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Concepts of Interpolation in Stratified Institutions

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Abstract: The extension of the (ordinary) institution theory of Goguen and Burstall, known as the theory of *stratified institutions*, is a general axiomatic approach to model theories where the satisfaction is parameterized by states of models. Stratified institutions cover a uniformly wide range of applications from various Kripke semantics to various automata theories and even model theories with partial signature morphisms. In this paper, we introduce two natural concepts of logical interpolation at the abstract level of stratified institutions and we provide some sufficient technical conditions in order to establish a causality relationship between them. In essence, these conditions amount to the existence of nominals structures, which are considered fully and abstractly.

Keywords: interpolation; institution theory; stratified institutions; model theory; abstract model theory; categorical model theory

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1. Introduction

1.1. Stratified Institutions

Institution theory is a general axiomatic approach to model theory that was originally introduced in computing science by Goguen and Burstall [1]. In institution theory, all three components of logical systems—namely the syntax, the semantics, and the satisfaction relation between them—are treated fully abstractly by relying heavily on category theory. This approach has significantly impacted both theoretical computing science [2] and model theory as such [3] (both mentioned monographs rather reflect the stage of development of institution theory and its applications at the moment they were published or even before that. In the meantime a lot of additional important developments have already taken place. At this moment, the literature around institution theory that has been developed over the course of four decades or so is rather vast.). In computing science, the concept of institution has emerged as the most fundamental mathematical structure of logic-based formal specifications, and a great deal of theory is being developed at the general level of abstract institutions. In model theory, the institution theoretic approach meant an axiomatic-driven redesign of core parts of model theory at a new level of generality—namely, that of abstract institutions—independently of any concrete logical system. Moreover, there is a strong interdependency between the two lines of development.

The institution theoretic approach to model theory has also been refined in order to directly address some important non-classical model theoretic aspects. One such direction is motivated by models with states, which appear in myriad forms in computing science and logic. The institution theory answer to this is given by the theory of stratified institutions introduced in [4,5] and further developed or invoked in works such as [6–10], etc. The concept of stratified institutions covers at least the following classes of examples:

- a wide variety of Kripke semantics like in [5,6,8,9];
- various automata theories;
- various model theories with partiality for signature morphisms [10], providing mathematical foundations to conceptual blending (see [11]).

1.2. Institution Theoretic Interpolation

Interpolation is a notoriously important logical property which is easy to understand but difficult to establish. It also has a number of important applications in computing science, especially in formal specification theory [12–17], but also in data bases (ontologies) [18], automated reasoning [19,20], type checking [21], model checking [22], structured theorem proving [23,24], etc. Computing science and model theoretical motivations have led to a very general approach to interpolation [25] within the theory of institutions proposed by Goguen and Burstall [1,3] that is completely independent of any concrete logical system. This direction of study and research has produced a substantial body of results reported in works such as [13,17,25–33].

The goal of the work reported in this paper is to take a first step towards the extension of institution theory interpolation to stratified institutions by considering the conceptual specificities of the latter.

1.3. Our Contributions

Interpolation is a property of a consequence relation. Stratified institutions admit two consequence relations, corresponding to "local" and "global" satisfaction, respectively. On this basis, we define two corresponding concepts of interpolation at the level of abstract stratified institution. Our second contribution is a study of the causal relationship between the two concepts of interpolation. Our main result is a proof that, under some technical conditions formulated at the general level, "local" interpolation causes "global" interpolation. This set of sufficient conditions essentially refers to the presence of a nominal structure (in the sense of the hybrid version of modal logic of [34–36], but in our work is considered axiomatically and abstractly and without any reference to Kripke semantics of any sort).

2. Preliminaries

This section is meant to introduce the reader to the mathematical background of our work in a way that makes the paper as self-contained as possible while still striking a right balance between the survey aspect and the developments of new concepts and results. As such, it can be treated as a basic introduction to the theory of stratified institutions. For this, we gradually recall from the literature basic concepts and terminology about

- category theory;
- institution theory; and
- stratified institution theory.

The latter part is much more substantial than the former two parts as it contains presentations of examples and of some more advanced concepts.

2.1. Categories

The mathematical structures in institution theory are category-theoretic. The following elementary category theory concepts are used in our work: opposite (dual) of a category $\mathbb{C}$ (denoted $\mathbb{C}^{\ominus}$), functor, natural transformation, lax natural transformation, and pushout. Familiarity with these concepts is a requirement for being able to follow this work.

We usually follow the terminology and notations of [37] with some few notable exceptions. One of them is the way we write compositions. Thus, we will use the diagrammatic notation for compositions of arrows in categories, i.e., if $f : A \to B$ and $g : B \to C$ are arrows then $f;g$ denotes their composition. For an arrow $f : A \to B$ we may denote its domain A by $\square f$ and its codomain B by $f\square$. Let **Set** denote the category of sets, **CAT** denote the "quasi-category" of categories, and $|\mathbf{CAT}|$ the collection of all categories. In general, for any category $\mathbb{C}$, by $|\mathbb{C}|$ we denote its class of objects. We use $\Rightarrow$ rather than $\stackrel{\cdot}{\to}$ for natural transformations.

2.2. Institutions

The original standard reference for institution theory is [1]. An *institution*

$$\mathcal{I} = (\mathrm{Sign}^{\mathcal{I}}, \mathrm{Sen}^{\mathcal{I}}, \mathrm{Mod}^{\mathcal{I}}, \models^{\mathcal{I}})$$

consists of

- a category $\mathrm{Sign}^{\mathcal{I}}$ whose objects are called *signatures*,
- a sentence functor $\mathrm{Sen}^{\mathcal{I}} : \mathrm{Sign}^{\mathcal{I}} \to \mathbf{Set}$ defining for each signature a set whose elements are called *sentences* over that signature and defining for each signature morphism a *sentence translation* function,
- a model functor $\mathrm{Mod}^{\mathcal{I}} : (\mathrm{Sign}^{\mathcal{I}})^{\ominus} \to \mathbf{CAT}$ defining for each signature Σ the category $\mathrm{Mod}^{\mathcal{I}}(\Sigma)$ of Σ-*models* and Σ-model homomorphisms, and for each signature morphism φ the *reduct*= functor $\mathrm{Mod}^{\mathcal{I}}(\varphi)$,
- for every signature Σ, a binary Σ-*satisfaction relation* $\models^{\mathcal{I}}_{\Sigma} \subseteq |\mathrm{Mod}^{\mathcal{I}}(\Sigma)| \times \mathrm{Sen}^{\mathcal{I}}(\Sigma)$,

such that for each signature morphism φ, the *satisfaction condition*

$$M' \models^{\mathcal{I}}_{\Sigma'} \mathrm{Sen}^{\mathcal{I}}(\varphi)\rho \quad \text{if and only if} \quad \mathrm{Mod}^{\mathcal{I}}(\varphi)M' \models^{\mathcal{I}}_{\Sigma} \rho \tag{1}$$

holds for each $M' \in |\mathrm{Mod}^{\mathcal{I}}(\varphi\Box)|$ and $\rho \in \mathrm{Sen}^{\mathcal{I}}(\Box\varphi)$. This can be expressed as the satisfaction relation $\models$ being a natural transformation:

$$\begin{tikzcd}
\Box\varphi \arrow[d, "\varphi"'] & \mathrm{Sen}^{\mathcal{I}}(\Box\varphi) \arrow[r, "\models^{\mathcal{I}}_{\Box\varphi}"] \arrow[d, "\mathrm{Sen}^{\mathcal{I}}(\varphi)"'] & {[|\mathrm{Mod}^{\mathcal{I}}(\Box\varphi)| \to 2]} \arrow[d, "{[|\mathrm{Mod}^{\mathcal{I}}(\varphi)| \to 2](=\mathrm{Mod}^{\mathcal{I}}(\varphi)^{-1})}"] \\
\varphi\Box & \mathrm{Sen}^{\mathcal{I}}(\varphi\Box) \arrow[r, "\models^{\mathcal{I}}_{\varphi\Box}"'] & {[|\mathrm{Mod}^{\mathcal{I}}(\varphi\Box)| \to 2]}
\end{tikzcd}$$

($[|\mathrm{Mod}(\Sigma)| \to 2]$ represents the "set" of the"subsets" of $|\mathrm{Mod}(\Sigma)|$).

