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Common Fixed-Point Theorems for Families of Compatible Mappings in Neutrosophic Metric Spaces

Umar Ishtiaq [1,*], Khaleel Ahmad [2], Farhan Ali [2], Moazzama Faraz [2] and Ioannis K. Argyros [3,*]

[1] Office of Research, Innovation and Commercialization, University of Management and Technology, Lahore 54770, Pakistan
[2] Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan; khalil7066616@gmail.com (K.A.); farhanali.numl.2016@gmail.com (F.A.); moazmafaraz@gmail.com (M.F.)
[3] Department of Computing and Mathematical Sciences, Cameron University, Lawton, OK 73505, USA
* Correspondence: umar.ishtiaq@umt.edu.pk (U.I.); iargyros@cameron.edu (I.K.A.)

Abstract: Sets, probability, and neutrosophic logic are all topics covered by neutrosophy. Moreover, the classical set, fuzzy set, and intuitionistic fuzzy set are generalized using the neutrosophic set. A neutrosophic set is a mathematical concept used to solve problems with inconsistent, ambiguous, and inaccurate data. In this article, we demonstrate some basic fixed-point theorems for any even number of compatible mappings in complete neutrosophic metric spaces. Our primary findings expand and generalize the findings previously established in the literature.

Keywords: neutrosophic metric spaces; common fixed-point theorems; compatible mappings; existence and uniqueness

MSC: 47H10; 54H25

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1. Introduction

We can quickly see the importance of fixed-point (fp) theorems by considering their applications in a variety of fields. The fp theorems demand that the function has at least one fp under specified conditions. As is evident, these conclusions usually benefit mathematics as a whole and are crucial for analyzing the existence and uniqueness of solutions to various mathematical models. Since then, numerous actual applications for handling uncertainty have utilized fuzzy sets (FSs) and fuzzy logic. The traditional FSs utilize one real value $\mu_{\mathcal{A}}(\varpi) \in [0,1]$ to constitute the class of community of a FS $\mathcal{A}$ defined in terms of the universe Ξ. Occasionally, $\mu_{\mathcal{A}}(\varpi)$ itself is unknown and difficult to define using an invigorating value. So, the concept of the interval valued as FS was proposed in [1] to capture the unpredictable nature of class of community.

FS uses an interval value $[\mu_{\mathcal{A}}\, L\,(\varpi),\ \mu_{\mathcal{A}}\, U(\varpi)]$ with $0 \leq \mu_{\mathcal{A}}\, L\,(\varpi)\mu_{\mathcal{A}}\, U(\varpi) \leq 1$ to constitute the class of the community of FS $\mathcal{A}$. For applications that consider authority structure, reliance system and information fusion, we should not only consider the truth community supported by the noticeable. An IFS can only hold insufficient details but not undefined details and inconsistent details, which commonly exist in reliance structures. In IFS, this detail is $1 - \tau_{\mathcal{A}}(\varpi) - f_{\mathcal{A}}(\varpi)$ by default. For instance, when we call on the support of a specialist to create a definite declaration, they may believe that the chances the declaration is true is 0.5, the chances that the declaration is wrong is 0.6, and level that it is dubious is 0.2.

In the neutrosophic set (NS), indeterminacy is quantified explicitly, and membership of the truth, indeterminacy, and falsehood classes are all independent. This presumption is crucial in many circumstances, including information fusion, which is the process of combining data derived from many sensors. Smarandache first introduced neutrosophy in 1995. The genesis, character, and range of neutralities, as well as how they interact with various ideational spectra, are studied in this area of philosophy [1]. The concept of the classic set, FS [2], interval-valued FS [3], IFS [4], etc., are all considered.

The Banach fp theorem, which Banach [5] initially proposed in 1922 and Caccioppoli [6] further derived in 1931 based on the framework of metric space (ms) fp theory, is covered in this paper. Several researchers established various conditions to examine fps. Through the help of Banach and Caccioppoli, the fp research community produced several good results. Utilizing the concept of FS theory, which Zadeh [2] developed in 1965, fixing real-world problems becomes undoubtedly simple because it helps to explain ambiguity and inaccuracy. Using the framework of a metric linear space, Arora and Sharma [7] derived the common fps through fuzzy mappings.

