_1) = \frac{\cos(a_1\lambda_3\lambda_1/o_1)}{\sum_{k_1=1}^{o_1} \cos(k_1\lambda_3\lambda_1/o_1)},$$

$$P_2(a_2|\lambda_1\lambda_2) = \frac{\cos(a_2\lambda_1\lambda_2/o_2)}{\sum_{k_2=1}^{o_2} \cos(k_2\lambda_1\lambda_2/o_2)},$$

$$P_3(a_3|\lambda_2\lambda_3) = \frac{\cos(a_3\lambda_2\lambda_3/o_3)}{\sum_{k_3=1}^{o_3} \cos(k_3\lambda_2\lambda_3/o_3)},$$

which are PDs of a_1, a_2, a_3, *respectively, and measurable w.r.t. Lebesgue measure* $(\gamma_1, \gamma_2, \gamma_3)$ *on* $\Lambda = [0,1]^3$. $\mathbf{P}_{\text{cube}}$ *is clearly a C-trilocal PT over* Ω_3 *using Proposition 1.*

Moreover, if we replace the space $\Lambda = [0,1]^3$ of hidden variables in Example 1 with $\Lambda = [-1,1]^3$ and take $p_i(\lambda_i) = \frac{1}{2}$ for $i = 1, 2, 3$, then the PT $\mathbf{P} = [\![P(a_1, a_2, a_3)]\!]$ defined by

$$P(a_1, a_2, a_3) = \iiint_{[-1,1]^3} p_1(\lambda_1)p_2(\lambda_2)p_3(\lambda_3)P_1(a_1|\lambda_3\lambda_1) P_2(a_2|\lambda_1\lambda_2) P_3(a_3|\lambda_2\lambda_3) \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \mathrm{d}\lambda_3 \tag{11}$$

is also C-trilocal.

Question 1. Consider the PT $\mathbf{P}_{\text{ball}} = [\![P(a_1, a_2, a_3)]\!]$ over Ω_3 given by Riemann integral

$$P(a_1, a_2, a_3) = \frac{3}{4\pi} \iiint_{\Lambda} P_1(a_1|\lambda_3\lambda_1) P_2(a_2|\lambda_1\lambda_2) P_3(a_3|\lambda_2\lambda_3) \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \mathrm{d}\lambda_3, \tag{12}$$

where Λ denotes the closed unit ball in $\mathbb{R}^3$ and the PDs $P_1(a_1|\lambda_3\lambda_1), P_2(a_2|\lambda_1\lambda_2)$ and $P_3(a_3|\lambda_2\lambda_3)$ are as in Example 1. An interesting question is whether $\mathbf{P}_{\text{ball}}$ is C-trilocal.

It is remarkable to mention that a C-triLHVM for a PT must be given by an integral that is taken over a *product space* $\Lambda_1 \times \Lambda_2 \times \Lambda_3$ due to the independence of the hidden variables λ_1, λ_2 and λ_3. It is also noted that the integrand must be a product of the three DFs of λ_1, λ_2 and λ_3 and the three PDs of a_1, a_2 and a_3 with parameters $(\lambda_3, \lambda_1), (\lambda_1, \lambda_2)$ and (λ_2, λ_3), respectively. Although the unit ball Λ in Question 1 is homeomorphic to the unit cube $[0, 1]^3$ or $[-1, 1]^3$, the integrand may be changed as the one that is not of the desired form. Thus, the answer to Question 1 may be very hard.

Definition 2. *A tripartite PT* $\mathbf{P} = \llbracket P(a_1, a_2, a_3) \rrbracket$ *over* Ω_3 *is said to be tri-quantum if there exists a* $\mathcal{TN}$ *with the state* $\rho_{\mathcal{TN}}$ *and a local POVM* $\mathcal{M} = M^{(1)} \otimes M^{(2)} \otimes M^{(3)}$ *such that* $\mathbf{P} = \mathbf{P}_{\mathcal{TN}}^{\mathcal{M}}$, *i.e.,*

$$P(a_1, a_2, a_3) = P_{\mathcal{TN}}^{\mathcal{M}}(a_1, a_2, a_3) := \text{tr}[(\otimes_{n=1}^3 M_{a_n}^{(n)})\widetilde{\rho_{\mathcal{TN}}}], \ \forall a_k \in [o_k]. \tag{13}$$

In particular, when the shares' states $\rho_{i,j}$ *can be chosen as separable states, we say that* $\mathbf{P}$ *is separable tri-quantum.*

Definition 3. *A triangle network* $\mathcal{TN}$ *given by Figure 1 is said to be C-trilocal (resp. D-trilocal) if, for every local POVM* $\mathcal{M} = M^{(1)} \otimes M^{(2)} \otimes M^{(3)}$, *where* $M^{(k)} = \{M_{a_k}^{(k)}\}_{a_k=1}^{d_k}$, *the generated PT* $\mathbf{P}_{\mathcal{TN}}^{\mathcal{M}} = \llbracket P_{\mathcal{TN}}(a_1, a_2, a_3) \rrbracket$ *is C-trilocal (resp. D-trilocal). It is said to be non C-trilocal (resp. non D-trilocal) if it is not C-trilocal (resp. non D-trilocal), i.e., there exists an* $\mathcal{M} = \{M^{(k)}\}_{k=1}^3$ *such that PT* $\mathbf{P}_{\mathcal{TN}}^{\mathcal{M}}$ *is non-C-trilocal (resp. non-D-trilocal), referring to Figure 3.*
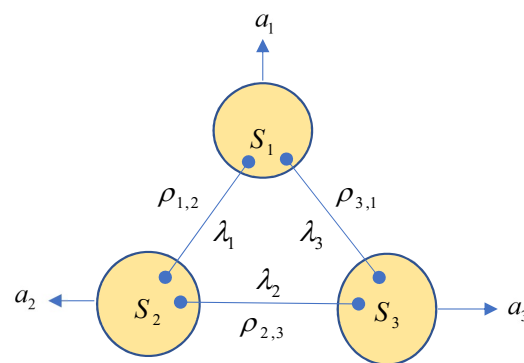

Figure 3. A trilocal triangle network.

Proposition 2. *Every separable (i.e., all shared states* $\rho_{i,j}$ *are separable) triangle network* $\mathcal{TN}$ *given by Figure 1 is D-trilocal.*

Proof. Suppose that the $\mathcal{TN}$ given by Figure 1 is separable. Then, the shared states $\rho_{s,t}$ are separable, i.e., there exist scalars $x_{\lambda_1}, y_{\lambda_2}, z_{\lambda_3} \in [0, 1]$ satisfying

$$\sum_{\lambda_1=1}^{n_1} x_{\lambda_1} = 1, \sum_{\lambda_2=1}^{n_2} y_{\lambda_2} = 1, \sum_{\lambda_3=1}^{n_3} z_{\lambda_3} = 1,$$

such that

$$\rho_{1,2} = \sum_{\lambda_1=1}^{n_1} x_{\lambda_1} \rho_1^{(1)}(\lambda_1) \otimes \rho_1^{(2)}(\lambda_1) \in \mathcal{D}(\mathcal{H}_1^{(1)} \otimes \mathcal{H}_1^{(2)}),$$

$$\rho_{2,3} = \sum_{\lambda_2=1}^{n_2} y_{\lambda_2} \rho_2^{(2)}(\lambda_2) \otimes \rho_1^{(3)}(\lambda_2) \in \mathcal{D}(\mathcal{H}_2^{(2)} \otimes \mathcal{H}_1^{(3)}),$$

$$\rho_{3,1} = \sum_{\lambda_3=1}^{n_3} z_{\lambda_3} \rho_2^{(3)}(\lambda_3) \otimes \rho_2^{(1)}(\lambda_3) \in \mathcal{D}(\mathcal{H}_2^{(3)} \otimes \mathcal{H}_2^{(1)}),$$

where $\rho_t^{(s)}(r) \in \mathcal{D}(\mathcal{H}_t^{(s)})$. Thus, the network state reads

$$\rho_{\mathcal{TN}} = \sum_{\lambda_1,\lambda_2,\lambda_3} x_{\lambda_1} y_{\lambda_2} z_{\lambda_3} \rho_1^{(1)}(\lambda_1) \otimes \rho_1^{(2)}(\lambda_1) \otimes \rho_2^{(2)}(\lambda_2) \otimes \rho_1^{(3)}(\lambda_2) \otimes \rho_2^{(3)}(\lambda_3) \otimes \rho_2^{(1)}(\lambda_3),$$

which is a state of system $\mathcal{H}_1^{(1)} \otimes \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)} \otimes \mathcal{H}_1^{(3)} \otimes \mathcal{H}_2^{(3)} \otimes \mathcal{H}_2^{(1)}$, and then the measurement state is

$$\widetilde{\rho_{\mathcal{TN}}} = \sum_{\lambda_1,\lambda_2,\lambda_3} x_{\lambda_1} y_{\lambda_2} z_{\lambda_3} (\rho_2^{(1)}(\lambda_3) \otimes \rho_1^{(1)}(\lambda_1)) \otimes (\rho_1^{(2)}(\lambda_1) \otimes \rho_2^{(2)}(\lambda_2)) \otimes (\rho_1^{(3)}(\lambda_2) \otimes \rho_2^{(3)}(\lambda_3)),$$

being a state of system

$$\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \mathcal{H}^{(3)} = (\mathcal{H}_2^{(1)} \otimes \mathcal{H}_1^{(1)}) \otimes (\mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)}) \otimes (\mathcal{H}_1^{(3)} \otimes \mathcal{H}_2^{(3)}).$$

For every local POVM measurement, $\mathcal{M} = M^{(1)} \otimes M^{(2)} \otimes M^{(3)}$ of system $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \mathcal{H}^{(3)}$, where $M^{(k)} = \{M_{a_k}^{(k)}\}_{a_k=1}^{d_k}$, we have

$$
\begin{aligned}
P_{\mathcal{TN}}^{\mathcal{M}}(a_1, a_2, a_3) &= \mathrm{tr}[(\otimes_{n=1}^3 M_{a_n}^{(n)})\widetilde{\rho_{\mathcal{TN}}}] \\
&= \sum_{\lambda_1,\lambda_2,\lambda_3} x_{\lambda_1} y_{\lambda_2} z_{\lambda_3} \mathrm{tr}[M_{a_1}^{(1)}(\rho_1^{(1)}(\lambda_1) \otimes \rho_2^{(1)}(\lambda_3))] \\
&\quad \times \mathrm{tr}[M_{a_2}^{(2)}(\rho_1^{(2)}(\lambda_1) \otimes \rho_2^{(2)}(\lambda_2))] \mathrm{tr}[M_{a_3}^{(3)}(\rho_1^{(3)}(\lambda_2) \otimes \rho_2^{(3)}(\lambda_3))] \\
&= \sum_{\lambda_1,\lambda_2,\lambda_3} q_1(\lambda_1) q_2(\lambda_2) q_3(\lambda_3) P_1(a_1|\lambda_3\lambda_1) P_2(a_2|\lambda_1\lambda_2) P_3(a_3|\lambda_2\lambda_3),
\end{aligned}
$$

for all $a_k \in [o_k]$, where $q_1(\lambda_1) = x_{\lambda_1}, q_2(\lambda_2) = y_{\lambda_2}, q_3(\lambda_3) = z_{\lambda_3}$ and

$$P_1(a_1|\lambda_3\lambda_1) = \mathrm{tr}[M_{a_1}^{(1)}(\rho_1^{(1)}(\lambda_1) \otimes \rho_2^{(1)}(\lambda_3))],$$

$$P_2(a_2|\lambda_1\lambda_2) = \mathrm{tr}[M_{a_2}^{(2)}(\rho_1^{(2)}(\lambda_1) \otimes \rho_2^{(2)}(\lambda_2))],$$

$$P_3(a_3|\lambda_2\lambda_3) = \mathrm{tr}[M_{a_3}^{(3)}(\rho_1^{(3)}(\lambda_2) \otimes \rho_2^{(3)}(\lambda_3))].$$

Clearly,

$$\{q_k(\lambda_k)\}_{\lambda_k \in [n_k]}, \{P_1(a_1|\lambda_3\lambda_1)\}_{a_1 \in [o_1]}, \{P_2(a_2|\lambda_1\lambda_2)\}_{a_2 \in [o_2]}, \{P_3(a_3|\lambda_2\lambda_3)\}_{a_3 \in [o_3]}$$

are PDs. It follows from Definition 3 that the triangle network $\mathcal{TN}$ given by Figure 1 is D-trilocal. The proof is completed. $\square$

Proposition 3. *A PT* **P** *over* Ω_3 *is D-trilocal if and only if it is separable tri-quantum.*