We may omit the superscripts or subscripts from the notations of the components of institutions when there is no risk of ambiguity. For instance, if the considered institution and signature are clear, we may denote $\models^{\mathcal{I}}_{\Sigma}$ just by $\models$. For $M = \mathrm{Mod}(\varphi)M'$, we say that M is the φ-*reduct* of M' and that M' is a φ-*expansion* of M. Another notational simplification consists, in the case of the sentence translations, of writing $\varphi(\rho)$ instead of $\mathrm{Sen}(\varphi)\rho$.

The literature (e.g., [1–3], etc.) shows a myriad of logical systems from computing or from mathematical logic captured as institutions. In fact, an informal thesis underlying institution theory is that any "logic" may be captured by the above definition. While this should be taken with a grain of salt, it certainly applies to any logical system based on satisfaction between sentences and models of any kind.

For any signature Σ in an institution if $E \subseteq \mathrm{Sen}(\Sigma)$ and $M \in |\mathrm{Mod}(\Sigma)|$ then $M \models E$ means that $M \models e$ for each $e \in E$. Then the relation $\models_{\Sigma}$ between any sets of Σ-sentences, called the *semantic consequence relation* of the respective institution, is defined by

$$E \models_{\Sigma} E' \quad \text{if and only if for any} \quad \Sigma\text{-model } M,\ M \models_{\Sigma} E \text{ implies } M \models_{\Sigma} E'.$$

The following are important well-known properties of the semantic consequence that have a very general nature. Let $\varphi : \Sigma \to \Sigma'$ be a signature morphism and let $\Gamma, \Gamma', \Gamma''$ any sets of Σ-sentences, and $(\Gamma_i)_{i\in\omega}$ a family of sets of Σ-sentences. Then

if $\Gamma \models_{\Sigma} \Gamma_i$ for each $i \in I$ then $\Gamma \models_{\Sigma} \bigcup_{i\in I} \Gamma_i$	*union*
if $\Gamma' \supseteq \Gamma$ then $\Gamma' \models_{\Sigma} \Gamma$	*monotonicity*
if $\Gamma \models_{\Sigma} \Gamma'$ and $\Gamma' \models_{\Sigma} \Gamma''$ then $\Gamma \models_{\Sigma} \Gamma''$	*transitivity*
if $\Gamma \models_{\Sigma} \Gamma'$ then $\varphi\Gamma \models_{\Sigma'} \varphi\Gamma'$	*translation*.

2.3. Stratified Institutions

Informally, the main idea behind the concept of the stratified institution as introduced in [4,5] is to enhance the concept of an institution with "states" for the models. Thus, each model M comes equipped with a set $[\![M]\!]$ that has to satisfy some structural axioms. The following definition has been given in [6] and represents an important upgrade of the original definition from [5], with the main purpose of making the definition of stratified institutions really usable for doing in-depth model theory. A slightly different upgrade, but very closely related to that of [6], has been proposed in [7].

A *stratified institution* $\mathcal{S}$ is a tuple $(\mathrm{Sign}^{\mathcal{S}}, \mathrm{Sen}^{\mathcal{S}}, \mathrm{Mod}^{\mathcal{S}}, [\![\_]\!]^{\mathcal{S}}, \models^{\mathcal{S}})$ consisting of:

- category $\mathrm{Sign}^{\mathcal{S}}$ of signatures,
- a sentence functor $\mathrm{Sen}^{\mathcal{S}} : \mathrm{Sign}^{\mathcal{S}} \to \mathbf{Set}$;
- a model functor $\mathrm{Mod}^{\mathcal{S}} : (\mathrm{Sign}^{\mathcal{S}})^{\ominus} \to \mathbf{CAT}$;
- a "stratification" lax natural transformation $[\![\_]\!]^{\mathcal{S}} : \mathrm{Mod}^{\mathcal{S}} \Rightarrow SET$, where $SET : \mathrm{Sign}^{\mathcal{S}} \to \mathbf{CAT}$ is a functor mapping each signature morphism to the identity functor on $\mathbf{Set}$; and
- a satisfaction relation between models and sentences which is parameterized by model states, $M\ (\models^{\mathcal{S}})^{w}_{\Sigma}\ \rho$ where $w \in [\![M]\!]^{\mathcal{S}}_{\Sigma}$ such that the following *satisfaction condition*

$$\mathrm{Mod}^{\mathcal{S}}(\varphi)M'\ \ (\models^{\mathcal{S}})^{[\![M']\!]_{\varphi}w}_{\Sigma}\ \ \rho\ \text{ if and only if }\ M'\ \ (\models^{\mathcal{S}})^{w}_{\Sigma'}\ \mathrm{Sen}^{\mathcal{S}}(\varphi)\rho \tag{2}$$

holds for any signature morphism φ, $M' \in |\mathrm{Mod}^{\mathcal{S}}(\varphi\Box)|$, $w \in [\![M']\!]^{\mathcal{S}}_{\varphi\Box}$, $\rho \in \mathrm{Sen}^{\mathcal{S}}(\Box\varphi)$.

As with ordinary institutions, when appropriate we shall also use simplified notations without superscripts or subscripts that are clear from the context.

The lax natural transformation property of $[\![\_]\!]$ is depicted in the diagram below

$$\begin{tikzcd}
\Sigma'' & & \mathrm{Mod}(\Sigma'') \arrow[r, "{[\![_]\!]_{\Sigma''}}"] \arrow[d, "\mathrm{Mod}(\varphi')"'] \arrow[dr, phantom, "\Leftarrow {[\![_]\!]_{\varphi'}}"] & \mathbf{Set} \arrow[d, "="] \\
\Sigma' \arrow[u, "\varphi'"] & & \mathrm{Mod}(\Sigma') \arrow[r, "{[\![_]\!]_{\Sigma'}}"] \arrow[d, "\mathrm{Mod}(\varphi)"'] \arrow[dr, phantom, "\Leftarrow {[\![_]\!]_{\varphi}}"] & \mathbf{Set} \arrow[d, "="] \\
\Sigma \arrow[u, "\varphi"] & & \mathrm{Mod}(\Sigma) \arrow[r, "{[\![_]\!]_{\Sigma}}"'] & \mathbf{Set}
\end{tikzcd}$$

with the following compositionality property for each Σ''-model M'':

$$[\![M'']\!]_{(\varphi;\varphi')} = [\![M'']\!]_{\varphi'}; [\![\mathrm{Mod}(\varphi')M'']\!]_{\varphi}. \tag{3}$$