Park [8], using the idea of IFS, defined the notion of IFMSs, with the support of continuous t-norms (CTN) and continuous t-conorms (CTCN) as a theory of fuzzy metric space (FMS), due to the work of George and Veeramani [9]. Sessa [10] describes a theory of fluctuation, which is called weak commutativity. Further, Jungck [11] established many theories of commutativity, which are called compatibility. Mishra et al. [12] gain common fp theorems for compatible maps based on FMS. Turkoglu et al. [13] worked out the definitions of compatible maps of class (α) and (β) in IFMS. Alaca et al. [14] established the theory of compatible mappings type (I) and (II) and satisfied common fp theorems for four mappings in IFMSs.

Kirişci et al. [15] established the NMSs. Ishtiaq et al. [16] established the concept of neutrosophic extended metric-like spaces and established few FP theorems. In neutrosophic extended metric-like spaces, the authors utilized the concept of neutrosophic sets, metric space, continuous triangular norms, and continuous conorms. Uddin et al. [17] defined the concept of neutrosophic double-controlled metric spaces as a generalization of NMSs. For more related results, see [18–23].

The main aim of this manuscript is to enhance a common fp theorem to any even number of mappings using a complete NMS. In the second part of this paper, we provide several basic definitions and results derived from the existing literature. In part 3, we establish the main theorems of this paper. In part 4, we satisfy a common fixed-point (CFP) theorem for four finite families of mappings using a complete NMS.

2. Preliminaries

In this section, we provide some definitions that are helpful for readers to understand the main section.

Definition 1 ([18]). *We suppose that a binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is said to be CTN if* $*$ *is fulfilling the following conditions:*

(T1) $*$ *is associative and commutative;*
(T2) $*$ *is continuous;*
(T3) $\varrho * 1 = \varrho \; \forall \; \varrho \in [0,1]$;
(T4) $\varrho * \pi \leq c * d$ *whenever* $\varrho \leq c$ *and* $\pi \leq d$*, and* $\varrho, \pi, c, d \in [0,1]$.

Definition 2 ([18]). *A binary operation* $\Diamond : [0,1] \times [0,1] \to [0,1]$ *is a CTCN if* $\Diamond$ *satisfies the (T1), (T2), and (T4) and also fulfills:*

(T5) $\varrho \Diamond 0 = \varrho \; \forall \; \varrho \in [0,1]$.

Definition 3 ([23]). *We suppose that* Ξ *is a nonempty set,* $*$ *is a CTN and* M *is a fuzzy set for* $\Xi \times \Xi \times [0,\infty)$*. Then,* M *is said to be a fuzzy metric on* Ξ *if for all* $\varpi, \omega, \sigma \in \Xi$*,* M *satisfies the following conditions:*

(f1) $M(\varpi, \omega, 0) = 0$;
(f2) $M(\varpi, \omega, \tau) = 1$ *for all* $\tau > 0$*, iff* $\mu = v$;
(f3) $M(\varpi, \omega, \tau) = M(\omega, \varpi, \tau)$;
(f4) $M(\varpi, \sigma, \tau + \lambda) \geq M(\varpi, \omega, \tau) * M(\omega, \sigma, \lambda)$ *for all* $\tau, \lambda > 0$;

(f5) $M(\varpi, \omega, .) : (0, \infty) \to [0, 1]$ *is left continuous and* $\lim_{\tau\to\infty} m(\varpi, \omega, \tau) = 1$;

Then, $(\Xi, M, *)$ *is called fuzzy metric space.*

Definition 4 ([20]). *A 5-tuple* $(\Xi, M_d, N_d, *, \diamondsuit)$ *is said to be an IFMS if* Ξ *is an arbitrary set,* $*$ *is a CTN,* $\diamondsuit$ *is a CTCN, and* M_d, N_d *are FS on* $\Xi^2 \times (0, \infty)$ *, satisfying the following conditions for all* $\varpi, \omega, \sigma \in \Xi$, $\lambda, \tau > 0$,