Proof. The sufficiency is given by Proposition 2. To show the necessity, we let $\mathbf{P} = \{P(a_1, a_2, a_3)\}$ be a D-trilocal PT over Ω_3. Then, it can be written as (8). Choose Hilbert spaces

$$\mathcal{H}_1^{(1)} = \mathcal{H}_1^{(2)} = \mathbb{C}^{n_1}, \mathcal{H}_2^{(2)} = \mathcal{H}_1^{(3)} = \mathbb{C}^{n_2}, \mathcal{H}_2^{(1)} = \mathcal{H}_2^{(3)} = \mathbb{C}^{n_3},$$

take their orthonormal bases $\{|\lambda_3\rangle\}_{\lambda_3=1}^{n_3}$, $\{|\lambda_1\rangle\}_{\lambda_1=1}^{n_1}$ and $\{|\lambda_2\rangle\}_{\lambda_2=1}^{n_2}$, respectively, and put

$$\mathcal{H}^{(1)} = \mathcal{H}_2^{(1)} \otimes \mathcal{H}_1^{(1)} = \mathbb{C}^{n_3} \otimes \mathbb{C}^{n_1}, \mathcal{H}^{(2)} = \mathcal{H}_1^{(2)} \otimes \mathcal{H}_2^{(2)} = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}, \mathcal{H}^{(3)} = \mathcal{H}_1^{(3)} \otimes \mathcal{H}_2^{(3)} = \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$$

and choose separable states

$$\rho_{1,2} = \sum_{\lambda_1=1}^{n_1} q_1(\lambda_1)|\lambda_1\rangle\langle\lambda_1| \otimes |\lambda_1\rangle\langle\lambda_1| \in \mathcal{D}(\mathcal{H}_1^{(1)}\otimes\mathcal{H}_1^{(2)}) = \mathcal{D}(\mathbb{C}^{n_1}\otimes\mathbb{C}^{n_1}),$$

$$\rho_{2,3} = \sum_{\lambda_2=1}^{n_2} q_2(\lambda_2)|\lambda_2\rangle\langle\lambda_2| \otimes |\lambda_2\rangle\langle\lambda_2| \in \mathcal{D}(\mathcal{H}_2^{(2)}\otimes\mathcal{H}_1^{(3)}) = \mathcal{D}(\mathbb{C}^{n_2}\otimes\mathbb{C}^{n_2}),$$

$$\rho_{3,1} = \sum_{\lambda_3=1}^{n_3} q_3(\lambda_3)|\lambda_3\rangle\langle\lambda_3| \otimes |\lambda_3\rangle\langle\lambda_3| \in \mathcal{D}(\mathcal{H}_2^{(3)}\otimes\mathcal{H}_2^{(1)}) = \mathcal{D}(\mathbb{C}^{n_3}\otimes\mathbb{C}^{n_3}),$$

then we obtain a triangle network $\mathcal{TN}$ with the network state

$$
\begin{aligned}
\rho_{\mathcal{TN}} &= \rho_{1,2}\otimes\rho_{2,3}\otimes\rho_{3,1} \\
&= \sum_{\lambda_1,\lambda_2,\lambda_3} q_1(\lambda_1)q_2(\lambda_2)q_3(\lambda_3) \\
&\quad \times |\lambda_1\rangle\langle\lambda_1| \otimes |\lambda_1\rangle\langle\lambda_1|\otimes|\lambda_2\rangle\langle\lambda_2| \otimes |\lambda_2\rangle\langle\lambda_2|\otimes|\lambda_3\rangle\langle\lambda_3| \otimes |\lambda_3\rangle\langle\lambda_3|,
\end{aligned}
$$

inducing the measurement state

$$
\begin{aligned}
\widetilde{\rho_{\mathcal{TN}}} &= \sum_{\lambda_1,\lambda_2,\lambda_3} q_1(\lambda_1)q_2(\lambda_2)q_3(\lambda_3) \\
&\quad \times (|\lambda_3\rangle\langle\lambda_3|\otimes|\lambda_1\rangle\langle\lambda_1|) \otimes (|\lambda_1\rangle\langle\lambda_1|\otimes|\lambda_2\rangle\langle\lambda_2|) \otimes (|\lambda_2\rangle\langle\lambda_2|\otimes|\lambda_3\rangle\langle\lambda_3|),
\end{aligned}
$$

in $\mathcal{D}(\mathcal{H}^{(1)}\otimes\mathcal{H}^{(2)}\otimes\mathcal{H}^{(3)})$. By defining separable positive operators:

$$M_{a_1}^{(1)} = \sum_{\lambda_3'=1}^{n_3} \sum_{\lambda_1'=1}^{n_1} P_1(a_1|\lambda_3'\lambda_1')|\lambda_3'\lambda_1'\rangle\langle\lambda_3'\lambda_1'|,$$

$$M_{a_2}^{(2)} = \sum_{\lambda_1'=1}^{n_1} \sum_{\lambda_2'=1}^{n_2} P_2(a_2|\lambda_1'\lambda_2')|\lambda_1'\lambda_2'\rangle\langle\lambda_1'\lambda_2'|,$$

$$M_{a_3}^{(3)} = \sum_{\lambda_2'=1}^{n_2} \sum_{\lambda_3'=1}^{n_3} P_3(a_3|\lambda_2'\lambda_3')|\lambda_2'\lambda_3'\rangle\langle\lambda_2'\lambda_3'|$$

on Hilbert spaces $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ and $\mathcal{H}^{(3)}$, respectively, we obtain POVMs $\{M_{a_k}^{(k)}\}_{a_k=1}^{o_k}$ of system $\mathcal{H}^{(k)}$ for each $k = 1, 2, 3$. Using (8) yields that

$$P(a_1, a_2, a_3) = \text{tr}[(\otimes_{n=1}^{3} M_{a_n}^{(n)})\widetilde{\rho_{\mathcal{TN}}}], \ \forall a_k \in [o_k].$$

This shows that **P** is separable tri-quantum. The proof is completed. □

Recently, Tavakoli et al. [33] said that, "in the triangle network with no inputs and binary outputs, the conjecture that the local and quantum sets are identical remains open". Proposition 3 above shows that D-trilocality and separable tri-quantum of a tripartite PT are equivalent. Renou et al. ([9] Theorem I) found a PT (they called a quantum distribution) $P_Q(a, b, c)$ that cannot be reproduced by any classical trilocal model (9) with deterministic response functions (DRFs) $P_1(a_1|\lambda_3\lambda_1), P_2(a_2|\lambda_1\lambda_2), P_3(a_3|\lambda_2\lambda_3)$. After a careful reading of their proof, we find that the proof of $X_0 \cap X_1 = \varnothing$ (for example) works well only for a D-triLHVM with DRFs. In fact, they proved that the $P_Q(a, b, c)$ cannot be reproduced by any D-triLHVM with DRFs. The following proposition shows that a D-triLHVM (8) can be assumed to be deterministic, i.e., the response functions are $\{0, 1\}$-valued. Thus, combining ([9] Theorem I), we see that the quantum distribution $P_Q(a, b, c)$ is not D-trilocal. This shows that a tri-quantum PT is not necessarily D-trilocal. Thus, an interesting question is whether the $P_Q(a, b, c)$ is C-trilocal.

Proposition 4. *A tripartite PT* $\mathbf{P} = [\![P(a_1, a_2, a_3)]\!]$ *over* Ω_3 *is D-trilocal if and only if it can be written as*

$$P(a_1, a_2, a_3) = \sum_{\mu_1, \mu_2, \mu_3} \pi_1(\mu_1) \pi_2(\mu_2) \pi_3(\mu_3) P_1(a_1|\mu_3\mu_1) P_2(a_2|\mu_1\mu_2) P_3(a_3|\mu_2\mu_3) \qquad (14)$$

for all $a_k \in [o_k]$, *where* $\{\pi_k(\mu_k)\}_{\mu_k \in D_k}$ *are PDs and*

$$\{P_1(a_1|\mu_3\mu_1)\}_{a_1 \in [o_1]}, \{P_2(a_2|\mu_1\mu_2)\}_{a_2 \in [o_2]}, \{P_3(a_3|\mu_2\mu_3)\}_{a_3 \in [o_3]}$$

are $\{0, 1\}$*-PDs for all* μ_k.

Proof. The sufficiency is clear. To show the necessity, we assume that $\mathbf{P}$ is D-trilocal. Then, it can be written as (8). Since matrices

$$[P_1(a_1|\lambda_3\lambda_1)] \in \mathbb{R}^{n_3 n_1 \times o_1}, [P_2(a_2|\lambda_1\lambda_2)] \in \mathbb{R}^{n_1 n_2 \times o_2} \text{ and } [P_3(a_3|\lambda_2\lambda_3)] \in \mathbb{R}^{n_3 n_1 \times o_3}$$

are row-stochastic (RS), they can be represented as convex combinations of all $\{0, 1\}$-RS matrices [34], i.e.,

$$P_1(a_1|\lambda_3\lambda_1) = \sum_{i=1}^{N_1} r_i \delta_{a_1, J_i(\lambda_3, \lambda_1)}, P_2(a_2|\lambda_1\lambda_2) = \sum_{j=1}^{N_2} s_j \delta_{a_2, K_j(\lambda_1, \lambda_2)}, P_3(a_3|\lambda_2\lambda_3) = \sum_{k=1}^{N_3} t_k \delta_{a_3, L_k(\lambda_2, \lambda_3)},$$

where $N_1 = (o_1)^{n_3 n_1}, N_2 = (o_2)^{n_1 n_2}, N_3 = (o_3)^{n_2 n_3}$, and $\{J_i\}_{i=1}^{N_1}, \{K_j\}_{j=1}^{N_2}$ and $\{L_k\}_{k=1}^{N_3}$ are the sets of all maps from $[n_3 n_1]$ into $[o_1]$, $[n_1 n_2]$ into $[o_2]$, and $[n_2 n_3]$ into $[o_3]$, respectively. Using (8) yields that

$$\begin{aligned}
P(a_1, a_2, a_3) &= \sum_{i,j,k} \sum_{\lambda_1, \lambda_2, \lambda_3} q_1(\lambda_1) q_2(\lambda_2) q_3(\lambda_3) r_i s_j t_k \delta_{a_1, J_i(\lambda_3, \lambda_1)} \delta_{a_2, K_j(\lambda_1, \lambda_2)} \delta_{a_3, L_k(\lambda_2, \lambda_3)} \\
&= \sum_{\mu_k \in D_k} \pi_1(\mu_1) \pi_2(\mu_2) \pi_3(\mu_3) P_1(a_1|\mu_3\mu_1) P_2(a_2|\mu_1\mu_2) P_3(a_3|\mu_2\mu_3),
\end{aligned}$$

where $D_1 = [N_2] \times [n_1], D_2 = [N_3] \times [n_2], D_3 = [N_1] \times [n_3]$, and

$$\mu_1 = (s_j, \lambda_1), \mu_2 = (t_k, \lambda_2), \mu_3 = (r_i, \lambda_3),$$

$$\pi_1(\mu_1) = q_1(\lambda_1) s_j, \pi_2(\mu_2) = q_2(\lambda_2) t_k, \pi_3(\mu_3) = q_3(\lambda_3) r_i,$$

$$P_1(a_1|\mu_3\mu_1) = \delta_{a_1, J_i(\lambda_3, \lambda_1)}, P_2(a_2|\mu_1\mu_2) = \delta_{a_2, K_j(\lambda_1, \lambda_2)}, P_3(a_3|\mu_2\mu_3) = \delta_{a_3, L_k(\lambda_2, \lambda_3)}.$$

Clearly, $\{\pi_k(\mu_k)\}_{\mu_k \in D_k} (k = 1, 2, 3)$ are PDs and for all μ_k,

$$\{P_1(a_1|\mu_3\mu_1)\}_{a_1 \in [o_1]}, \{P_2(a_2|\mu_1\mu_2)\}_{a_2 \in [o_2]}, \{P_3(a_3|\mu_2\mu_3)\}_{a_3 \in [o_3]}$$

are $\{0, 1\}$-PDs. Equation (14) follows, and the proof is completed. $\square$

To discuss geometric and topological properties of C-trilocal and D-trilocal PTs, we have to put them into a topological space. A natural way is to consider the real Hilbert space $\mathcal{P}(\Omega_3)$ consisting of all tensors $\mathbf{P} = [\![P(a_1, a_2, a_3)]\!]$ over Δ_3 defined by functions $P : \Omega_3 \to \mathbb{R}$, in which the operations and inner products are given by

$$s\mathbf{P} + t\mathbf{Q} = [\![sP(a_1, a_2, a_3) + tQ(a_1, a_2, a_3)]\!], \langle \mathbf{P}|\mathbf{Q} \rangle = \sum_{a_i} P(a_1, a_2, a_3) Q(a_1, a_2, a_3)$$

for all $s, t \in \mathbb{R}$ and all elements $\mathbf{P}$ and $\mathbf{Q}$ of $\mathcal{P}(\Delta_3)$. The norm induced by the inner product reads

$$\|\mathbf{P}\| = \left(\sum_{a_i} |P(a_1, a_2, a_3)|^2 \right)^{\frac{1}{2}}$$

and then a sequence $\{\mathbf{P}_n\}_{n=1}^{\infty} = \{\llbracket P_n(a_1, a_2, a_3) \rrbracket\}_{n=1}^{\infty}$ is convergent (in norm) to $\mathbf{P} = \llbracket P(a_1, a_2, a_3) \rrbracket$ if and only if

$$\lim_{n \to \infty} P_n(a_1, a_2, a_3) = P(a_1, a_2, a_3), \ \forall a_i \in [o_i] (i = 1, 2, 3).$$

Thus, the set $\mathcal{PT}(\Delta_3)$ of all PTs over Ω_3 forms a compact convex set in the Hilbert space $\mathcal{P}(\Omega_3)$.

Since the hidden variables in a C-triLHVM or a D-triLHVM for a PT are assumed to be independent, the sets $\mathcal{PT}^{\text{C-trilocal}}(\Omega_3)$ and $\mathcal{PT}^{\text{D-trilocal}}(\Omega_3)$ are not necessarily convex. However, we have the following.