Moreover the natural transformation property of each $[\![\_]\!]_{\varphi}$ is given by the commutativity of the following diagram:

$$\begin{tikzcd}
M' \arrow[d, "h'"'] & & {[\![M']\!]_{\Sigma'}} \arrow[r, "{[\![M']\!]_{\varphi}}"] \arrow[d, "{[\![h']\!]_{\Sigma'}}"'] & {[\![\mathrm{Mod}(\varphi)M']\!]_{\Sigma}} \arrow[d, "{[\![\mathrm{Mod}(\varphi)h']\!]_{\Sigma}}"] \\
N' & & {[\![N']\!]_{\Sigma'}} \arrow[r, "{[\![N']\!]_{\varphi}}"'] & {[\![\mathrm{Mod}(\varphi)N']\!]_{\Sigma}}
\end{tikzcd} \tag{4}$$

The satisfaction relation can be presented as a natural transformation

$$\models\ :\ \mathrm{Sen} \Rightarrow [\![\mathrm{Mod}(\_) \to \mathbf{Set}]\!]$$

where the functor $[\![\mathrm{Mod}(\_) \to \mathbf{Set}]\!] : \mathrm{Sign} \to \mathbf{Set}$ is defined by

– for each signature $\Sigma \in |\mathrm{Sign}|$, $[\![\mathrm{Mod}(\Sigma) \to \mathbf{Set}]\!]$ denotes the set of all the mappings $f : |\mathrm{Mod}(\Sigma)| \to \mathbf{Set}$ such that $f(M) \subseteq [\![M]\!]_{\Sigma}$; and

– for each signature morphism $\varphi : \ \Sigma \to \Sigma'$

$$[\![\mathrm{Mod}(\varphi) \to \mathbf{Set}]\!](f)(M') = [\![M']\!]_{\varphi}^{-1}(f(\mathrm{Mod}(\varphi)M')).$$

A straightforward check reveals that the satisfaction condition (2) appears exactly as the naturality property of $\models$:

$$\begin{tikzcd} \Sigma \arrow[d, "\varphi"'] & & \mathrm{Sen}(\Sigma) \arrow[r, "\models_\Sigma"] \arrow[d, "\mathrm{Sen}(\varphi)"'] & {[\![\mathrm{Mod}(\Sigma) \to \mathbf{Set}]\!]} \arrow[d, "{[\![\mathrm{Mod}(\varphi)\to\mathbf{Set}]\!]}"] \\ \Sigma' & & \mathrm{Sen}(\Sigma') \arrow[r, "\models_{\Sigma'}"'] & {[\![\mathrm{Mod}(\Sigma') \to \mathbf{Set}]\!]} \end{tikzcd}$$

Ordinary institutions are the stratified institutions for which $[\![M]\!]_\Sigma$ is always a singleton set. In the upgraded definition, we have removed the surjectivity condition on $[\![M']\!]_\varphi$ from the definition of the stratified institutions of [5] and will, instead, make it explicit when necessary. This is motivated by the fact that most of the results developed do not depend upon this condition which, nonetheless, holds in all examples known by us. On the one hand, in many important concrete situations (Kripke semantics, automata, etc.) $[\![M']\!]_\varphi$ are even identities, which makes $[\![\_]\!]$ a strict rather than a lax natural transformation. However, on the other hand, there are interesting examples when the stratification is properly lax, such as in the $\mathcal{OFOL}$ example below or the representation of 3/2-institutions as stratified institutions developed in [10].

The following very expected property does not follow from the axioms of stratified institutions, hence we impose it explicitly.

Assumption 1. *In all considered stratified institutions the satisfaction is preserved by model isomorphisms, i.e., for each Σ-model isomorphism $h : \ M \to N$, each $w \in [\![M]\!]_\Sigma$, and each Σ-sentence ρ,*

$$M \models^w \rho \ \text{ if and only if } \ N \models^{[\![h]\!]w} \rho.$$

The literature on stratified institutions shows many model theories that are captured as stratified institutions. Here, we recall some of them in a very succint form; in a more detailed form one may find them in [6,9,10].

1. In *modal propositional logic* ($\mathcal{MPL}$) the category of the signatures is **Set**, $\mathrm{Sen}(P)$ is the set of the usual modal sentences formed with the atomic propositions from P, and the P-models are the Kripke structures (W, M) where $W = (|W|, W_\lambda)$ consists of a set of "possible worlds" $|W|$ and an accessibility relation $W_\lambda \subseteq |W| \times |W|$, and $M : \ |W| \to 2^P$. The stratification is given by $[\![(W, M)]\!] = |W|$.
2. In *first order modal logic* ($\mathcal{MFOL}$) the signatures are first-order logic ($\mathcal{FOL}$) signatures consisting of sets of operation and relation symbols structured by their arities. The sentences extend the usual construction of $\mathcal{FOL}$ sentences with the modal connectives $\Box$ and $\Diamond$. The models for a signature Σ are Kripke structures (W, M) where W is like in $\mathcal{MPL}$ but $M : \ |W| \to |\mathrm{Mod}^{\mathcal{FOL}}(\Sigma)|$ subject to the constraint that the carrier sets and the interpretations of the constants are shared across the possible worlds. The stratification is like in $\mathcal{MPL}$.
3. *Hybrid logics* refine modal logics by adding explicit syntax for the possible worlds such as *nominals* and @. Stratified institutions of hybrid logics upgrade the syntactic and the semantic components of the stratified institutions of modal logics accordingly. For instance, in the stratified institution of hybrid propositional logic ($\mathcal{HPL}$) the signatures are pairs of sets (Nom, P), the (Nom, P)-models are Kripke structures (W, M) like in $\mathcal{MPL}$, but where W adds interpretations of the nominals, i.e., $W = (|W|, (W_i)_{i\in\mathrm{Nom}}, W_\lambda)$, and at the level of the syntax, for each $i \in \mathrm{Nom}$ we have a new sentence i-sen, a new unary connective $@_i$, and existential quantifications over nominals variables. Then $((W, M) \models^w i\text{-sen}) = (W_i = w)$, $((W, M) \models^w @_i\rho) = ((W, M) \models^{W_i} \rho)$, etc.

4. *Multi-modal logics* exhibit several modalities instead of only the traditional $\Diamond$ and $\Box$ and, moreover, these may have various arities. If one considers the sets of modalities to be variable then they have to be considered as part of the signatures. Each of the stratified institutions discussed in the previous examples admit an upgrade to the multi-modal case.
5. In a series of works on *modalization of institutions* [38–40] modal logic and Kripke semantics are developed by abstracting away details that do not belong to modality, such as sorts, functions, predicates, etc. This is achieved by extensions of abstract institutions (in the standard situations meant in principle to encapsulate the atomic part of the logics) with the essential ingredients of modal logic and Kripke semantics. The result of this process, when instantiated to various concrete logics (or to their atomic parts only) generate a uniformly wide range of hierarchical combinations between various flavours of modal logic and various other logics. Concrete examples discussed in [38–40] include various modal logics over non-conventional structures of relevance in computing science, such as partial algebra, preordered algebra, etc. Various constraints on the respective Kripke models, many of them having to do with the underlying non-modal structures, have also been considered. All of these arise as examples of stratified institutions like the examples presented above in the paper. An interesting class of examples that has emerged quite smoothly out of the general works on *hybridization* (i.e., modalization including also hybrid logic features) of institutions is that of multi-layered hybrid logics that provide a logical base for specifying hierarchical transition systems (see [41]).
6. *Open first order logic* ($\mathcal{OFOL}$). This is a $\mathcal{FOL}$ instance of $St(\mathcal{I})$, the "internal stratification" abstract example developed in [5]. An $\mathcal{OFOL}$ signature is a pair (Σ, X) consisting of $\mathcal{FOL}$ signature Σ and a finite block of variables. To any $\mathcal{OFOL}$ signature (Σ, X) it corresponds a $\mathcal{FOL}$ signature $\Sigma + X$ that adjoins X to Σ as new constants. Then, $\mathrm{Sen}^{\mathcal{OFOL}}(\Sigma, X) = \mathrm{Sen}^{\mathcal{FOL}}(\Sigma + X)$, $\mathrm{Mod}^{\mathcal{OFOL}}(\Sigma, X) = \mathrm{Mod}^{\mathcal{FOL}}(\Sigma)$, $[\![M]\!]_{\Sigma,X} = M^X$, i.e the set of the "valuations" of X to M and for each (Σ, X)-model M, each $w \in M^X$, and each (Σ, X)-sentence ρ we define $(M(\models^{\mathcal{OFOL}}_{\Sigma,X})^w\rho) = (M^w \models^{\mathcal{FOL}}_{\Sigma+X} \rho)$ where M^w is the expansion of M to $\Sigma + X$ such that $M^w_X = w$ (i.e., the new constants of X are interpreted in M^w according to the "valuation" w).
7. Various kinds of automata theories can be presented as stratified institutions. For instance, the deterministic automata (for regular languages) have the set of the input symbols as signatures, the automata A are the models, and the words are the sentences. Then, $[\![A]\!]$ is the set of the states of A and $A \models^s \alpha$ if and only if α is recognized by A from the state s.
8. In [10], there is a development of a general representation theorem of 3/2-institutions as stratified institutions. The theory of 3/2-institutions [11] is an extension of ordinary institution theory that accommodates the partiality of the signature morphisms and its syntactic and semantic effects, motivated by applications to conceptual blending and software evolution. The representation theorem is based, for each $\varphi\Box$-model M, on setting $[\![M]\!]$ to the *set* its φ-reducts. This is possible because in 3/2-institutions, unlike in ordinary institution theory, a model may have more than one reduct with respect to a fixed signature morphism, this being the semantic effect of the (implicit) partiality of the signature morphisms.

2.4. Flattening Stratified Institutions to Ordinary Institutions

We have already seen that ordinary institutions are trivial stratified institutions (i.e., with singleton stratifications). The other way around, meaning reducing proper stratified institutions to ordinary institutions as a non-trivial enterprise, can be achieved in two different ways as will be described.

The following construction from [6] was later on presented in [9] in the form of an adjunction. Its importance resides in the possibility of transferring and interpreting concepts and results from ordinary institution theory to stratified institution theory.

Given any stratified institution $\mathcal{S} = (\mathrm{Sign}, \mathrm{Sen}, \mathrm{Mod}, [\![\_]\!], \models)$ we define an institution $\mathcal{S}^\sharp = (\mathrm{Sign}, \mathrm{Sen}, \mathrm{Mod}^\sharp, \models^\sharp)$ (called the *local institution of* $\mathcal{S}$) by:

– the objects of $\mathrm{Mod}^\sharp(\Sigma)$ are the pairs (M, w) such that $M \in |\mathrm{Mod}(\Sigma)|$ and $w \in [\![M]\!]_\Sigma$;
– the Σ-homomorphisms $(M, w) \to (N, v)$ are the pairs (h, w) such that $h : M \to N$ and $[\![h]\!]_\Sigma w = v$;
– for any signature morphism $\varphi : \Sigma \to \Sigma'$ and any Σ'-model (M', w')

$$\mathrm{Mod}^\sharp(\varphi)(M', w') = (\mathrm{Mod}(\varphi)M', [\![M']\!]_\varphi w');$$

– for each Σ-model M, each $w \in [\![M]\!]_\Sigma$, and each $\rho \in \mathrm{Sen}(\Sigma)$

$$((M, w) \models^\sharp_\Sigma \rho) = (M \models^w_\Sigma \rho). \tag{5}$$

The following second interpretation of stratified institutions as ordinary institutions has already been given in [5]. Note that unlike $\mathcal{S}^\sharp$ above, $\mathcal{S}^*$ below shares with $\mathcal{S}$ the model functor. However, this comes with a slight technical cost. For any stratified institution $\mathcal{S} = (\mathrm{Sign}, \mathrm{Sen}, \mathrm{Mod}, [\![\_]\!], \models)$ we say that $[\![\_]\!]$ *is surjective* when for each signature morphism $\varphi : \Sigma \to \Sigma'$ and each Σ'-model M', $[\![M']\!]_\varphi : [\![M']\!]_{\Sigma'} \to [\![\mathrm{Mod}(\varphi)M']\!]_\Sigma$ is surjective. Then each stratified institution $\mathcal{S} = (\mathrm{Sign}, \mathrm{Sen}, \mathrm{Mod}, [\![\_]\!], \models)$ with $[\![\_]\!]$ surjective determines an (ordinary) institution $\mathcal{S}^* = (\mathrm{Sign}, \mathrm{Sen}, \mathrm{Mod}, \models^*)$ (called the *global institution of* $\mathcal{S}$) by defining

$$(M \models^*_\Sigma \rho) = \bigwedge\{M \models^w_\Sigma \rho \mid w \in [\![M]\!]_\Sigma\}.$$

From now on whenever we invoke an institution $\mathcal{S}^*$ we tacitly assume that $[\![\_]\!]^\mathcal{S}$ is surjective.

The institutions $\mathcal{S}^\sharp$ and $\mathcal{S}^*$ represent generalizations of the concepts of local and global satisfaction, respectively, from modal logic (e.g., [42]). While $\mathcal{S}^*$ "forgets" the stratification of $\mathcal{S}$, $\mathcal{S}^\sharp$ fully retains it (but in an implicit form). This is the reason why $\mathcal{S}^\sharp$ rather than $\mathcal{S}^*$ can be used for reflecting concepts and results from ordinary institution theory in stratified institutions. It is important to avoid a possible confusion regarding $\mathcal{S}^\sharp$, namely that through the flattening represented by the $\sharp$ construction, stratified institution theory gets reduced to ordinary institution theory. This cannot be the case, because although $\mathcal{S}^\sharp$ is an ordinary institution it retains a particular character induced by the stratified structure of $\mathcal{S}$. This means that many general institution theory concepts are not refined enough to properly reflect the stratification aspects.

2.5. Nominals in Stratified Institutions

The abstract nominals structures introduced in [6] and subsequently used in [8,9] play an important role in our work. Nominals are a distinctive feature of the hybrid variations of modal logics [34–36], but in principle, they can be defined for any logic that has models with states, not necessarily in a Kripke semantics context. In [6] this has been achieved axiomatically at the level of abstract stratified institutions. In what follows, we recall the corresponding definitions from there.

Let *SETC* denote the "sub-institution" of first order logic that is determined by the signatures that contain only symbols of constants (hence no sentences and the empty satisfaction relation). Given a stratified institution $\mathcal{S}$, a *nominals extraction* is a pair (N, Nm) consisting of a functor $N : \mathrm{Sign}^\mathcal{S} \to \mathrm{Sign}^{SETC}$ and a lax natural transformation $Nm : \mathrm{Mod}^\mathcal{S} \Rightarrow N^{\mathrm{op}}; \mathrm{Mod}^{SETC}$ such that for each signature Σ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Mod}(\Sigma) \arrow[r, "{[\![_]\!]_\Sigma}"] \arrow[dr, "Nm_\Sigma"'] & \mathbf{Set} \\
 & \mathrm{Mod}^{SETC}(N(\Sigma)) \arrow[u, "\text{forgetful } (|_|)"']
\end{tikzcd}$$

More explicitly, for each signature Σ, $N(\Sigma)$ is a signature of constants while Nm_Σ maps each Σ-model M to a model of that signature such that its underlying set is just $[\![M]\!]$. So the constants of $N(\Sigma)$ are interpreted as elements of $[\![M]\!]$. We can say that the constants of $N(\Sigma)$ are the "nominals of Σ" which are interpreted as elements of $[\![M]\!]$.