Proposition 5. *Both $\mathcal{PT}^{\text{C-trilocal}}(\Omega_3)$ and $\mathcal{PT}^{\text{D-trilocal}}(\Omega_3)$ are path-connected sets in the Hilbert space $\mathcal{P}(\Omega_3)$.*

Proof. Let $\mathbf{P} = \llbracket P(a_1, a_2, a_3) \rrbracket$ and $\mathbf{Q} = \llbracket Q(a_1, a_2, a_3) \rrbracket$ be any two elements of $\mathcal{PT}^{\text{C-trilocal}}(\Omega_3)$. Then, $\mathbf{P}$ and $\mathbf{Q}$ have C-trLHVMs:

$$P(a_1, a_2, a_3) = \int_{\Lambda} p_1(\lambda_1) p_2(\lambda_2) p_3(\lambda_3) P_1(a_1|\lambda_3\lambda_1) P_2(a_2|\lambda_1\lambda_2) P_3(a_3|\lambda_2\lambda_3) \mathrm{d}\mu(\lambda),$$

$$Q(a_1, a_2, a_3) = \int_{\Gamma} q_1(\xi_1) q_2(\xi_2) q_3(\xi_3) Q_1(a_1|\xi_3\xi_1) Q_2(a_2|\xi_1\xi_2) Q_3(a_3|\xi_2\xi_3) \mathrm{d}\gamma(\xi),$$

for all possible a_1, a_2, a_3. Put $P_0(a_1, a_2, a_3) \equiv \frac{1}{o_1 o_2 o_3}$; then, $\mathbf{P}_0 := \llbracket P_0(a_1, a_2, a_3) \rrbracket$ is a D-trilocal (and then C-trilocal) CT over Ω_3. For every $t \in [0, 1/2]$, set

$$P_1^t(a_1|\lambda_3\lambda_1) = (1 - 2t)P_1(a_1|\lambda_3\lambda_1) + 2t\frac{1}{o_1};$$

$$P_2^t(a_2|\lambda_1\lambda_2) = (1 - 2t)P_2(a_2|\lambda_1\lambda_2) + 2t\frac{1}{o_2};$$

$$P_3^t(a_3|\lambda_2\lambda_3) = (1 - 2t)P_3(a_3|\lambda_2\lambda_3) + 2t\frac{1}{o_3},$$

which are clearly PDs of a_1, a_2 and a_3, respectively. Putting

$$P^t(a_1, a_2, a_3) = \int_{\Lambda} q_1(\lambda_1) q_2(\lambda_2) q_3(\lambda_3) P_1^t(a_1|\lambda_3\lambda_1) P_2^t(a_2|\lambda_1\lambda_2) P_3^t(a_3|\lambda_2\lambda_3) \mathrm{d}\mu(\lambda),$$

then $\mathbf{P}(t) := \llbracket P^t(a_1, a_2, a_3) \rrbracket$ is a C-trilocal CT for all $t \in [0, 1/2]$ with $\mathbf{P}(0) = \mathbf{P}$ and $\mathbf{P}(1/2) = \mathbf{P}_0$. Obviously, the map $t \mapsto \mathbf{P}(t)$ from $[0, 1/2]$ into $\mathcal{PT}^{\text{C-trilocal}}(\Omega_3)$ is continuous.

For every $t \in [1/2, 1]$, set

$$Q_1^t(a_1|\xi_3\xi_1) = (2t - 1)Q_1(a_1|\xi_3\xi_1) + 2(1 - t)\frac{1}{o_1};$$

$$Q_2^t(a_2|\xi_1\xi_2) = (2t - 1)Q_2(a_2|\xi_1\xi_2) + 2(1 - t)\frac{1}{o_2};$$

$$Q_3^t(a_3|\xi_2\xi_3) = (2t - 1)Q_3(a_3|\xi_2\xi_3) + 2(1 - t)\frac{1}{o_3},$$

which are clearly PDs of a_1, a_2 and a_3, respectively. Putting

$$Q^t(a_1, a_2, a_3) = \int_{\Gamma} q_1(\xi_1) q_2(\xi_2) q_3(\xi_3) Q_1^t(a_1|\xi_3\xi_1) Q_2^t(a_2|\xi_1\xi_2) Q_3^t(a_3|\xi_2\xi_3) \mathrm{d}\gamma(\xi),$$

then $\mathbf{Q}(t) := \llbracket Q^t(a_1, a_2, a_3) \rrbracket$ is a C-trilocal CT for all $t \in [1/2, 1]$ with $\mathbf{Q}(1/2) = \mathbf{P}_0$ and $\mathbf{Q}(1) = \mathbf{Q}$. Obviously, the map $t \mapsto \mathbf{Q}(t)$ from $[1/2, 1]$ into $\mathcal{PT}^{\text{C-trilocal}}(\Omega_3)$ is continuous.

Next, we define a mapping $f : [0,1] \to \mathcal{PT}^{\text{C-trilocal}}(\Omega_3)$ by

$$f(t) = \begin{cases} \mathbf{P}(t), & t \in [0,1/2]; \\ \mathbf{Q}(t), & t \in (1/2,1]. \end{cases}$$

Clearly, f is continuous everywhere and and then induces a path in $\mathcal{PT}^{\text{C-trilocal}}(\Omega_3)$, connecting $\mathbf{P}$ and $\mathbf{Q}$. This shows that $\mathcal{PT}^{\text{C-trilocal}}(\Omega_3)$ is path-connected. Similarly, $\mathcal{PT}^{\text{D-trilocal}}(\Omega_3)$ is also path-connected. The proof is completed. $\square$

Clearly, if a PT is D-trilocal, then it must be C-trilocal with a C-triLHVM given by counting measures on $\Lambda_j (j = 1, 2, 3)$. We can not show that the converse of this implication, but we obtain the following approximation result.

Proposition 6. *Suppose that* $\mathbf{P} = [\![P(a_1, a_2, a_3)]\!]$ *is a C-trilocal PT over* Ω_3 *with a C-triLHVM given by three-hold Riemann integral over* $\Lambda = [r_1, s_1] \times [r_2, s_2] \times [r_3, s_3]$*; then,* $\mathbf{P}$ *is in the closure of* $\mathcal{PT}^{\text{D-trilocal}}(\Omega_3)$ *in the Hilbert space* $\mathcal{P}(\Omega_3)$*.*

Proof. Suppose that

$$\begin{aligned} P(a_1, a_2, a_3) &= \iiint_{\Lambda} q_1(\lambda_1)q_2(\lambda_2)q_3(\lambda_3)P_1(a_1|\lambda_3\lambda_1)P_2(a_2|\lambda_1\lambda_2) \\ &\quad \times P_3(a_3|\lambda_2\lambda_3)\mathrm{d}\lambda_1\mathrm{d}\lambda_2\mathrm{d}\lambda_3 \end{aligned} \tag{15}$$

for all $a_k \in [o_k](k = 1, 2, 3)$, where $q_k(\lambda_k) \geq 0(\forall \lambda_k \in \Lambda_k := [r_k, s_k])$ with $\int_{r_k}^{s_k} q_k(\lambda_k)\mathrm{d}\lambda_k = 1(k = 1, 2, 3)$. Let us show that there exists a sequence $\{\mathbf{P}_n\}_{n=1}^{+\infty}$ of D-trilocal PTs over Ω_3 such that $\mathbf{P}_n \to \mathbf{P}$ as $n \to \infty$.

Dividing each interval $[r_k, s_k]$ into n small equal-length intervals:

$$I_j^{(k)} := [r_k + (s_k - r_k)(j-1)/n, r_k + (s_k - r_k)j/n](j = 1, 2, \ldots, n),$$

we obtain a partition T^n of Λ:

$$T^n = \{T_{j_1,j_2,j_3}^n := I_{j_1}^{(1)} \times I_{j_2}^{(2)} \times I_{j_3}^{(3)} | 1 \leq j_k \leq n(k = 1, 2, 3)\}.$$

For each $(j_1, j_2, j_3) \in [n]^3$, by taking a point $c_{j_1,j_2,j_3}^n = (\xi_{j_1}^{(n)}, \xi_{j_2}^{(n)}, \xi_{j_3}^{(n)}) \in T_{j_1,j_2,j_3}^n$ and letting

$$f_{n,k} = \sum_{i_k \in [n]} q_k(\xi_{i_k}^{(n)}), \quad \pi_k^{(n)}(j_k) = \begin{cases} \dfrac{q_k(\xi_{j_k}^{(n)})}{f_{n,k}}, & \text{if } f_{n,k} > 0; \\ \dfrac{1}{n}, & \text{if } f_{n,k} = 0, \end{cases}$$

we obtain a PD $\{\pi_k^{(n)}(j_k)\}_{j_k \in [n]}$ such that

$$q_k(\xi_{j_k}^{(n)}) = f_{n,k}\pi_k^{(n)}(j_k) = \pi_k^{(n)}(j_k) \sum_{i_k \in [n]} q_k(\xi_{i_k}^{(n)}). \tag{16}$$

Put

$$P_1^{(n)}(a_1|j_3j_1) = P_1(a_1|\xi_{j_3}^{(n)}\xi_{j_1}^{(n)}), P_2^{(n)}(a_2|j_1j_2) = P_2(a_2|\xi_{j_1}^{(n)}\xi_{j_2}^{(n)}), P_3^{(n)}(a_3|j_2j_3) = P_3(a_3|\xi_{j_2}^{(n)}\xi_{j_3}^{(n)}),$$

$$P_n(a_1, a_2, a_3) = \sum_{j_1,j_2,j_3=1}^{n} \pi_1^{(n)}(j_1)\pi_2^{(n)}(j_2)\pi_3^{(n)}(j_3)P_1^{(n)}(a_1|j_3j_1)P_2^{(n)}(a_2|j_1j_2)P_3^{(n)}(a_3|j_2j_3).$$

Clearly, $\mathbf{P}_n := [\![P_n(a_1, a_2, a_3)]\!](n = 1, 2, \ldots)$ are D-trilocal PTs over Ω_3. We see from the property of Riemann integral that

$$\lim_{n \to +\infty} \frac{s_k - r_k}{n} \sum_{i_k \in [n]} q_k(\xi_{i_k}^{(n)}) = \int_{r_k}^{s_k} q_k(\lambda_k) \mathrm{d}\lambda_k = 1 (k = 1, 2, 3). \tag{17}$$

Thus, by using Equations (17), (16) and the property of Riemann integral as well as Equation (15), we obtain that, for each $a_k \in [o_k](k = 1, 2, 3)$,

$$\begin{aligned}
&\lim_{n \to +\infty} P_n(a_1, a_2, a_3) \\
=\ & \lim_{n \to +\infty} \sum_{j_1, j_2, j_3 = 1}^{n} \pi_1^{(n)}(j_1) \pi_2^{(n)}(j_2) \pi_3^{(n)}(j_3) P_1^{(n)}(a_1|j_3 j_1) P_2^{(n)}(a_2|j_1 j_2) P_3^{(n)}(a_3|j_2 j_3) \\
=\ & \lim_{n \to +\infty} \frac{(s_1 - r_1)(s_2 - r_2)(s_3 - r_3)}{n^3} \sum_{i_1 \in [n]} q_1(\xi_{i_1}^{(n)}) \sum_{i_2 \in [n]} q_2(\xi_{i_2}^{(n)}) \sum_{i_3 \in [n]} q_3(\xi_{i_3}^{(n)}) \\
& \times \sum_{j_1, j_2, j_3 = 1}^{n} \pi_1^{(n)}(j_1) \pi_2^{(n)}(j_2) \pi_3^{(n)}(j_3) P_1^{(n)}(a_1|j_3 j_1) P_2^{(n)}(a_2|j_1 j_2) P_3^{(n)}(a_3|j_2 j_3) \\
=\ & \lim_{n \to +\infty} \frac{(s_1 - r_1)(s_2 - r_2)(s_3 - r_3)}{n^3} \sum_{j_1, j_2, j_3 = 1}^{n} q_1(\xi_{j_1}^{(n)}) q_2(\xi_{j_2}^{(n)}) q_3(\xi_{j_3}^{(n)}) \\
& \times P_1(a_1|\xi_{j_3}^{(n)} \xi_{j_1}^{(n)}) P_2(a_2|\xi_{j_1}^{(n)} \xi_{j_2}^{(n)}) P_3(a_3|\xi_{j_2}^{(n)} \xi_{j_3}^{(n)}) \\
=\ & \iiint_\Lambda q_1(\lambda_1) q_2(\lambda_2) q_3(\lambda_3) P_1(a_1|\lambda_3 \lambda_1) P_2(a_2|\lambda_1 \lambda_2) P_3(a_3|\lambda_2 \lambda_3) \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \mathrm{d}\lambda_3 \\
=\ & P(a_1, a_2, a_3).
\end{aligned}$$

This shows that $\mathbf{P}_n \to \mathbf{P}$ as $n \to \infty$. The proof is completed. $\square$

This conclusion implies that, if the set of all a D-trilocal PTs $\mathbf{P} = [\![P(a_1, a_2, a_3)]\!]$ over Ω_3 is closed, then the PT given by Equation (15) is D-trilocal.

In addition, when a PT $\mathbf{P}$ is given by Equation (15) where $\Lambda = [s_1, +\infty) \times [s_2, +\infty) \times [s_3, +\infty)$, DFs q_i and RFs $P_i(a_i|\cdot\cdot)$ are Riemann integrable on any $[s_i, S_i]$ and $[s_1, S_1] \times [s_2, S_2] \times [s_3, S_3]$, respectively, it is C-trilocal with a C-triLHVM (15) given by Lebesgue measure on Λ. In this case, the Levi's lemma yields that

$$\begin{aligned}
P(a_1, a_2, a_3) =\ & \lim_{n \to +\infty} \iiint_{\Lambda_n} q_1(\lambda_1) q_2(\lambda_2) q_3(\lambda_3) P_1(a_1|\lambda_3 \lambda_1) \\
& \times P_2(a_2|\lambda_1 \lambda_2) P_3(a_3|\lambda_2 \lambda_3) \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \mathrm{d}\lambda_3
\end{aligned} \tag{18}$$

for all $a_k \in [o_k](k = 1, 2, 3)$, where $\Lambda_n = [s_1, s_1 + n] \times [s_2, s_2 + n] \times [s_3, s_3 + n]$. Put

$$q_i^{(n)}(\lambda_i) = \frac{q_i(\lambda_i)}{\int_{[s_i, s_i + n]} q_i(t_i) \mathrm{d}t_i} (n = 1, 2, \ldots),$$

then $\lim_{n \to +\infty} \int_{[s_i, s_i + n]} q_i(t_i) \mathrm{d}t_i = \int_{[s_i, +\infty)} q_i(t_i) \mathrm{d}t_i = 1$ as $n \to +\infty$, and

$$q_i^{(n)}(\lambda_i) \geq 0, \forall \lambda_i \in [s_i, s_i + n], \int_{[s_i, s_i + n]} q_i^{(n)}(\lambda_i) \mathrm{d}\lambda_i = 1.$$

For each $n = 1, 2, \ldots$, letting

$$\begin{aligned}
P_n(a_1, a_2, a_3) =\ & \iiint_{\Lambda_n} q_1^{(n)}(\lambda_1) q_2^{(n)}(\lambda_2) q_3^{(n)}(\lambda_3) P_1(a_1|\lambda_3 \lambda_1) \\
& \times P_2(a_2|\lambda_1 \lambda_2) P_3(a_3|\lambda_2 \lambda_3) \mathrm{d}\lambda_1 \mathrm{d}\lambda_2 \mathrm{d}\lambda_3,
\end{aligned} \tag{19}$$

we obtain a C-trilocal PT $\mathbf{P}_n = [\![P_n(a_1, a_2, a_3)]\!]$ over Ω_3 with a C-triLHVM (19) in terms of Riemann integral over Λ_n. Proposition 6 yields that $\mathbf{P}_n \in \overline{\mathcal{PT}^{\text{D-trilocal}}}(\Omega_3)$ for all n. Equation (18) implies that $\mathbf{P} = \lim_{n \to +\infty} \mathbf{P}_n$. It follows that $\mathbf{P} \in \overline{\mathcal{PT}^{\text{D-trilocal}}}(\Omega_3)$.