In [6], there are several concrete examples of nominals extraction. For instance in $\mathcal{HPL}$ we may define $N(\mathrm{Nom}, P) = \mathrm{Nom}$ and $Nm_{(\mathrm{Nom},P)}(W, M) = (|W|, (W_i)_{i \in \mathrm{Nom}})$.

Let $\mathcal{S}$ be a stratified institution endowed with a nominals extraction N, Nm. For any $i \in N(\Sigma)$

- a Σ-sentence i-sen is an *i-sentence* when

$$(M \models^w i\text{-sen}) = ((Nm_\Sigma M)_i = w);$$

- for any Σ-sentence ρ, a Σ-sentence $@_i\rho$ is the *satisfaction of ρ at i* when

$$(M \models^w @_i\rho) = (M \models^{(Nm_\Sigma M)_i} \rho)$$

for each Σ-model M and for each $w \in [\![M]\!]_\Sigma$.

The stratified institution $\mathcal{S}$ has *explicit local satisfaction* when there exists a satisfaction at i for each sentence and each appropriate i. For instance, $\mathcal{HPL}$ has explicit local satisfaction as i-sen and $@_i$ in $\mathcal{HPL}$ are examples of i-sentences and of satisfaction at i, respectively.

2.6. Quantifications in Stratified Institutions

The institution theoretic approach to quantifications (introduced in [43]; see also [3], etc.) crucially exploits the multi-signature aspect of the concept of institution. The quantification variables are assimilated to the signature extensions obtained by adding the variables as new syntactic entities to the respective signature, and consequently the valuations of the variables are assimilated to model expansions. Thus, for any signature morphism $\chi : \Sigma \to \Sigma'$ and any Σ'-sentence ρ', a Σ-sentence ρ is a *universal χ-quantification* of ρ' if and only if for each Σ-model M,

$M \models_\Sigma \rho$ if and only if $M' \models_{\Sigma'} \rho'$ for each χ-expansion M' of M.

In [6], this has been extended to stratified institutions as follows. For any signature morphism $\chi : \Sigma \to \Sigma'$, a Σ-sentence ρ is a *universal χ-quantification* when for any Σ-model M and each $w \in [\![M]\!]_\Sigma$,

$$(M \models^w_\Sigma \rho) = \bigwedge_{\mathrm{Mod}(\chi)(M')=M} \Big(\bigwedge_{w' \in [\![M']\!]^{-1}_\chi(w)} (M' \models^{w'}_{\Sigma'} \rho') \Big).$$

In [6], it has also been noted that ρ is a universal χ-quantification of ρ' in a stratified institution $\mathcal{S}$ if and only if it is in $\mathcal{S}^\sharp$ in the ordinary institution theoretic acceptation. Designated universal χ-quantifications ρ are usually denoted as $(\forall\chi)\rho'$.

2.7. Concepts of Model Amalgamation in Stratified Institutions

Model amalgamation is one of the most important concepts and properties in institution theory, the corresponding literature containing numerous works where model amalgamation is used decisively. References [2,13] are representative for computing science works, especially in the area of foundations of software modularization, while in [3], and many articles, one may find an abundance of uses of model amalgamation in institution-independent model theory. In particular, works on interpolation, many of them collected in the dedicated chapter of [3], reveal a strong causality relationship between model amalgamation and interpolation. This is also the case in our work.

The following definition from [9] extends the concept of model amalgamation [2,3,13,44–47], etc., from ordinary institution theory to stratified institutions. This introduces two concepts. The first one represents just the ordinary institution theoretic concept of model amalgamation formulated for stratified institutions (it does not involve the stratification structure). The second one is specific to stratified institutions. Consider a stratified institution $\mathcal{S}$ and a commutative square of signature morphisms like below:

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma_2 \arrow[r, "\theta_2"'] & \Sigma' \end{tikzcd} \tag{6}$$

Then this square:

- is a *model amalgamation square* when for each Σ_k-model M_k, $k = 1,2$ such that $\mathrm{Mod}(\varphi_1)M_1 = \mathrm{Mod}(\varphi_2)M_2$ there exists an unique Σ'-model M' such that $\mathrm{Mod}(\theta_k)M' = M_k$, $k = 1,2$, and
- is a *stratified model amalgamation square* when for each Σ_k-model M_k and each $w_k \in [\![M_k]\!]_{\Sigma_k}$, $k = 1,2$, such that $\mathrm{Mod}(\varphi_1)M_1 = \mathrm{Mod}(\varphi_2)M_2$ and $[\![M_1]\!]_{\varphi_1} w_1 = [\![M_2]\!]_{\varphi_2} w_2$ there exists an unique Σ'-model M' and a unique $w' \in [\![N']\!]_{\Sigma'}$ such that $\mathrm{Mod}(\theta_k)M' = M_k$ and $[\![M']\!]_{\theta_k} w' = w_k$, $k = 1,2$.

The model M' is called the *(stratified) amalgamation of M_1 and M_2*. When all pushout squares of signature morphisms are (stratified) model amalgamation squares we say that $\mathcal{S}$ is *(stratified) semi-exact*.

The following straightforward fact reduces stratified model amalgamation to ordinary model amalgamation.

Fact 1 ([9]). *A commutative square of signature morphisms like Equation (6) is a stratified model amalgamation square in $\mathcal{S}$ if and only if it is a model amalgamation square in $\mathcal{S}^\sharp$.*

Note also that when the stratification is strict (as natural transformation), then any model amalgamation square is a trivially a stratified model amalgamation square.

3. Two Concepts of Interpolation

In this section, we clarify the concept of interpolation in stratified institutions. Interpolation is a property of the semantic consequence relation on (sets of) sentences, and stratified institutions admit two different semantic consequence relations that correspond to the two possible flattenings of the stratified institutions to ordinary institutions. This means that there are two concepts of interpolation that emerge from the respective concepts of semantic consequence. In what follows, we will discuss the two concepts of semantic consequence in stratified institutions and the general relationship between them, and then we will define the two concepts of Craig interpolation.

3.1. Two Semantic Consequence Relations

Definition 1. *For any stratified institution $\mathcal{S}$*

- *the* local semantic consequence *relation $\models^\sharp$ is the semantic consequence relation of the institution $\mathcal{S}^\sharp$, and*
- *the* global semantic consequence *relation $\models^*$ is the semantic consequence relation of the institution $\mathcal{S}^*$.*

Fact 2. *For any stratified institution $\mathcal{S}$, any $\mathcal{S}$-signature Σ, any set E of Σ-sentences, and any Σ-sentence e*

- *$E \models^\sharp e$ if and only if for each Σ-model M and each $w \in [\![M]\!]_\Sigma$,*

$$M \models^w E \ \text{ implies } \ M \models^w e,$$

- *$E \models^* e$ if and only if for each Σ-model M,*

$$\text{for each } \ w \in [\![M]\!]_\Sigma,\ M \models^w E \ \text{ implies that for each } \ w \in [\![M]\!]_\Sigma,\ M \models^w e.$$

If we interchanged Fact 2 and Definition 1 then we gain more theoretical generality for the definition of the semantic consequence relation $\models^*$ because we do not rely anymore on the institution $\mathcal{S}^*$, which means that we do not have to assume the surjectivity of $[\![\_]\!]$. However, this higher generality would come at the expense of the loss of the *translation* property of $\models^*$, namely that for each signature morphism $\varphi : \ \Sigma \to \Sigma'$

$$E \models^* e \ \text{ implies } \ \varphi E \models^*_{\Sigma'} \varphi e.$$

Since this property is very important for interpolation we cannot afford to lose it.

The following result shows that local semantic consequence is stronger than the global semantic consequence.

Proposition 1. *$E \models^\sharp e$ implies $E \models^* e$.*

Proof. Let us suppose that $E \models^\sharp e$ and consider a model M such that $M \models^* E$. Let $w \in [\![M]\!]$. From $E \models^\sharp e$ it follows that $M \models^w e$. Since w has been considered arbitrary, we conclude that $M \models^* e$. □

The implication of Proposition 1 is proper as shown by the following very simple example. Let $\mathcal{S}$ consists of only one signature, only one model M such that $[\![M]\!] = \{w_1, w_2\}$ and only two sentences ρ_1 and ρ_2 such that

$$M \models^{w_i} \rho_j \ \text{ if and only if } \ i = j.$$

Then $\rho_1 \models^* \rho_2$ but $\rho_1 \not\models^\sharp \rho_2$.

3.2. Local Versus Global Interpolation in Stratified Institutions

The interpolation concepts in stratified institutions are based on classical institution theoretic interpolation. Yet, the former concepts refine the latter.

Proposition 2. *Consider any institution $\mathcal{I}$ and a commutative square of signature morphisms like below:*

$$\begin{tikzcd} \Sigma \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & \Sigma_1 \arrow[d, "\theta_1"] \\ \Sigma_2 \arrow[r, "\theta_2"'] & \Sigma' \end{tikzcd} \tag{7}$$

Let E_k be sets of Σ_k-sentences, $k = 1, 2$. If there exists a set E of Σ-sentences such that

$$E_1 \models \varphi_1 E \ \text{ and } \ \varphi_2 E \models E_2$$

then $\theta_1 E_1 \models \theta_2 E_2$.

Proof. By the *translation* property $\theta_1 E_1 \models \theta_1(\varphi_1 E)$ and $\theta_2(\varphi_2 E) \models \theta_2 E_2$. Since $\theta_1(\varphi_1 E) = \theta_2(\varphi_2 E)$ (by the functoriality of Sen because $\varphi_1;\theta_1 = \varphi_2;\theta_2$), by *transitivity* it follows that $\theta_1 E_1 \models \theta_2 E_2$. □

The conclusion of Proposition 2 is simple and general. Its reversal represents the interpolation property. The following definition formulates it directly for abstract stratified institutions.

Definition 2 (Craig interpolation in stratified institutions)**.** *Let $\mathcal{S}$ be any stratified institution. A commutative square of signature morphisms like in diagram (7) is a* local/global Craig interpolation square *when the reversal of the implication of Proposition 2 holds in $\mathcal{S}^\sharp$ / $\mathcal{S}^*$, respectively, by assuming that E, E_k are finite.*

Given classes of signature morphisms $\mathcal{L}$ and $\mathcal{R}$ we say that $\mathcal{S}$ has local/global Craig $(\mathcal{L}, \mathcal{R})$-interpolation *when each pushout square like in diagram (7) with $\varphi_1 \in \mathcal{L}$ and $\varphi_2 \in \mathcal{R}$ is a local/global Craig interpolation square.*

Note that when $\mathcal{S}$ is an ordinary institution, i.e., for each model M its stratification $[\![M]\!]$ is a singleton set, both concepts of local and global interpolation collapse to the well-established ordinary concept of institution theoretic interpolation. It is possible to have an infinitary version of interpolation by dropping the finiteness condition on the sets of sentences E, E_k. However in the applications finitary interpolation is more meaningful.

The relationship between institution theoretic interpolation and the common concept of logical interpolation has been explained extensively in many publications. Institution theoretic interpolation has been introduced in [25] and further refined in [13,17,29]. Even when instantiated to concrete logics it introduces several new layers of generality with respect to the common concept of interpolation. In essence, these have much to do with how interpolation is used in formal specification theory, but they may be highly relevant from a logic perspective. Let us recall here briefly some of the main aspects of the relationship between the institution theoretic and common interpolation.

1. With ordinary interpolation, the diagram (7) is restricted to an intersection-union square consisting of signature inclusions, i.e., $\Sigma = \Sigma_1 \cap \Sigma_2$ and $\Sigma' = \Sigma_1 \cup \Sigma_2$.
2. Concrete interpolation considers single sentences rather than sets of sentences. In the presence of logical conjunctions, there is no difference between these two approaches. However, there are important logics that lack conjunctions, such as equational and Horn logics, and many more. In these logics, interpolation is traditionally considered as failed. But works such as [29,48,49] show that this is a misunderstanding due to the rather faulty import of the single-sentence formulation of interpolation from some very prominent logics (such as classical propositional or first order logic) to other logical systems.

4. When Local Interpolation Causes Global Interpolation

In general, it is not possible to establish any causality relationship between local and global interpolation. In the light of the sharpness of Proposition 1, there is hope for establishing a set of general sufficient conditions only for the former to be a cause of the latter. This is what we will achieve in this section.

4.1. Signature Extensions with Nominals

The main envisaged condition is the presence of a nominals extraction. However this needs an enhancement with a slight technicality that, in the applications, amounts to the possibility of extending signatures with nominals in a way that enables quantifications over nominals at the level of abstract stratified institutions. The following definition does that.

Definition 3 (signature extensions with nominals)**.** *Any stratified institution has* signature extensions with nominals *when it has a nominals extraction* (N, Nm) *and for each signature* Σ *there exists a signature morphism* $\iota : \Sigma \to \Sigma'$*—called the* signature extension of Σ with the nominal i*—such that*

1. $N(\iota) : N(\Sigma) \to N(\Sigma') = N(\Sigma) \cup \{i\}$ *is the extension of* $N(\Sigma)$ *with one new constant* i*, and,*
2. *for each* Σ*-model* M *and each* $w \in [\![M]\!]_\Sigma$ *there exists a* ι*-expansion* M' *of* M *such that*
$$[\![M']\!]_\iota (Nm_{\Sigma'} M')_i = w.$$
3. *For each signature morphism* $\theta_1 : \Sigma_1 \to \Sigma'$ *and each signature extension* $\iota' : \Sigma' \to \Sigma''$ *with one nominal* i' *there exists a signature extension* $\iota_1 : \Sigma_1 \to \Sigma_1'$ *with one nominal* i *and signature morphism* $\theta_1' : \Sigma_1' \to \Sigma''$ *such that*
$$\begin{tikzcd} \Sigma_1 \arrow[r, "\iota_1"] \arrow[d, "\theta_1"'] & \Sigma_1' \arrow[d, "\theta_1'"] \\ \Sigma' \arrow[r, "\iota'"'] & \Sigma'' \end{tikzcd} \tag{8}$$

is a stratified model amalgamation square.

These are the three specific conditions/properties that makes Definition 3 have the following informal meaning. The former two conditions say that it is possible to achieve a signature "extension" with a single nominal only, both syntactically and semantically. The latter condition is more technical and says that, in the case of a codomain of a signature morphism such "extensions" with nominals can be traced back to a signature "extension" of the domain in a way that is "stable" under model amalgamation. In many concrete situations of interest, we can substitute in Definition 3 "model amalgamation" with "pushout", which would give this "stability" aspect a purely syntactic meaning since in stratified semi-exact institutions the latter implies the former. The following is a most typical example for the abstract concept of signature extensions with nominals introduced by Definition 3.