Similarly, one can check that the PT $\mathbf{P}$ over Ω_3 defined by infinite series

$$P(a_1, a_2, a_3) = \sum_{\lambda_1 = s_1}^{+\infty} \sum_{\lambda_2 = s_2}^{+\infty} \sum_{\lambda_3 = s_3}^{+\infty} q_1(\lambda_1) q_2(\lambda_2) q_3(\lambda_3) P_1(a_1 | \lambda_3 \lambda_1) P_2(a_2 | \lambda_1 \lambda_2) P_3(a_3 | \lambda_2 \lambda_3)$$

is also C-trilocal and in the closure $\overline{\mathcal{PT}^{\text{D-trilocal}}}(\Omega_3)$ of $\mathcal{PT}^{\text{D-trilocal}}(\Omega_3)$.

3. Trilocality of Tripartite CTs

In this section, we aim to discuss two types of trilocality of a tripartite correlation tensor (CTs) [35]: $\mathbf{P} = [\![P(a_1 a_2 a_3 | x_1 x_2 x_3)]\!]$ over an index set

$$\Delta_3 = [o_1] \times [o_2] \times [o_3] \times [m_1] \times [m_2] \times [m_3],$$

which is a nonnegative tensor with index set Δ_3 such that

$$\sum_{a_i \in [o_i]} P(a_1 a_2 a_3 | x_1 x_2 x_3) = 1, \quad \forall x_i \in [m_i] (i = 1, 2, 3).$$

We use $\mathcal{CT}(\Delta_3)$ to denote the sets of CTs over Δ_3.

Definition 4. *Let* $\mathbf{P} = [\![P(a_1 a_2 a_3 | x_1 x_2 x_3)]\!]$ *be a CT over* Δ_3.

(1) $\mathbf{P}$ is said to *C-trilocal* if it has a C-triLHVM:

$$\begin{aligned}
P(a_1 a_2 a_3 | x_1 x_2 x_3) &= \int_{\Lambda} q_1(\lambda_1) q_2(\lambda_2) q_3(\lambda_3) P_1(a_1 | x_1, \lambda_3 \lambda_1) \\
&\quad \times P_2(a_2 | x_2, \lambda_1 \lambda_2) P_3(a_3 | x_3, \lambda_2 \lambda_3) \mathrm{d}\mu(\lambda)
\end{aligned} \tag{20}$$

for a product measure space

$$(\Lambda, \Omega, \mu) = (\Lambda_1 \times \Lambda_2 \times \Lambda_3, \Omega_1 \times \Omega_2 \times \Omega_3, \mu_1 \times \mu_2 \times \mu_3),$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $q_j(\lambda_j)$ is a DF of λ_j, $P_1(a_1 | x_1, \lambda_3 \lambda_1), P_2(a_2 | x_2, \lambda_1 \lambda_2)$ and $P_3(a_3 | x_3, \lambda_2 \lambda_3)$, called response functions (RSs) at nodes 1, 2 and 3, are nonnegative Ω-measurable on Λ for all x_i, a_i and PDs of outcomes a_1, a_2 and a_3, respectively, for all λ_1, λ_2 and λ_3.

(2) $\mathbf{P}$ is said to be *D-trilocal* if it has a D-triLHVM:

$$\begin{aligned}
P(a_1 a_2 a_3 | x_1 x_2 x_3) &= \sum_{\lambda_1 = 1}^{n_1} \sum_{\lambda_2 = 1}^{n_2} \sum_{\lambda_3 = 1}^{n_3} q_1(\lambda_1) q_2(\lambda_2) q_3(\lambda_3) P_1(a_1 | x_1, \lambda_3 \lambda_1) \\
&\quad \times P_2(a_2 | x_2, \lambda_1 \lambda_2) P_3(a_3 | x_3, \lambda_2 \lambda_3)
\end{aligned} \tag{21}$$

for all $x_k \in [m_k], a_k \in [o_k] (k = 1, 2, 3)$, where

$$q_k(\lambda_k), P_1(a_1 | x_1, \lambda_3 \lambda_1), P_2(a_2 | x_2, \lambda_1 \lambda_2), P_3(a_3 | x_3, \lambda_2 \lambda_3)$$

are PDs of λ_k, a_1, a_2, a_3, respectively.

(3) $\mathbf{P}$ is said to be *C-nontrilocal* (resp. *D -nontrilocal*) if it is not C-trilocal (resp. not D-trilocal).

We use $\mathcal{CT}^{\text{C-trilocal}}(\Delta_3)$ and $\mathcal{CT}^{\text{D-trilocal}}(\Delta_3)$ to denote the sets of all C-trilocal and D-trilocal CTs over Δ_3, respectively. Clearly, $\mathcal{CT}^{\text{C-trilocal}}(\Delta_3) \supset \mathcal{CT}^{\text{D-trilocal}}(\Delta_3)$.

Similar to the analysis before Proposition 1, we can obtain the following.

Proposition 7. *A CT* $\mathbf{P} = [\![P(a_1a_2a_3|x_1x_2x_3)]\!]$ *over* Δ_3 *is C-trilocal if and only if it admits a C-triLHVM:*

$$P(a_1a_2a_3|x_1x_2x_3) \;=\; \int_\Lambda P_1(a_1|x_1,\lambda_3\lambda_1)P_2(a_2|x_2,\lambda_1\lambda_2)P_3(a_3|x_3,\lambda_2\lambda_3)\mathrm{d}\gamma(\lambda) \quad (22)$$

for some product probability space

$$(\Lambda,\Sigma,\gamma) = (\Lambda_1 \times \Lambda_2 \times \Lambda_3, \Sigma_1 \times \Sigma_2 \times \Sigma_3, \gamma_1 \times \gamma_2 \times \gamma_3).$$

It is obvious that different C-trilocal CTs over the same index set Δ_3 have their C-triLHVMs that are given by product measure spaces that may be different. However, the following result shows that a finite number of C-trilocal CTs $\mathbf{P}_k(k = 1, 2, \ldots, m)$ over Δ_3 have C-triLHVMs based on a common product measure space.

Proposition 8. *Let* $\mathbf{P}_k = [\![P_k(a_1a_2a_3|x_1x_2x_3)]\!](k = 1, 2, \ldots, m)$ *be m C-trlocal CTs over* Δ_3. *Then, there is a product measure space*

$$(S_1 \times S_2 \times S_3, T_1 \times T_2 \times T_3, \gamma_1 \times \gamma_2 \times \gamma_3)$$

and three DFs $f_i(s_i)$ *of* $s_i(i = 1, 2, 3)$ *such that*

$$P_k(a_1a_2a_3|x_1x_2x_3) \;=\; \iiint_{S_1 \times S_2 \times S_3} f_1(s_1)f_2(s_2)f_3(s_3)P_1^{(k)}(a_1|x_1,s_3s_1)P_2^{(k)}(a_2|x_2,s_1s_2)$$
$$\times P_3^{(k)}(a_3|x_3,s_2s_3)\mathrm{d}\gamma_1(s_1)\mathrm{d}\gamma_2(s_2)\mathrm{d}\gamma_3(s_3), \;\forall k \in [m], \quad (23)$$

for all a_i, x_i.

Proof. By Definition 4, each $\mathbf{P}_k$ can be represented as

$$P_k(a_1a_2a_3|x_1x_2x_3) \;=\; \iiint_{\Lambda_1^{(k)} \times \Lambda_2^{(k)} \times \Lambda_3^{(k)}} q_1^{(k)}(\lambda_1^{(k)})q_2^{(k)}(\lambda_2^{(k)})q_3^{(k)}(\lambda_3^{(k)})P_A^{(k)}(a_1|x_1,\lambda_3^{(k)}\lambda_1^{(k)})$$
$$\times P_B^{(k)}(a_2|x_2,\lambda_1^{(k)}\lambda_2^{(k)})P_C^{(k)}(a_3|x_3,\lambda_2^{(k)}\lambda_3^{(k)})$$
$$\times \mathrm{d}\mu_1^{(k)}(\lambda_1^{(k)})\mathrm{d}\mu_2^{(k)}(\lambda_2^{(k)})\mathrm{d}\mu_3^{(k)}(\lambda_3^{(k)}) \quad (24)$$

for some product measure space

$$(\Lambda_1^{(k)} \times \Lambda_2^{(k)} \times \Lambda_3^{(k)}, \Omega_1^{(k)} \times \Omega_2^{(k)} \times \Omega_3^{(k)}, \mu_1^{(k)} \times \mu_2^{(k)} \times \mu_3^{(k)}).$$

Putting

$$S_i = \prod_{k=1}^m \Lambda_i^{(k)}, T_i = \prod_{k=1}^m \Omega_i^{(k)}, \gamma_i = \prod_{k=1}^m \mu_i^{(k)},$$

$$s_i = (\lambda_i^{(1)}, \lambda_i^{(2)}, \ldots, \lambda_i^{(m)}), f_i(s_i) = \prod_{k=1}^m q_i^{(k)}(\lambda_i^{(k)})(i = 1, 2, 3)$$

produces a product measure space

$$(S_1 \times S_2 \times S_3, T_1 \times T_2 \times T_3, \gamma_1 \times \gamma_2 \times \gamma_3)$$

and three DFs $f_i(s_i)$ of $s_i(i = 1, 2, 3)$. By letting

$$P_1^{(k)}(a_1|x_1,s_3s_1) = P_A^{(k)}(a_1|x_1,\lambda_3^{(k)}\lambda_1^{(k)}),$$

$$P_2^{(k)}(a_2|x_2,s_1s_2)) = P_B^{(k)}(a_2|x_2,\lambda_1^{(k)}\lambda_2^{(k)}),$$

$$P_3^{(k)}(a_3|x_3, s_2 s_3) = P_C^{(k)}(a_3|x_3, \lambda_2^{(k)} \lambda_3^{(k)}),$$

for all $s_i = (\lambda_i^{(1)}, \lambda_i^{(2)}, \dots, \lambda_i^{(m)})$ in S_i, we obtain (23) using Equation (24). The proof is completed.

Using Definitions 1 and 4, we see that, when a CT $\mathbf{P} = [\![P(a_1 a_2 a_3 | x_1 x_2 x_3)]\!]$ over Δ_3 is C-trilocal (resp. D-trilocal), the induced PTs $\mathbf{P}_{x_1 x_2 x_3} := [\![P(a_1 a_2 a_3 | x_1 x_2 x_3)]\!]$ over Ω_3 must be C-trilocal (resp. D-trilocal) for all (x_1, x_2, x_3) in $[m_1] \times [m_2] \times [m_3]$. Equivalently, if the PT $\mathbf{P}_{x_1^0 x_2^0 x_3^0}$ is non-C-trilocal (resp. non-D-trilocal) for some (x_1^0, x_2^0, x_3^0) in $[m_1] \times [m_2] \times [m_3]$, then the CT $\mathbf{P} = [\![P(a_1 a_2 a_3 | x_1 x_2 x_3)]\!]$ must be non-C-trilocal (resp. non-D-trilocal). In this sense, we can say that the non-trilocality of PTs is stronger than that of CTs. Furthermore, let $\mathbf{P} = [\![P(a_1 a_2 a_3 | x_1 x_2 x_3)]\!]$ be a C-trilocal CT. Then, it has a C-triLHVM (20). By letting

$$P_1(a_1|x_1, \lambda_1) = \int_{\Lambda_3} q_3(\lambda_3) P_1(a_1|x_1, \lambda_3 \lambda_1) \mathrm{d}\mu_3(\lambda_3);$$

$$P_2(a_2|x_2, \lambda_1) = \int_{\Lambda_2} q_2(\lambda_2) P_2(a_2|x_2, \lambda_1 \lambda_2) \mathrm{d}\mu_2(\lambda_2),$$

we see from (20) that the marginal distribution of $\mathbf{P}$ on the subsystem $S_1 S_2$ reads

$$P_{12}(a_1 a_2 | x_1 x_2) = \sum_{a_3} P(a_1 a_2 a_3 | x_1 x_2 x_3) = \int_{\Lambda_1} q_1(\lambda_1) P_1(a_1|x_1, \lambda_1) P_2(a_2|x_2, \lambda_1) \mathrm{d}\mu_1(\lambda_1) \quad (25)$$

for all possible x_1, x_2, a_1, a_2. Thus, $\mathbf{P}_{12} = [\![P_{12}(a_1 a_2 | x_1 x_2)]\!]$ becomes a Bell local CT [35] over $[o_1] \times [o_2] \times [m_1] \times [m_2]$. Similarly, the marginal distributions $\mathbf{P}_{23} = [\![P_{23}(a_2 a_3 | x_2 x_3)]\!]$ and $\mathbf{P}_{13} = [\![P_{13}(a_1 a_3 | x_1 x_3)]\!]$ are Bell local CTs over $[o_2] \times [o_3] \times [m_2] \times [m_3]$ and $[o_1] \times [o_3] \times [m_1] \times [m_3]$, respectively. This analysis leads to the following necessary condition for a CT to be C-trilocal. $\square$