Let us consider $\mathcal{HPL}$, the stratified institution of hybrid propositional logic discussed in Section 2.3. Recall that in $\mathcal{HPL}$, the nominals extraction (N, Nm) is defined by $N(\mathrm{Nom}, P) = \mathrm{Nom}$ and $Nm_{\mathrm{Nom},P}(W, M) = (|W|, (W_i)_{i \in \mathrm{Nom}})$. Then, for any signature (Nom, P) and $i \notin \mathrm{Nom}$ the inclusion signature morphism $\iota : (\mathrm{Nom}, P) \to (\mathrm{Nom} \cup \{i\}, P)$ satisfies the axioms of Definition 3 as follows:

1. $N(\iota)$ is the set inclusion $\mathrm{Nom} \subseteq \mathrm{Nom} \cup \{i\}$.
2. Let (W, M) be any (Nom, P)-model. For any $w \in |W|$ we define the ι-expansion (W', M) of (W, M) by $W'_i = w$.
3. Let $\theta_1 : (\mathrm{Nom}_1, P_1) \to (\mathrm{Nom}', P')$, $\iota' : (\mathrm{Nom}', P') \to (\mathrm{Nom}' \cup \{i'\}, P')$. Then we consider $\iota_1 : (\mathrm{Nom}_1, P_1) \to (\mathrm{Nom}_1 \cup \{i\}, P_1)$, $i \notin \mathrm{Nom}_1$, and $\theta'_1 : (\mathrm{Nom}_1 \cup \{i\}, P_1) \to (\mathrm{Nom}', \cup\{i'\}, P')$ that just extends θ_1 by letting $\theta'_1 i = i'$. Note that in this case the commutative square Equation (8) yields a pushout square in $\mathrm{Sign}^{\mathcal{HPL}}$. Since $\mathcal{HPL}$ is semi-exact (cf. [9]) this is a model amalgamation square. Furthermore, since $\mathcal{HPL}$ is strictly stratified this is also a stratified model amalgamation square.

4.2. Interpolation in Stratified Institutions with Universal Nominals

With the following definition we give a name to the collected conditions that are sufficient for local interpolation to be a cause for global interpolation.

Definition 4 (Universal nominals)**.** *We say that a stratified institution $\mathcal{S}$ has* universal nominals *when:*

- *it has signature extensions with nominals;*
- *it has universal quantifications corresponding to the signature extensions with nominals; and*
- *it has explicit local satisfaction.*

Notation 1. *Consider a stratified institution $\mathcal{S}$ with universal nominals. For any signature extension $\iota : \Sigma \to \Sigma'$ with a nominal i and for each Σ-sentence e, the Σ-sentence $(\forall \iota) @_i (\iota e)$ is denoted by e^ι. Moreover, this convention extends to sets of sentences, i.e., $E^\iota = \{e^\iota \mid e \in E\}$.*

The following list of semantic properties of e^ι constitute technical support for the subsequent results leading to the concluding result of this section.

Proposition 3. *Let $\mathcal{S}$ be any stratified institution with universal nominals. Let M be any Σ-model, $w \in [\![M]\!]_\Sigma$ be any "state" of M, and E be any set of Σ-sentences and e be any Σ-sentence. Let ι be an extension of Σ with a nominal i. Then:*

1. $M \models^* e$ *if and only if* $(M, w) \models^\sharp e^\iota$.
2. $M \models^* e$ *if and only if* $M \models^* e^\iota$.
3. $E \models^* e$ *implies* $E^\iota \models^\sharp e^\iota$.
4. $E^\iota \models^* e$ *implies* $E \models^* e$.
5. $E \models^* e^\iota$ *implies* $E \models^* e$.

Proof.

1. For the implication from the left to the right we consider that $M \models^* e$. Let M' be any ι-expansion of M and let $w' \in [\![M']\!]_{\Sigma'}$ such that $[\![M']\!]_\iota w' = w$. We have to prove that $M' \models^{w'} @_i(\iota e)$. We have that

$$\begin{array}{rcll} M' \models^{w'} @_i(\iota e) & = & M' \models^{(Nm_{\Sigma'} M')_i} \iota e & \text{definition of local explicit satisfaction} \\ & = & M \models^{[\![M']\!]_\iota (Nm_{\Sigma'} M')_i} e & \text{satisfaction condition in } \mathcal{S}. \end{array}$$

The latter satisfaction holds because $M \models^* e$.

For the implication from the right to the left we assume $M \models^w (\forall \iota)@_i(\iota e)$. Let $v \in [\![M]\!]_\Sigma$ be arbitrary. We have to prove that $M \models^v e$. Let us consider M' a ι-expansion of M such that

$$[\![M']\!]_\iota (Nm_{\Sigma'} M')_i = v.$$

Then

$$\begin{array}{rcll} M \models^w (\forall \iota)@_i(\iota e) \Rightarrow M' \models^{w'} @_i(\iota e) \text{ for any } w' \in [\![M']\!]_\iota^{-1} & & & \text{definition of univ. quantification} \\ & = & M' \models^{(Nm_{\Sigma'} M')_i} \iota e & \text{definition of local explicit satisfaction} \\ & = & M \models^v e & \text{satisfaction condition.} \end{array}$$

2. Follows from 1. because $w \in [\![M]\!]_\Sigma$ is arbitrary.
3. Let $M \models^v E^\iota$. By 1. it follows that $M \models^* E$ hence $M \models^* e$. By 1. (again) we obtain $M \models^v e^\iota$.
4. Let $M \models^* E$. By 2. it follows that $M \models^* E^\iota$. Hence $M \models^* e$.
5. Let $M \models^* E$. Then $M \models^* e^\iota$. By 2. it follows $M \models^* e$.

□

The following technical result is instrumental in proving the main result of this section that relates local to global interpolation in stratified institutions.

Proposition 4. *Let $\mathcal{S}$ be any stratified institution with universal nominals and let*

$$\begin{tikzcd} \Sigma_1 \arrow[r, "\iota_1"] \arrow[d, "\theta_1"'] & \Sigma'_1 \arrow[d, "\theta'_1"] \\ \Sigma' \arrow[r, "\iota'"'] & \Sigma'' \end{tikzcd} \tag{9}$$

be a stratified model amalgamation square such that both ι_1 and ι' are signature extensions with one nominal. Then for each Σ'-model M' and each $w' \in [\![M']\!]_{\Sigma'}$, for each Σ_1-sentence e_1 we have that

$$(M', w') \models^\sharp (\theta_1 e_1)^{\iota'} \textit{ if and only if } (M', w') \models^\sharp \theta_1(e_1^{\iota_1}).$$

Proof. This proof involves some heavy formulas. In order to support its readability we adopt the following notational simplification: if $\zeta : \Omega \to \Omega'$ is a signature morphism and N' is a Ω'-model then by $\zeta N'$ we denote its reduct $\mathrm{Mod}(\zeta)N'$.

Let $N(\Sigma'_1) = N(\Sigma_1) \cup \{i_1\}$ and $N(\Sigma'') = N(\Sigma') \cup \{i'\}$.