Proposition 9. *The three bipartite marginal distributions of a tripartite C-trilocal CT are Bell local.*

Remark 1. *In particular, when Λ_3 is a singleton $\{\lambda_3\}(\lambda_3 = 1)$ and $q_3(\lambda_3) = \mu_3(\{\lambda_3\}) = 1$, Equation (20) becomes*

$$
\begin{aligned}
P(a_1 a_2 a_3 | x_1 x_2 x_3) = {} & \iint_{\Lambda_1 \times \Lambda_2} q_1(\lambda_1) q_2(\lambda_2) P_1(a_1|x_1, \lambda_1) P_2(a_2|x_2, \lambda_1 \lambda_2) \\
& \times P_3(a_3|x_3, \lambda_2) \mathrm{d}\mu_1(\lambda_1) \mathrm{d}\mu_2(\lambda_2).
\end{aligned}
\quad (26)
$$

In this case, $\mathbf{P}$ is said to be C-bilocal, shortly bilocal [20,21,36] and Equation (26) is called a C-biLHVM of $\mathbf{P}$. In addition, when Λ_2 and Λ_3 can be chosen as finite sets, $\mathbf{P}$ is said to be D-bilocal. We use $\mathcal{CT}^{\text{C-bilocal}}(\Delta_3)$ and $\mathcal{CT}^{\text{D-bilocal}}(\Delta_3)$ to denote the sets of all C-bilocal and D-bilocal CTs over Δ_3, respectively. Conversely, when $\mathbf{P}$ is a C-bilocal over Δ_3, it has a C-biLHVM (26), which can be written as (20) with Λ_3 being a singleton $\{\lambda_3\}$ with $\lambda_3 = 1$ and $q_3(\lambda_3) = \mu_3(\{\lambda_3\}) = 1$. Thus,

$$\mathcal{CT}^{\text{C-bilocal}}(\Delta_3) \subset \mathcal{CT}^{\text{C-trilocal}}(\Delta_3), \ \mathcal{CT}^{\text{D-bilocal}}(\Delta_3) \subset \mathcal{CT}^{\text{D-trilocal}}(\Delta_3).$$

It is proved in ([36] Theorem 2.1) that

$$\mathcal{CT}^{\text{C-bilocal}}(\Delta_3) = \mathcal{CT}^{\text{D-bilocal}}(\Delta_3) := \mathcal{CT}^{\text{bilocal}}(\Delta_3).$$

Definition 5. *A tripartite CT $\mathbf{P} = [\![P(a_1 a_2 a_3 | x_1 x_2 x_3)]\!]$ over Δ_3 is said to be tri-quantum if there exists a $\mathcal{TN}$ with the state $\rho_{\mathcal{TN}}$ and a set of local POVMs*

$$\mathcal{M} = \{\mathcal{M}_{x_1 x_2 x_3} | x_k \in [m_k]\} = \{M_{x_1}^{(1)} \otimes M_{x_2}^{(2)} \otimes M_{x_3}^{(3)} | x_k \in [m_k]\}, \quad (27)$$

with $M_{x_k}^{(k)} = \{M_{a_k|x_k}^{(k)}\}_{a_k=1}^{o_k}$ such that $\mathbf{P} = \mathbf{T}_{TN}^{\mathcal{M}}$, where

$$T_{TN}^{\mathcal{M}}(a_1 a_2 a_3 | x_1 x_2 x_3) = \text{tr}[(\otimes_{n=1}^{3} M_{a_n|x_k}^{(n)})\widetilde{\rho_{\mathcal{TN}}}], \ \forall a_k \in [o_k] \tag{28}$$

for all possible x_k, a_k. In particular, when the shares states $\rho_{i,j}$ can be chosen as separable states, we say that P is separable tri-quantum.

Definition 6. *A triangle network $\mathcal{TN}$ given by Figure 1 is said to be strongly trilocal if, for any set $\mathcal{M}$ of local POVMs of the form (27), the resulting CT $\mathbf{T}_{TN}^{\mathcal{M}}$ is D-trilocal.*

Using Proposition 9, we see that, when one of the three marginal distributions is Bell nonlocal, $\mathbf{P}$ must be neither C-trilocal nor D-trilocal. Since every entangled pure state is Bell nonlocal [37], when one of the shared states $\rho_{i,j}$ in the triangle network given by Figure 1 is an entangled pure state, there are a set of local POVMs (27) such that the resulting CT $\mathbf{P} = \mathbf{T}_{TN}^{\mathcal{M}}$ is not C-trilocal and then not D-trilocal. Thus, the network is not strongly trilocal. Conversely, we have the following.

Proposition 10. *Every separable (i.e., all shared states $\rho_{i,j}$ are separable) triangle network $\mathcal{TN}$ given by Figure 1 is strongly trilocal.*

Proof. Suppose that the $\mathcal{TN}$ given by Figure 1 is separable. Then, the shared states $\rho_{s,t}$ are separable, i.e., there exist PDs $\{q_1(\lambda_1)\}_{\lambda_1=1}^{n_1}$, $\{q_2(\lambda_2)\}_{\lambda_2=1}^{n_2}$ and $\{q_3(\lambda_3)\}_{\lambda_3=1}^{n_3}$ such that

$$\rho_{1,2} = \sum_{\lambda_1=1}^{n_1} q_1(\lambda_1)\rho_1^{(1)}(\lambda_1)\otimes\rho_1^{(2)}(\lambda_1) \in \mathcal{D}(\mathcal{H}_1^{(1)}\otimes\mathcal{H}_1^{(2)}),$$

$$\rho_{2,3} = \sum_{\lambda_2=1}^{n_2} q_2(\lambda_2)\rho_2^{(2)}(\lambda_2)\otimes\rho_1^{(3)}(\lambda_2) \in \mathcal{D}(\mathcal{H}_2^{(2)}\otimes\mathcal{H}_1^{(3)}),$$

$$\rho_{3,1} = \sum_{\lambda_3=1}^{n_3} q_3(\lambda_3)\rho_2^{(3)}(\lambda_3)\otimes\rho_2^{(1)}(\lambda_3) \in \mathcal{D}(\mathcal{H}_2^{(3)}\otimes\mathcal{H}_2^{(1)}),$$

where $\rho_t^{(s)}(r) \in \mathcal{D}(\mathcal{H}_t^{(s)})$. Thus, the network state reads

$$\rho_{\mathcal{TN}} = \sum_{\lambda_1,\lambda_2,\lambda_3} q_1(\lambda_1)q_2(\lambda_2)q_3(\lambda_3)\rho_1^{(1)}(\lambda_1)\otimes\rho_1^{(2)}(\lambda_1)\otimes\rho_2^{(2)}(\lambda_2)\otimes\rho_1^{(3)}(\lambda_2)\otimes\rho_2^{(3)}(\lambda_3)\otimes\rho_2^{(1)}(\lambda_3),$$

being a state of system $\mathcal{H}_1^{(1)}\otimes\mathcal{H}_1^{(2)}\otimes\mathcal{H}_2^{(2)}\otimes\mathcal{H}_1^{(3)}\otimes\mathcal{H}_2^{(3)}\otimes\mathcal{H}_2^{(1)}$. Then, the measurement state is

$$\widetilde{\rho_{\mathcal{TN}}} = \sum_{\lambda_1,\lambda_2,\lambda_3} q_1(\lambda_1)q_2(\lambda_2)q_3(\lambda_3)(\rho_2^{(1)}(\lambda_3)\otimes\rho_1^{(1)}(\lambda_1))\otimes(\rho_1^{(2)}(\lambda_1)\otimes\rho_2^{(2)}(\lambda_2))\otimes(\rho_1^{(3)}(\lambda_2)\otimes\rho_2^{(3)}(\lambda_3)).$$

being a state of system

$$\mathcal{H}^{(1)}\otimes\mathcal{H}^{(2)}\otimes\mathcal{H}^{(3)} = (\mathcal{H}_2^{(1)}\otimes\mathcal{H}_1^{(1)})\otimes(\mathcal{H}_1^{(2)}\otimes\mathcal{H}_2^{(2)})\otimes(\mathcal{H}_1^{(3)}\otimes\mathcal{H}_2^{(3)}).$$

for any set $\mathcal{M}$ of local POVMs of the form (27) of system $\mathcal{H}^{(1)}\otimes\mathcal{H}^{(2)}\otimes\mathcal{H}^{(3)}$, we compete that

$$
\begin{aligned}
T_{\mathcal{TN}}^{\mathcal{M}}(a_1a_2a_3|x_1x_2x_3) &= \mathrm{tr}[(\otimes_{n=1}^{3}M_{a_n}^{(n)})\widetilde{\rho_{\mathcal{TN}}}] \\
&= \sum_{\lambda_1,\lambda_2,\lambda_3} q_1(\lambda_1)q_2(\lambda_2)q_3(\lambda_3)\mathrm{tr}[M_{a_1|x_1}^{(1)}(\rho_1^{(1)}(\lambda_1)\otimes\rho_2^{(1)}(\lambda_3))] \\
&\quad \times \mathrm{tr}[M_{a_2|x_2}^{(2)}(\rho_1^{(2)}(\lambda_1)\otimes\rho_2^{(2)}(\lambda_2))]\mathrm{tr}[M_{a_3|x_3}^{(3)}(\rho_1^{(3)}(\lambda_2)\otimes\rho_2^{(3)}(\lambda_3))] \\
&= \sum_{\lambda_k\in[n_k]} q_1(\lambda_1)q_2(\lambda_2)q_3(\lambda_3)P_1(a_1|x_1,\lambda_3\lambda_1)P_2(a_2|x_2,\lambda_1\lambda_2)P_3(a_3|x_3,\lambda_2\lambda_3),
\end{aligned}
$$

for all $a_k \in [o_k]$, where

$$
P_1(a_1|x_1,\lambda_3\lambda_1) = \mathrm{tr}[M_{a_1|x_1}^{(1)}(\rho_1^{(1)}(\lambda_1)\otimes\rho_2^{(1)}(\lambda_3))],
$$

$$
P_2(a_2|x_2,\lambda_1\lambda_2) = \mathrm{tr}[M_{a_2|x_2}^{(2)}(\rho_1^{(2)}(\lambda_1)\otimes\rho_2^{(2)}(\lambda_2))],
$$

$$
P_3(a_3|x_3,\lambda_2\lambda_3) = \mathrm{tr}[M_{a_3|x_3}^{(3)}(\rho_1^{(3)}(\lambda_2)\otimes\rho_2^{(3)}(\lambda_3))].
$$

Clearly, $\{q_k(\lambda_k)\}_{\lambda_k\in[n_k]}, \{P_1(a_1|x_1,\lambda_3\lambda_1)\}_{a_1\in[o_1]}, \{P_2(a_2|x_2,\lambda_1\lambda_2)\}_{a_2\in[o_2]}$, and $\{P_3(a_3|x_3,\lambda_2\lambda_3)\}_{a_3\in[o_3]}$ are PDs of λ_k, a_1, a_2 and a_3, respectively. This shows that $\mathbf{T}_{\mathcal{TN}}^{\mathcal{M}}$ is D-trilocal. It follows from Definition 6 that the triangle network $\mathcal{TN}$ given by Figure 1 is strongly trilocal. The proof is completed. $\square$

Theorem 1. (Realization). *A CT* $\mathbf{P}$ *over* Δ_3 *is D-trilocal if and only if it is separable tri-quantum.*

Proof. The sufficiency is given by Proposition 10. To show the necessity, we let $\mathbf{P} = \{P(a_1a_2a_3|x_1x_2x_3)\}$ be a D-trilocal PT over Δ_3. Then, it can be written as the form of (21):

$$
\begin{aligned}
P(a_1a_2a_3|x_1x_2x_3) &= \sum_{\lambda_1=1}^{n_1}\sum_{\lambda_2=1}^{n_2}\sum_{\lambda_3=1}^{n_3} q_1(\lambda_1)q_2(\lambda_2)q_3(\lambda_3)P_1(a_1|x_1,\lambda_3\lambda_1) \\
&\quad \times P_2(a_2|x_2,\lambda_1\lambda_2)P_3(a_3|x_3,\lambda_2\lambda_3)
\end{aligned} \tag{29}
$$

for all $a_k \in [o_k](k=1,2,3)$, where

$$
\{q_k(\lambda_k)\}_{\lambda_k\in[n_k]}, \{P_1(a_1|x_1,\lambda_3\lambda_1)\}_{a_1\in[o_1]}, \{P_2(a_2|x_2,\lambda_1\lambda_2)\}_{a_2\in[o_2]}, \{P_3(a_3|x_3,\lambda_2\lambda_3)\}_{a_3\in[o_3]}
$$

are PDs for all possible x_k, λ_j. Define

$$
\mathcal{H}_1^{(1)} = \mathcal{H}_1^{(2)} = \mathbb{C}^{n_1}, \mathcal{H}_2^{(2)} = \mathcal{H}_1^{(3)} = \mathbb{C}^{n_2}, \mathcal{H}_2^{(1)} = \mathcal{H}_2^{(3)} = \mathbb{C}^{n_3},
$$

take their orthonormal bases $\{|\lambda_3\rangle\}_{\lambda_3=1}^{n_3}$, $\{|\lambda_1\rangle\}_{\lambda_1=1}^{n_1}$ and $\{|\lambda_2\rangle\}_{\lambda_2=1}^{n_2}$, respectively, and put

$$
\mathcal{H}^{(1)} = \mathcal{H}_2^{(1)}\otimes\mathcal{H}_1^{(1)} = \mathbb{C}^{n_3}\otimes\mathbb{C}^{n_1}, \mathcal{H}^{(2)} = \mathcal{H}_1^{(2)}\otimes\mathcal{H}_2^{(2)} = \mathbb{C}^{n_1}\otimes\mathbb{C}^{n_2}, \mathcal{H}^{(3)} = \mathcal{H}_1^{(3)}\otimes\mathcal{H}_2^{(3)} = \mathbb{C}^{n_2}\otimes\mathbb{C}^{n_3}
$$

and choose separable states

$$
\rho_{1,2} = \sum_{\lambda_1} q_1(\lambda_1)|\lambda_1\rangle\langle\lambda_1| \otimes |\lambda_1\rangle\langle\lambda_1| \in \mathcal{D}(\mathcal{H}_1^{(1)}\otimes\mathcal{H}_1^{(2)}) = \mathcal{D}(\mathbb{C}^{n_1}\otimes\mathbb{C}^{n_1}),
$$

$$
\rho_{2,3} = \sum_{\lambda_2=1}^{n_2} q_2(\lambda_2)|\lambda_2\rangle\langle\lambda_2| \otimes |\lambda_2\rangle\langle\lambda_2| \in \mathcal{D}(\mathcal{H}_2^{(2)}\otimes\mathcal{H}_1^{(3)}) = \mathcal{D}(\mathbb{C}^{n_2}\otimes\mathbb{C}^{n_2}),
$$

$$
\rho_{3,1} = \sum_{\lambda_3=1}^{n_3} q_3(\lambda_3)|\lambda_3\rangle\langle\lambda_3| \otimes |\lambda_3\rangle\langle\lambda_3| \in \mathcal{D}(\mathcal{H}_2^{(3)}\otimes\mathcal{H}_2^{(1)}) = \mathcal{D}(\mathbb{C}^{n_3}\otimes\mathbb{C}^{n_3}),
$$

then we obtain a triangle network $\mathcal{TN}$ with the network state

$$
\begin{aligned}
\rho_{\mathcal{TN}} &= \rho_{1,2} \otimes \rho_{2,3} \otimes \rho_{3,1} \\
&= \sum_{\lambda_1,\lambda_2,\lambda_3} q_1(\lambda_1) q_2(\lambda_2) q_3(\lambda_3) |\lambda_1\rangle\langle\lambda_1| \otimes |\lambda_1\rangle\langle\lambda_1| \otimes |\lambda_2\rangle\langle\lambda_2| \otimes |\lambda_2\rangle\langle\lambda_2| \otimes |\lambda_3\rangle\langle\lambda_3| \otimes |\lambda_3\rangle\langle\lambda_3|,
\end{aligned}
$$

inducing the measurement state

$$
\widetilde{\rho_{\mathcal{TN}}} = \sum_{\lambda_1,\lambda_2,\lambda_3} q_1(\lambda_1) q_2(\lambda_2) q_3(\lambda_3) (|\lambda_3\rangle\langle\lambda_3| \otimes |\lambda_1\rangle\langle\lambda_1|) \otimes (|\lambda_1\rangle\langle\lambda_1| \otimes |\lambda_2\rangle\langle\lambda_2|) \otimes (|\lambda_2\rangle\langle\lambda_2| \otimes |\lambda_3\rangle\langle\lambda_3|),
$$

in $\mathcal{D}(\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \mathcal{H}^{(3)})$. By defining positive operators:

$$
M^{(1)}_{a_1|x_1} = \sum_{\lambda'_3=1}^{n_3} \sum_{\lambda'_1=1}^{n_1} P_1(a_1|x_1, \lambda'_3\lambda'_1) |\lambda'_3\lambda'_1\rangle\langle\lambda'_3\lambda'_1|,
$$

$$
M^{(2)}_{a_2|x_2} = \sum_{\lambda'_1=1}^{n_1} \sum_{\lambda'_2=1}^{n_2} P_2(a_2|x_2, \lambda'_1\lambda'_2) |\lambda'_1\lambda'_2\rangle\langle\lambda'_1\lambda'_2|,
$$

$$
M^{(3)}_{a_3|x_3} = \sum_{\lambda'_2=1}^{n_2} \sum_{\lambda'_3=1}^{n_3} P_3(a_3|x_3, \lambda'_2\lambda'_3) |\lambda'_2\lambda'_3\rangle\langle\lambda'_2\lambda'_3|
$$

on $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ and $\mathcal{H}^{(3)}$, respectively, we obtain POVMs $\{M^{(k)}_{a_k}\}_{a_k=1}^{o_k}$ of system $\mathcal{H}^{(k)}$ for each $k = 1, 2, 3$. It is easy to check that

$$
P(a_1 a_2 a_3 | x_1 x_2 x_3) = \mathrm{tr}[(\otimes_{n=1}^3 M^{(n)}_{a_n|x_n}) \widetilde{\rho_{\mathcal{TN}}}], \ \forall a_k \in [o_k], x_k \in [m_k].
$$

This shows that **P** is separable tri-quantum. The proof is completed. □

To discuss geometric and topological properties of C-trilocal and D-trilocal CTs, we have to put them into a topological space. A natural way is to consider the real Hilbert space $\mathcal{T}(\Delta_3)$ consisting of all correlation-type tensors [35] $\mathbf{P} = [\![P(a_1 a_2 a_3 | x_1 x_2 x_3)]\!]$ over Δ_3 defined by functions $P : \Delta_3 \to \mathbb{R}$, in which the operations and inner products are given by

$$
s\mathbf{P} + t\mathbf{Q} = [\![sP(a_1 a_2 a_3 | x_1 x_2 x_3) + tQ(a_1 a_2 a_3 | x_1 x_2 x_3)]\!],
$$

$$
\langle \mathbf{P} | \mathbf{Q} \rangle = \sum_{a_i, x_i} P(a_1 a_2 a_3 | x_1 x_2 x_3) Q(a_1 a_2 a_3 | x_1 x_2 x_3)
$$

for all $s, t \in \mathbb{R}$ and all elements **P** and **Q** of $\mathcal{T}(\Delta_3)$. The norm induced by the inner product reads

$$
\|\mathbf{P}\| = \left(\sum_{a_i, x_i} |P(a_1 a_2 a_3 | x_1 x_2 x_3)|^2 \right)^{\frac{1}{2}}
$$

and then a sequence $\{\mathbf{P}_n\}_{n=1}^\infty = \{[\![P_n(a_1 a_2 a_3 | x_1 x_2 x_3)]\!]\}_{n=1}^\infty$ in $\mathcal{T}(\Delta_3)$ is convergent (in norm) to $\mathbf{P} = [\![P(a_1 a_2 a_3 | x_1 x_2 x_3)]\!]$ if and only if

$$
\lim_{n\to\infty} P_n(a_1 a_2 a_3 | x_1 x_2 x_3) = P(a_1 a_2 a_3 | x_1 x_2 x_3), \ \forall x_i \in [m_i], a_i \in [o_i] (i = 1, 2, 3).
$$

Thus, the set $\mathcal{CT}(\Delta_3)$ of all CTs over Δ_3 forms a compact convex set in $\mathcal{T}(\Delta_3)$. Since the hidden variables in a C-triLHVM or a D-triLHVM are assumed to be independent, the sets $\mathcal{CT}^{\text{C-trilocal}}(\Delta_3)$ and $\mathcal{CT}^{\text{D-trilocal}}(\Delta_3)$ are not necessarily convex. However, we have the following.

Theorem 2. (Path-connectedness). *Both $\mathcal{CT}^{\text{C-trilocal}}(\Delta_3)$ and $\mathcal{CT}^{\text{D-trilocal}}(\Delta_3)$ are path-connected sets in the Hilbert space $\mathcal{T}(\Delta_3)$.*

Proof. Let $\mathbf{P} = [\![P(a_1a_2a_3|x_1x_2x_3)]\!]$ and $\mathbf{Q} = [\![Q(a_1a_2a_3|x_1x_2x_3)]\!]$ be any two elements of $\mathcal{CT}^{\text{C-trilocal}}(\Delta_3)$. Then, $\mathbf{P}$ and $\mathbf{Q}$ have C-trLHVMs:

$$P(a_1a_2a_3|x_1x_2x_3) = \int_\Lambda p_1(\lambda_1)p_2(\lambda_2)p_3(\lambda_3)P_1(a_1|x_1,\lambda_3\lambda_1)P_2(a_2|x_2,\lambda_1\lambda_2)P_3(a_3|x_3,\lambda_2\lambda_3)\mathrm{d}\mu(\lambda),$$

and

$$Q(a_1a_2a_3|x_1x_2x_3) = \int_\Gamma q_1(\xi_1)q_2(\xi_2)q_3(\xi_3)Q_1(a_1|x_1,\xi_3\xi_1)Q_2(a_2|x_2,\xi_1\xi_2)Q_3(a_3|x_3,\xi_2\xi_3)\mathrm{d}\gamma(\xi)$$

for all possible a_1, a_2, a_3. Put

$$P_0(a_1a_2a_3|x_1x_2x_3) \equiv \frac{1}{o_1o_2o_3}, \mathbf{P}_0 := [\![P_0(a_1a_2a_3|x_1x_2x_3)]\!],$$

then $\mathbf{P}_0$ is a D-trilocal (and then C-trilocal) CT over Δ_3. For every $t \in [0, 1/2]$, set

$$P_1^t(a_1|x_1,\lambda_3\lambda_1) = (1-2t)P_1(a_1|x_1,\lambda_3\lambda_1) + 2t\frac{1}{o_1};$$

$$P_2^t(a_2|x_2,\lambda_1\lambda_2) = (1-2t)P_2(a_2|x_2,\lambda_1\lambda_2) + 2t\frac{1}{o_2};$$

$$P_3^t(a_3|x_3,\lambda_2\lambda_3) = (1-2t)P_3(a_3|x_3,\lambda_2\lambda_3) + 2t\frac{1}{o_3},$$

which are clearly PDs of a_1, a_2 and a_3, respectively. Putting

$$P^t(a_1a_2a_3|x_1x_2x_3) = \int_\Lambda q_1(\lambda_1)q_2(\lambda_2)q_3(\lambda_3)P_1^t(a_1|x_1,\lambda_3\lambda_1)P_2^t(a_2|x_2,\lambda_1\lambda_2)P_3^t(a_3|x_3,\lambda_2\lambda_3)\mathrm{d}\mu(\lambda),$$

then $\mathbf{P}(t) := [\![P^t(a_1a_2a_3|x_1x_2x_3)]\!]$ is a C-trilocal CT over Δ_3 for every $t \in [0, 1/2]$ with $\mathbf{P}(0) = \mathbf{P}$ and $\mathbf{P}(1/2) = \mathbf{P}_0$. Obviously, the map $t \mapsto \mathbf{P}(t)$ from $[0, 1/2]$ into $\mathcal{PT}^{\text{C-trilocal}}(\Omega_3)$ is continuous.

Similarly, for every $t \in [1/2, 1]$, set

$$Q_1^t(a_1|x_1,\xi_3\xi_1) = (2t-1)Q_1(a_1|x_1,\xi_3\xi_1) + 2(1-t)\frac{1}{o_1};$$

$$Q_2^t(a_2|x_2,\xi_1\xi_2) = (2t-1)Q_2(a_2|x_2,\xi_1\xi_2) + 2(1-t)\frac{1}{o_2};$$

$$Q_3^t(a_3|x_3,\xi_2\xi_3) = (2t-1)Q_3(a_3|x_3,\xi_2\xi_3) + 2(1-t)\frac{1}{o_3},$$

which are clearly PDs of a_1, a_2 and a_3, respectively. Putting

$$Q^t(a_1a_2a_3|x_1x_2x_3) = \int_\Gamma q_1(\xi_1)q_2(\xi_2)q_3(\xi_3)Q_1^t(a_1|x_1,\xi_3\xi_1)Q_2^t(a_2|x_2,\xi_1\xi_2)Q_3^t(a_3|x_3,\xi_2\xi_3)\mathrm{d}\gamma(\xi),$$

then $\mathbf{Q}(t) := [\![Q^t(a_1a_2a_3|x_1x_2x_3)]\!]$ is a C-trilocal CT over Δ_3 for every $t \in [1/2, 1]$ with $\mathbf{Q}(1/2) = \mathbf{P}_0$ and $\mathbf{Q}(1) = \mathbf{Q}$. Obviously, the map $t \mapsto \mathbf{Q}(t)$ from $[1/2, 1]$ into $\mathcal{PT}^{\text{C-trilocal}}(\Delta_3)$ is continuous.

Define a mapping $f : [0, 1] \to \mathcal{CT}^{\text{C-trilocal}}(\Delta_3)$ by

$$f(t) = \begin{cases} \mathbf{P}(t), & t \in [0, 1/2]; \\ \mathbf{Q}(t), & t \in (1/2, 1], \end{cases}$$

then f is continuous everywhere and and then induces a path in $\mathcal{CT}^{\text{C-trilocal}}(\Delta_3)$, connecting $\mathbf{P}$ and $\mathbf{Q}$. This shows that $\mathcal{CT}^{\text{C-trilocal}}(\Delta_3)$ is path-connected. Similarly, $\mathcal{CT}^{\text{D-trilocal}}(\Delta_3)$ is also path-connected. The proof is completed. □

For $k = 1, 2, 3$, taking a CT $\mathbf{E_k} = [\![E_k(a_k|x_k)]\!]$ over $[o_k] \times [m_k]$ and defining

$$S_1(a_1 a_2 a_3 | x_1 x_2 x_3) = E_1(a_1 | x_1) \times \frac{1}{o_2} \times \frac{1}{o_3},$$

$$S_2(a_1 a_2 a_3 | x_1 x_2 x_3) = \frac{1}{o_1} \times E_2(a_2 | x_2) \times \frac{1}{o_3},$$

$$S_3(a_1 a_2 a_3 | x_1 x_2 x_3) = \frac{1}{o_1} \times \frac{1}{o_2} \times E_3(a_3 | x_3),$$

we obtain three CTs $\mathbf{S}_k := [\![S_k(a_1 a_2 a_3 | x_1 x_2 x_3)]\!]$ over Δ_3 with

$$\sum_{a_i (i \neq k)} S_k(a_1 a_2 a_3 | x_1 x_2 x_3) = E_k(a_k | x_k)$$

for $k = 1, 2, 3$. Clearly, $\mathbf{S}_k$ is D-trilocal and then C-trilocal CT over Δ_3 for each k. Put

$$\mathcal{CT}_{\mathbf{E}_k}^{\text{C-trilocal}}(\Delta_3) = \{\mathbf{P} \in \mathcal{CT}^{\text{C-trilocal}}(\Delta_3) : \mathbf{P}_k = \mathbf{E}_k\},$$

where

$$P_k(a_k | x_k) := \sum_{a_i (i \neq k)} P(a_1 a_2 a_3 | x_1 x_2 x_3)$$

denotes the marginal distribution of $P(a_1 a_2 a_3 | x_1 x_2 x_3)$ on the k-th node.