- On the one hand, $(M', w') \models^\sharp (\theta_1 E_1)^{\iota'}$ means

  $$(M', w') \models^\sharp (\forall \iota')@_{i'}(\iota'(\theta_1 E_1))$$

  which by the definition of universal ι'-quantification means that *for all ι'-expansions (M'', w'') of (M', w')*

  $$(M'', w'') \models^\sharp @_{i'}(\iota'(\theta_1 E_1))$$

  which is successively equivalent to:

1	$(M'', (Nm_{\Sigma''}M'')_{i'}) \models^\sharp \iota'(\theta_1 E_1)$	definition of explicit local satisfaction
2	$(\theta_1;\iota')(M'', (Nm_{\Sigma''}M'')_{i'}) \models^\sharp E_1$	satisfaction condition in $\mathcal{S}^\sharp$
3	$(\theta_1 M', [\![M'']\!]_{\theta_1;\iota'}(Nm_{\Sigma''}M'')_{i'}) \models^\sharp E_1$	definition of reducts in $\mathcal{S}^\sharp$
4	$(\theta_1 M', [\![M'']\!]_{\iota_1;\theta'_1}(Nm_{\Sigma''}M'')_{i'}) \models^\sharp E_1$	$\theta_1;\iota' = \iota_1;\theta'$
5	$(\theta_1 M', [\![\theta'_1 M'']\!]_{\iota_1}([\![M'']\!]_{\theta'_1}(Nm_{\Sigma''}M'')_{i'})) \models^\sharp E_1$	definition of reducts in $\mathcal{S}^\sharp$.

Because $Nm_{\theta'_1}M'' : \ N(\theta'_1)(Nm_{\Sigma''}M'') \to Nm_{\Sigma'_1}(\theta'_1 M'')$ is a homomorphism of *SETC*-models, because its underlying function is $[\![M'']\!]_{\theta'_1}$, and because $N(\theta'_1)i_1 = i'$, we have that

$$[\![M'']\!]_{\theta'_1}(Nm_{\Sigma''}M'')_{i'} = (Nm_{\Sigma'_1}(\theta'_1 M''))_{i_1}$$

which makes the satisfaction 5 above equivalent to

6	$(\theta_1 M', [\![\theta'_1 M'']\!]_{\iota_1}(Nm_{\Sigma'_1}(\theta'_1 M''))_{i_1}) \models^\sharp E_1.$	

- On the other hand, $(M', w') \models^\sharp \theta_1(E_1^{\iota_1})$ is equivalent to the following satisfactions:

7	$\theta_1(M', w') \models^\sharp E_1^{\iota_1}$	satisfaction condition in $\mathcal{S}^\sharp$
8	$(\theta_1 M', [\![M']\!]_{\theta_1} w') \models^\sharp E_1{}^{\iota_1}$	definition of reducts in $\mathcal{S}^\sharp$
9	$(\theta_1 M', [\![M']\!]_{\theta_1} w') \models^\sharp (\forall \iota_1)@_{i_1}(\iota_1 E_1)$	definition of $E_1^{\iota_1}$.

By the definition of universal ι_1-quantifications the satisfaction 9 means that *for all ι_1-expansions (M'_1, w'_1) of $(\theta_1 M', [\![M']\!]_{\theta_1} w')$*

10	$(M'_1, w'_1) \models^\sharp @_{i_1}(\iota_1 E_1)$	

which is successively equivalent to the following satisfactions:

11	$(M'_1, (Nm_{\Sigma'_1}M'_1)_{i_1}) \models^\sharp \iota_1 E_1$	definition of explicit local satisfaction
12	$\iota_1(M'_1, (Nm_{\Sigma'_1}M'_1)_{i_1}) \models^\sharp E_1$	satisfaction condition in $\mathcal{S}^\sharp$
13	$(\iota_1 M'_1, [\![M'_1]\!]_{\iota_1}(Nm_{\Sigma'_1}M'_1)_{i_1}) \models^\sharp E_1$	definition of reducts in $\mathcal{S}^\sharp$.

The satisfactions 6 and 13 above are both universally quantified by the respective model expansions, i.e., the ι'-expansions (M'', w'') of (M', w') and the ι_1-expansions (M'_1, w'_1) of $\theta_1(M', w')$. By the stratified model amalgamation property of Equation (9) there is a bijective correspondence between the two sets of expansions, which is determined by the relation $(M'_1, w'_1) = \theta'_1(M'', w'')$. Note also that in this case we have that

$$(\iota_1 M'_1, [\![M'_1]\!]_{\iota_1}(Nm_{\Sigma'_1}M'_1)_{i_1}) = (\theta_1 M', [\![\theta'_1 M'']\!]_{\iota_1}(Nm_{\Sigma'_1}(\theta'_1 M''))_{i_1}).$$

Hence, the satisfaction relations 6 and 13 are equivalent, which completes the proof of the proposition. □

The following is the main result of the paper.

Theorem 1. *Let $\mathcal{S}$ be any stratified institution with universal nominals. Then any local Craig interpolation square is a global Craig interpolation square too.*

Proof. Let us consider a commutative square of signature morphisms

$$\begin{tikzcd}
\Sigma_1 \arrow[r, "\iota_1"] \arrow[d, "\theta_1"'] & \Sigma'_1 \arrow[d, "\theta'_1"] \\
\Sigma' \arrow[r, "\iota'"'] & \Sigma''
\end{tikzcd}$$

that is a local Craig interpolation square and let $E_k \subseteq \mathrm{Sen}(\Sigma_k)$, $k = 1,2$, such that $\theta_1 E_1 \models^* \theta_2 E_2$. By Proposition 3, it follows that $(\theta_1 E_1)^{\iota'} \models^\sharp (\theta_2 E_2)^{\iota'}$ for some signature extension with one nominal $\iota' : \Sigma' \to \Sigma''$. For each $k = 1,2$ let $\iota_k : \Sigma_k \to \Sigma'_k$ be a signature extension with one nominal such that

$$\begin{tikzcd}
\Sigma_k \arrow[r, "\iota_k"] \arrow[d, "\theta_k"'] & \Sigma'_k \arrow[d, "\theta'_k"] \\
\Sigma' \arrow[r, "\iota'"'] & \Sigma''
\end{tikzcd}$$

is a stratified model amalgamation square. By Proposition 4, from $(\theta_1 E_1)^{\iota'} \models^\sharp (\theta_2 E_2)^{\iota'}$ we have that, it follows

$$\theta_1(E_1^{\iota_1}) \models^\sharp \theta_2(E_2^{\iota_2}).$$

According to the local Craig interpolation hypothesis there exists an interpolant $E \subseteq \mathrm{Sen}(\Sigma)$ such that

$$E_1{}^{\iota_1} \models^\sharp \varphi_1 E \ \text{ and } \ \varphi_2 E \models^\sharp E_2{}^{\iota_2}.$$

By Proposition 1, it follows that

$$E_1{}^{\iota_1} \models^* \varphi_1 E \ \text{ and } \ \varphi_2 E \models^* E_2{}^{\iota_2}.$$

By Proposition 3, it follows that

$$E_1 \models^* \varphi_1 E \ \text{ and } \ \varphi_2 E \models^* E_2$$

hence E is an interpolant for E_1 and E_2 with respect to $\models^*$. $\square$

Based on the examples above of how some of our concepts apply to $\mathcal{HPL}$, which can be easily extended to other forms of Kripke semantics with nominals and universal quantifications over those we can formulate the following concrete consequence of our main result.

Corollary 1. *$\mathcal{HPL}$ and its first order, multi-modal extensions enjoy the property that local Craig interpolation implies global Craig interpolation.*

5. Conclusions

In this paper, we first extended the concept of ordinary institution-theoretic interpolation to stratified institutions by interpreting it for the local and for the global semantic consequence relations. Then, we formulated a set of applicable sufficient conditions such that local interpolation implies global interpolation. These conditions imply a certain nominal infrastructure that also includes universal quantifications over nominals, but not necessarily a Kripke relational structure. All of these developments have been achieved at the level of fully abstract stratified institutions.

Our work can be regarded as a first step towards a more elaborated theory of interpolation for stratified institutions. On this basis, we plan a further in-depth study of interpolation in this context similar to what has been achieved in ordinary institution theory, but which addresses the general specificity given by the “models with states” paradigm.

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