Theorem 3. (Partial star-convexity). *The set* $\mathcal{CT}_{\mathbf{E}_k}^{\text{C-trilocal}}(\Delta_3)$ *is star-convex with a sun* $\mathbf{S}_k$ *for each* $k = 1, 2, 3$, *i.e.*,

$$t\mathbf{S}_k + (1-t)\mathcal{CT}_{\mathbf{E}_k}^{\text{C-trilocal}}(\Delta_3) \subset \mathcal{CT}_{\mathbf{E}_k}^{\text{C-trilocal}}(\Delta_3), \ \forall t \in [0, 1]. \tag{30}$$

Proof. Let $\mathbf{P} = [\![P(a_1 a_2 a_3 | x_1 x_2 x_3)]\!] \in \mathcal{CT}_{\mathbf{E}_1}^{\text{C-trilocal}}(\Delta_3)$. Then, $\mathbf{P}$ has a C-triLHVM:

$$
\begin{aligned}
P(a_1 a_2 a_3 | x_1 x_2 x_3) &= \int_\Lambda p_1(\lambda_1) p_2(\lambda_2) p_3(\lambda_3) P_1(a_1 | x_1, \lambda_3 \lambda_1) \\
&\quad \times P_2(a_2 | x_2, \lambda_1 \lambda_2) P_3(a_3 | x_3, \lambda_2 \lambda_3) \mathrm{d}\mu(\lambda),
\end{aligned} \tag{31}
$$

where $(\Lambda, \Omega, \mu) = (\Lambda_1 \times \Lambda_2 \times \Lambda_3, \Omega_1 \times \Omega_2 \times \Omega_3, \mu_1 \times \mu_2 \times \mu_3)$ is a product measure space with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. Thus,

$$
\begin{aligned}
E(a_1 | x_1) = P_1(a_1 | x_1) &:= \sum_{a_2, a_3} P(a_1 a_2 a_3 | x_1 x_2 x_3) \\
&= \int_{\Lambda_1 \times \Lambda_3} p_1(\lambda_1) p_3(\lambda_3) P_1(a_1 | x_1, \lambda_3 \lambda_1) \mathrm{d}\mu_1(\lambda_1) \mathrm{d}\mu_3(\lambda_3).
\end{aligned} \tag{32}
$$

Put $P(\{0, 1\}) = \{\varnothing, \{0\}, \{1\}, \{0, 1\}\}$, which is a σ-algebra on $\{0, 1\}$, and set

$$\Lambda_2 = \Lambda \times \{0, 1\}, \Omega_2' = \Omega_2 \times P(\{0, 1\}), \lambda_2' = (\lambda_2, s), \mu_2' = \mu_2 \times c,$$

where c denotes the counting measure on $\{0, 1\}$. Then, we obtain a product measure space

$$(\Lambda_1 \times \Lambda_2' \times \Lambda_3, \Omega_1 \times \Omega_2' \times \Omega_3, \mu_1 \times \mu_2' \times \mu_3).$$

For every $t \in [0, 1]$ and every $\lambda_2' = (\lambda_2, s)$, set

$$f(\lambda_2') = \begin{cases} p_2(\lambda_2)(1 - t), & s = 0; \\ p_2(\lambda_2)t, & s = 1, \end{cases}$$

which is a DF of λ_2'; define

$$P_2(a_2|x_2, \lambda_1\lambda_2') = \begin{cases} \frac{1}{o_2}, & s = 0; \\ P_B(a_2|x_2, \lambda_1\lambda_2), & s = 1, \end{cases}$$

$$P_3(a_3|x_3, \lambda_2'\lambda_3) = \begin{cases} \frac{1}{o_3}, & s = 0; \\ P_C(a_3|x_3, \lambda_2\lambda_3), & s = 1, \end{cases}$$

which are PDs of a_2 and a_3, respectively. For all $x_1, x_2, x_3, a_1, a_2, a_3$, we see from (32) and (31) that

$$
\begin{aligned}
& \int_{\Lambda_1 \times \Lambda_2' \times \Lambda_3} p_1(\lambda_1) f(\lambda_2, s) p_3(\lambda_3) P_1(a_1|x_1, \lambda_3\lambda_1) \\
& \times P_2(a_2|x_2, \lambda_1\lambda_2') P_3(a_3|x_3, \lambda_2'\lambda_3) d\mu_1(\lambda_1) d\mu_2'(\lambda_2') d\mu_3(\lambda_3) \\
= & \int_{\Lambda_1 \times \Lambda_2 \times \Lambda_3} p_1(\lambda_1) p_2(\lambda_2) p_3(\lambda_3)(1-t) P_1(a_1|x_1, \lambda_3\lambda_1) \\
& \times \frac{1}{o_2} \frac{1}{o_3} d\mu_1(\lambda_1) d\mu_2(\lambda_2) d\mu_3(\lambda_3) \\
& + \int_{\Lambda_1 \times \Lambda_2 \times \Lambda_3} p_1(\lambda_1) p_2(\lambda_2) p_3(\lambda_3) t P_1(a_1|x_1, \lambda_3\lambda_1) \\
& \times P_2(a_2|x_2, \lambda_1\lambda_2) P_3(a_3|x_3, \lambda_2\lambda_3) d\mu_1(\lambda_1) d\mu_2(\lambda_2) d\mu_3(\lambda_3) \\
= & (1-t) S(a_1 a_2 a_3|x_1 x_2 x_3) + t P(a_1 a_2 a_3|x_1 x_2 x_3).
\end{aligned}
$$

This shows that $(1-t)\mathbf{S}_1 + t\mathbf{P}$ is C-trilocal with $\mathbf{S}_1 = \mathbf{E}_1$ and then an element of $\mathcal{CT}_{\mathbf{E}_1}^{\text{C-trilocal}}(\Delta_3)$. Thus,

$$t\mathbf{S}_1 + (1-t)\mathcal{CT}_{\mathbf{E}_1}^{\text{C-trilocal}}(\Delta_3) \subset \mathcal{CT}_{\mathbf{E}_1}^{\text{C-trilocal}}(\Delta_3)$$

for all $t \in [0,1]$. That is, $\mathcal{CT}_{\mathbf{E}_1}^{\text{C-trilocal}}(\Delta_3)$ is star-convex with a sun $\mathbf{S}_1$. Similarly, $\mathcal{CT}_{\mathbf{E}_k}^{\text{C-trilocal}}(\Delta_3)$ is star-convex with a sun $\mathbf{S}_k$ for $k = 2, 3$. The proof is completed. $\square$

Remark 2. *Let* $\mathbf{p} = [\![p(i,j,k)]\!]$ *be a C-trilocal PT over a finite set* $I \times J \times K$ *with a C-triLHVM:*

$$p(i,j,k) = \int_\Lambda q_1(\lambda_1) q_2(\lambda_2) q_3(\lambda_3) P_1(i|\lambda_3\lambda_1) P_2(j|\lambda_1\lambda_2) P_3(k|\lambda_2\lambda_3) d\mu(\lambda),$$

where $q_j(\lambda_j)$ *is a DF of* λ_j, $P_1(i|\lambda_3\lambda_1), P_2(j|\lambda_1\lambda_2), P_3(k|\lambda_2\lambda_3)$ *are PDs of* λ_j, i, j *and* k, *respectively. Suppose that* $\{P_i(a_1|x_1)\}_{a_1 \in [o_1]}, \{P_j(a_2|x_2)\}_{a_2 \in [o_2]}$ *and* $\{P_k(a_3|x_3)\}_{a_3 \in [o_3]}$ *are PDs of* a_1, a_2 *and* a_3, *respectively, Thus, the CT* $\mathbf{P}$ *defined by*

$$P(a_1 a_2 a_3|x_1 x_2 x_3) = \sum_{i,j,k} p(i,j,k) P_i(a_1|x_1) P_j(a_2|x_2) P_k(a_3|x_3) \tag{33}$$

can be written as

$$
\begin{aligned}
P(a_1 a_2 a_3|x_1 x_2 x_3) &= \sum_{i,j,k} p(i,j,k) P_i(a_1|x_1) P_j(a_2|x_2) P_k(a_3|x_3) \\
&= \int_\Lambda q_1(\lambda_1) q_2(\lambda_2) q_3(\lambda_3) P_1(a_1|x_1, \lambda_3\lambda_1) \\
&\quad \times P_2(a_2|x_2, \lambda_1\lambda_2) P_3(a_3|x_3, \lambda_2\lambda_3) d\mu(\lambda),
\end{aligned}
$$

where

$$P_1(a_1|x_1, \lambda_3\lambda_1) = \sum_{i \in I} P_1(i|\lambda_3\lambda_1) P_i(a_1|x_1),$$

$$P_2(a_2|x_2, \lambda_1\lambda_2) = \sum_{j \in J} P_2(j|\lambda_1\lambda_2) P_j(a_2|x_2),$$

$$P_3(a_3|x_3, \lambda_2\lambda_3) = \sum_{k \in K} P_3(k|\lambda_2\lambda_3) P_k(a_3|x_3),$$

which are PDs of a_1, a_2 and a_3, respectively. Thus, $\mathbf{P}$ is a C-trilocal CT over Δ_3. In particular, when

$$N_i = o_i^{m_i} (i = 1, 2, 3), \Gamma_3 = [N_1] \times [N_2] \times [N_3], \mathbf{p} = [\![p(i,j,k)]\!] \in \mathcal{PT}^{C\text{-trilocal}}(\Gamma_3),$$

we obtain that $\mathbf{P} := \sum_{i,j,k} p(i,j,k)\mathbf{D}_{ijk}$ is a C-trilocal CT over Δ_3, where

$$\mathbf{D}_{ijk} = [\![D_{ijk}(a_1 a_2 a_3 | x_1 x_2 x_3)]\!] = [\![\delta_{a_1, J_i(x_1)} \delta_{a_2, K_j(x_2)} \delta_{a_3, L_k(x_3)}]\!],$$

in which

$$\{J_1, J_2, \ldots, J_{N_1}\} = \{J | J : [m_1] \to [o_1]\},$$
$$\{K_1, K_2, \ldots, K_{N_2}\} = \{K | K : [m_2] \to [o_2]\},$$
$$\{L_1, L_2, \ldots, L_{N_3}\} = \{L | L : [m_3] \to [o_3]\}.$$

Clearly, $\mathbf{D}_{ijk}$'s are D-trilocal CTs over Δ_3. This shows that

$$\mathcal{CT}^{C\text{-trilocal}}(\Delta_3) \supset \left\{ \sum_{i,j,k} p(i,j,k)\mathbf{D}_{ijk} : \mathbf{p} = [\![p(i,j,k)]\!] \in \mathcal{PT}^{C\text{-trilocal}}(\Gamma_3) \right\}. \tag{34}$$

Similarly,

$$\mathcal{CT}^{D\text{-trilocal}}(\Delta_3) \supset \left\{ \sum_{i,j,k} p(i,j,k)\mathbf{D}_{ijk} : \mathbf{p} = [\![p(i,j,k)]\!] \in \mathcal{PT}^{D\text{-trilocal}}(\Gamma_3) \right\}. \tag{35}$$

Next, we aim to show that Equations (34) and (35) are indeed equalities. To do this, we recall that an $m \times n$ function matrix $B(\lambda) = [b_{ij}(\lambda)]$ on Λ is said to be row-stochastic (RS) means that, for each $\lambda \in \Lambda$, $b_{ij}(\lambda) \geq 0$ for all i, j and $\sum_{j=1}^{n} b_{ij}(\lambda) = 1$ for all $i \in [m]$. It is clear that every $m \times n$ $\{0,1\}$-row statistics matrix $T = [T_{ij}]$ corresponds uniquely a mapping $F : [m] \to [n]$ so that $T_{ij} = \delta_{j, F(i)}$. Thus, the sets of all $\{0,1\}$-row-stochastic matrices of orders $m_1 \times o_1$, $m_2 \times o_2$, and $m_3 \times o_3$ can be written as

$$RSM_{m_1 \times o_1}^{(0,1)} = \{R_i := [\delta_{a_1, J_i(x_1)}]_{x_1, a_1} : i = 1, 2, \ldots, N_1\},$$

$$RSM_{m_2 \times o_2}^{(0,1)} = \{K_j := [\delta_{a_2, K_j(x_2)}]_{x_2, a_2} : j = 1, 2, \ldots, N_2\},$$

$$RSM_{m_3 \times o_3}^{(0,1)} = \{L_k := [\delta_{a_3, L_k(x_3)}]_{x_3, a_3} : k = 1, 2, \ldots, N_3\},$$

respectively.

Lemma 1 ([36]). *Let (Λ, Ω, μ) be a measure space. Then, every $m \times n$ function RS matrix $B(\lambda) = [b_{ij}(\lambda)]$ on Λ whose entries are Ω-measurable on Λ can be written as a convex combination of all $\{0,1\}$-RS matrices R_k's:*

$$B(\lambda) = \sum_{k=1}^{n^m} \alpha_k(\lambda) R_k, \ \forall \lambda \in \Lambda, \tag{36}$$

where $\alpha_k (k = 1, 2, \ldots, n^m)$ are all nonnegative and Ω-measurable functions on Λ.

Using ([35] Theorem 5.1) implies that

$$\mathcal{CT}^{\text{Bell-local}}(\Delta_3) = \left\{ \sum_{i,j,k} p(i,j,k)\mathbf{D}_{ijk} : \mathbf{p} = [\![p(i,j,k)]\!] \in \mathcal{PT}(\Gamma_3) \right\}, \tag{37}$$

where $\mathcal{PT}(\Gamma_3)$ denotes the set of all PTs over Γ_3. Based this lemma, we can show the following conclusion, which say that a CT over Δ_3 is C-trilocal (resp. D-trilocal) if and only if it can be written as a convex combination of local deterministic CTs $\mathbf{D}_{ijk}$'s with C-trilocal (resp. D-trilocal) coefficients.

Theorem 4.

$$\mathcal{CT}^{\text{C-trilocal}}(\Delta_3) = \left\{ \sum_{i,j,k} p(i,j,k) \mathbf{D}_{ijk} : \mathbf{p} = [\![p(i,j,k)]\!] \in \mathcal{PT}^{\text{C-trilocal}}(\Gamma_3) \right\}, \tag{38}$$

$$\mathcal{CT}^{\text{D-trilocal}}(\Delta_3) = \left\{ \sum_{i,j,k} p(i,j,k) \mathbf{D}_{ijk} : \mathbf{p} = [\![p(i,j,k)]\!] \in \mathcal{PT}^{\text{D-trilocal}}(\Gamma_3) \right\}. \tag{39}$$

Proof. Suppose that $\mathbf{P}$ is C-trilocal; then, it has a C-triLHVM (20). Since matrices

$$M(\lambda_3, \lambda_1) := [P_1(a_1|x_1, \lambda_3\lambda_1)]_{x_1, a_1} \in \mathbb{R}^{m_1 \times o_1},$$

$$M(\lambda_1, \lambda_2) := [P_2(a_2|x_2, \lambda_1\lambda_2)]_{x_2, a_2} \in \mathbb{R}^{m_2 \times o_2},$$

$$M(\lambda_2, \lambda_3) := [P_3(a_2|x_3, \lambda_2\lambda_3)]_{x_3, a_3} \in \mathbb{R}^{m_3 \times o_3}$$

are row-stochastic with measurable entries, we see from Lemma 1 that they have the following decompositions:

$$P_1(a_1|x_1, \lambda_3\lambda_1) = \sum_{i=1}^{N_1} P_1(i|\lambda_3\lambda_1)\delta_{a_1, J_i(x_1)}, \tag{40}$$

$$P_2(a_2|x_2, \lambda_1\lambda_2) = \sum_{j=1}^{N_2} P_2(j|\lambda_1\lambda_2)\delta_{a_2, K_j(x_2)}, \tag{41}$$

$$P_3(a_3|x_3, \lambda_2\lambda_3) = \sum_{k=1}^{N_3} P_3(k|\lambda_2\lambda_3)\delta_{a_3, L_k(x_3)}, \tag{42}$$

where $P_1(i|\lambda_3\lambda_1)$, $P_2(j|\lambda_1\lambda_2)$ and $P_3(k|\lambda_2\lambda_3)$ are PDs of i, j and k, respectively, and measurable w.r.t. (λ_3, λ_1), (λ_1, λ_2) and (λ_2, λ_3), respectively. Hence,

$$P(a_1a_2a_3|x_1x_2x_3) = \sum_{i,j,k} p(i,j,k)\delta_{a_1, J_i(x_1)}\delta_{a_2, K_j(x_2)}\delta_{a_3, L_k(x_3)}, \tag{43}$$

where

$$p(i,j,k) = \int_\Lambda q_1(\lambda_1)q_2(\lambda_2)q_3(\lambda_3)P_1(i|\lambda_3\lambda_1)P_2(j|\lambda_1\lambda_2)P_3(k|\lambda_2\lambda_3)\mathrm{d}\mu(\lambda), \tag{44}$$

which forms a C-trilocal PT $\mathbf{p} = [\![p(i,j,k)]\!]$ over Γ_3, satisfying

$$\mathbf{P} = \sum_{i,j,k} p(i,j,k)\mathbf{D}_{ijk}.$$

Conversely, if $\mathbf{p} = [\![p(i,j,k)]\!]$ is a C-trilocal PT over Γ_3, then it has a C-triLVHM (44), and so the CT $\mathbf{P} = [\![P(a_1a_2a_3|x_1x_2x_3)]\!]$ defined by (43) has a C-triLHVM (20) in light of (40)–(42). Thus, $\mathbf{P}$ becomes a C-trilocal CT over Δ_3 and Equation (38) follows. Similarly, (39) is also valid. The proof is completed. $\square$

Theorem 4 implies that both D-trilocal and C-trilocal CTs over Δ_3 are Bell local. It also yields that every C-trilocal CT $\mathbf{P}$ over Δ_3 can be written as a convex combination (43) of the deterministic D-bilocal CTs $\mathbf{D}_{ijk}$ over Δ_3.

Corollary 1.

$$\mathcal{CT}^{\text{C-trilocal}}(\Delta_3) \subset \text{conv}(\mathcal{CT}^{\text{D-bilocal}}(\Delta_3)) = \mathcal{CT}^{\text{Bell-local}}(\Delta_3). \tag{45}$$

Let $\mathcal{CT}_R^{\text{C-trilocal}}(\Delta_3)$ be the set of all C-trilocal CTs over Δ_3 with C-triLHVMs given by three-hold Riemann integrals over a product region $\Lambda_1 \times \Lambda_2 \times \Lambda_3$.

Theorem 5.
$$\mathcal{CT}^{\text{D-trilocal}}(\Delta_3) \subset \mathcal{CT}_R^{\text{C-trilocal}}(\Delta_3) \subset \overline{\mathcal{CT}^{\text{D-trilocal}}(\Delta_3)}, \tag{46}$$

where $\overline{\mathcal{CT}^{\text{D-trilocal}}(\Delta_3)}$ *denotes the closure of* $\mathcal{CT}^{\text{D-trilocal}}(\Delta_3)$ *in the Hilbert space* $\mathcal{T}(\Delta_3)$.

Proof. The second inclusion can be checked in a way similar to the proof of Proposition 6. To check the first inclusion, we let $\mathbf{P} \in \mathcal{CT}^{\text{D-trilocal}}(\Delta_3)$. Then, it can be written as (21):

$$
\begin{aligned}
P(a_1 a_2 a_3 | x_1 x_2 x_3) &= \sum_{\lambda_1=1}^{n_1} \sum_{\lambda_2=1}^{n_2} \sum_{\lambda_3=1}^{n_3} q_1(\lambda_1) q_2(\lambda_2) q_3(\lambda_3) P_1(a_1 | x_1, \lambda_3 \lambda_1) \\
&\quad \times P_2(a_2 | x_2, \lambda_1 \lambda_2) P_3(a_3 | x_3, \lambda_2 \lambda_3)
\end{aligned}
$$

for all $x_k \in [m_k], a_k \in [o_k] (k = 1, 2, 3)$, where

$$q_k(\lambda_k), P_1(a_1 | x_1, \lambda_3 \lambda_1), P_2(a_2 | x_2, \lambda_1 \lambda_2), P_3(a_3 | x_3, \lambda_2 \lambda_3)$$

are PDs of λ_k, a_1, a_2, a_3, respectively. By using the characteristic function of a set S:

$$\chi_S(x) = \begin{cases} 1, & x \in S; \\ 0, & x \notin S, \end{cases}$$

we define functions:

$$p_k(t_k) = \sum_{\lambda_k} q_k(\lambda_k) \chi_{[\lambda_k - 1, \lambda_k)}(t_k) (\forall t_k \in [0, n_k)), \; p_k(n_k) = 0, k = 1, 2, 3,$$

$$Q_1(a_1 | x_1, t_3 t_1) = \sum_{\lambda_3, \lambda_1} P_1(a_1 | x_1, \lambda_3 \lambda_1) \chi_{[\lambda_1 - 1, \lambda_1) \times [\lambda_3 - 1, \lambda_3)}(t_1, t_3)$$

if $(t_1, t_3) \in [0, n_1) \times [0, n_3)$; $Q_1(a_1 | x_1, t_3 t_1) = \frac{1}{o_1}$, otherwise;

$$Q_2(a_2 | x_2, t_1 t_2) = \sum_{\lambda_1, \lambda_2} P_2(a_2 | x_2, \lambda_1 \lambda_2) \chi_{[\lambda_1 - 1, \lambda_1) \times [\lambda_2 - 1, \lambda_2)}(t_1, t_2)$$

if $(t_1, t_2) \in [0, n_1) \times [0, n_2)$; $Q_2(a_2 | x_2, t_1 t_2) = \frac{1}{o_2}$, otherwise;

$$Q_3(a_3 | x_3, t_2 t_3) = \sum_{\lambda_2, \lambda_3} P_3(a_3 | x_3, \lambda_2 \lambda_3) \chi_{[\lambda_2 - 1, \lambda_2) \times [\lambda_3 - 1, \lambda_3)}(t_2, t_3)$$

if $(t_2, t_3) \in [0, n_2) \times [0, n_3)$; $Q_3(a_3 | x_3, t_2 t_3) = \frac{1}{o_3}$, otherwise. Clearly, $p_k(t_k)$ is a DF of $t_k \in [0, n_k] (k = 1, 2, 3)$, $Q_1(a_1 | x_1, t_3 t_1), Q_2(a_2 | x_2, t_1 t_2)$ and $Q_3(a_3 | x_3, t_2 t_3)$ are PDs of a_1, a_2 and a_3, respectively, for all $x_k \in [m_k]$ and all $t_k \in [0, n_k]$. It is easy to check that

$$
\begin{aligned}
P(a_1 a_2 a_3 | x_1 x_2 x_3) &= \int_0^{n_1} \int_0^{n_2} \int_0^{n_3} p_1(t_1) p_2(t_2) p_3(t_3) Q_1(a_1 | x_1, t_3 t_1) \\
&\quad \times Q_2(a_2 | x_2, t_1 t_2) Q_3(a_3 | x_3, t_2 t_3) dt_1 dt_2 dt_3
\end{aligned}
$$

for all possible x_i, a_i. Thus, $\mathbf{P} \in \mathcal{CT}_R^{\text{C-trilocal}}(\Delta_3)$. This completes the proof. $\quad\square$

4. Conclusions and Questions

When a triangle network is locally measured one run or many runs, a probability tensor (PT) $\mathbf{P} = [\![P(a_1 a_2 a_3)]\!]$ over Ω_3 or a correlation tensor (CT) $\mathbf{P} = [\![P(a_1 a_2 a_3 | x_1 x_2 x_3)]\!]$ over Δ_3 is obtained. In this work, we have introduced and discussed C-trilocality and D-trilocality of PTs and CTs according to their descriptions of continuous (integral) and

discrete (sum) trilocal hidden variable models (C-triLHVMs and D-triLHVMs). We named that a PT (or CT) **P** is C-trilocal (resp. D-trilocal) if it can be described by a C-triLHVM (resp. D-triLHVM). With these definitions, the following conclusions have been proved:

(1) A PT (resp. CT) is D-trilocal if and only if it can be realized in a triangle network by three shared separable states and a local POVM (resp. a set of local POVMs);

(2) A CT is C-trilocal (resp. D-trilocal) if and only if it can be written as a convex combination of the product deterministic CTs with a C-trilocal (resp. D-trilocal) PT as coefficient tensor;

(3) When one of the shared states $\rho_{i,j}$ in the triangle network is Bell nonlocal (especially, a pure entangled state), the network must be C-nontrilocal and then D-nontrilocal;

(4) The sets $\mathcal{PT}^{\text{C-trilocal}}(\Omega_3)$, $\mathcal{PT}^{\text{D-trilocal}}(\Omega_3)$, $\mathcal{CT}^{\text{C-trilocal}}(\Delta_3)$ and $\mathcal{CT}^{\text{D-trilocal}}(\Delta_3)$ are path-connectedness and have partial star-convexity.

However, the following questions are interesting and needed to be discussed further.

Question 2.

(Q2.1) $\mathcal{CT}^{\text{C-trilocal}}(\Delta_3) = \mathcal{CT}^{\text{D-trilocal}}(\Delta_3)$?

(Q2.2) $\mathcal{PT}^{\text{C-trilocal}}(\Omega_3) = \mathcal{PT}^{\text{D-trilocal}}(\Omega_3)$?

Question 3.

(Q3.1) $\overline{\mathcal{CT}^{\text{D-trilocal}}(\Delta_3)} = \mathcal{CT}^{\text{D-trilocal}}(\Delta_3)$?

(Q3.2) $\overline{\mathcal{PT}^{\text{D-trilocal}}(\Omega_3)} = \mathcal{PT}^{\text{D-trilocal}}(\Omega_3)$?

Question 4.

(Q4.1) $\overline{\mathcal{CT}^{\text{C-trilocal}}(\Delta_3)} = \mathcal{CT}^{\text{C-trilocal}}(\Delta_3)$?

(Q4.2) $\overline{\mathcal{PT}^{\text{C-trilocal}}(\Omega_3)} = \mathcal{PT}^{\text{C-trilocal}}(\Omega_3)$?

Theorem 4 implies that (Qi.1) and (Qi.2) are equivalent for each $i = 2, 3, 4$.

Author Contributions: The work of this paper was accomplished by S.X., H.C., Z.G. and K.H. Moreover, all authors have read the paper carefully and approved the research contents that were written in the final manuscript. All authors have read and agreed to the published version of the manuscript.

Institutional Review Board Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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