(a) $M_d(\varpi, \omega, \tau) + N_d(\varpi, \omega, \tau) \leq 1$;
(b) $M_d(\varpi, \omega, 0) = 0$;
(c) $M_d(\varpi, \omega, \tau) = 1 \, \forall \, \tau > 0 \iff \varpi = \omega$;
(d) $M_d(\varpi, \omega, \tau) = M_d(\omega, \varpi, \tau)$;
(e) $M_d(\varpi, \omega, \tau) * M_d(\omega, \sigma, \lambda) \leq M_d(\varpi, \sigma, \tau + \lambda) \, \forall \, \varpi, \omega, \sigma \in \Xi, \lambda, \tau > 0$;
(f) $M_d(\varpi, \omega, \cdot) : (0, \infty) \to [0, 1]$ *is left continuous;*
(g) $\lim_{\tau\to\infty} M_d(\varpi, \omega, \sigma) = 1 \, \forall \, \varpi, \omega \in \Xi$;
(h) $N_d(\varpi, \omega, 0) = 1$;
(i) $N_d(\varpi, \omega, \tau) = 0 \, \forall \, \tau > 0 \iff \varpi = \omega$;
(j) $N_d(\varpi, \omega, \tau) = N_d(\omega, \varpi, \tau)$;
(k) $N_d(\varpi, \omega, \tau) \diamondsuit N_d(\omega, \sigma, \lambda) \geq N_d(\varpi, \sigma, \tau + \lambda) \, \forall \, \varpi, \omega, \sigma \in \Xi, \lambda, \tau > 0$;
(l) $N_d(\varpi, \omega, \cdot) : (0, \infty) \to [0, 1]$ *is right continuous;*
(m) $\lim_{\tau\to\infty} N_d(\varpi, \omega, \tau) = 0, \, \forall \, \varpi, \omega \in \Xi$.

Then, (M, N_d) *is said to be an IFM on* Ξ.

Example 1 ([9]). *We suppose that* (Ξ, d) *is a ms. Let* $\varrho * \pi = \varrho\pi$ *and* $\varrho \diamondsuit \pi = min\{1, \varrho + \pi\}$, $\forall \, \varrho, \pi \in [0, 1]$ *and let* M_d *and* N_d *be FSs on* $\Xi^2 \times (0, \infty)$, *specifying the following conditions:*

$$M_d(\varpi, \omega, \tau) = \frac{h\tau^n}{h\tau^n + md(\varpi, \omega)},$$

$$N_d(\varpi, \omega, \tau) = \frac{d(\varpi, \omega)}{\lambda\tau^n + md(\varpi, \omega)},$$

for all $h, \lambda, m, n \in \mathbb{N}$. *Then,* $(\Xi, M_d, N_d, *, \diamondsuit)$ *is an IFMS.*

Remark 1. *Note that the above example holds even with the CTN* $\varrho * \pi = min\{\varrho, \pi\}$ *and the CTCN* $\varrho * \pi = max\{\varrho, \pi\}$ *; hence,* (M, N) *is an IFMS with respect to any CTN and CTCN. In the above example, by taking* $h = \lambda = m = n = 1$,

$$N_d(\varpi, \omega, \tau) = \frac{d(\varpi, \omega)}{\tau + d(\varpi, \omega)}, \quad M_d(\varpi, \omega, \tau) = \frac{\tau}{\tau + d(\varpi, \omega)}.$$

Theorem 1 ([22]). *Let* $(\Xi, M_d, N_d, *, \diamondsuit)$ *be a complete intuitionistic fuzzy metric space, and let* $\mathcal{A}, B, S, T, p$ *and* Q *be mappings from* Ξ *into itself such that the following conditions are satisfied:*

1. $p(\Xi) \subset ST(\Xi)$, $Q(\Xi) \subset \mathcal{A}B(\Xi)$.
2. $\mathcal{A}B = B\mathcal{A}, ST = TS, pB = Bp, QT = TQ$.
3. *Either* p *or* AB *is continuous.*
4. $(p, \mathcal{A}B)$ *is compatible of type* (β) *and* (Q, ST) *is semi-compatible.*
5. *There exists* $\lambda \in (0, 1)$ *such that for every* $\varpi, \omega \in \Xi$, $\alpha \in (0, 2)$ *and* $\tau > 0$

$$M_d(p\varpi, Q\omega, \lambda\tau) \geq \min\{M_d(\mathcal{A}B\varpi, Q\omega, (2 - \alpha)\tau), M_d(\mathcal{A}B\varpi, ST\omega, \tau), M_d(ST\omega, Q\omega, \tau)\}$$

$$N_d(p\varpi, Q\omega, \lambda\tau) \leq \max\{N_d(\mathcal{A}B\varpi, Q\omega, (2 - \alpha)\tau), N_d(\mathcal{A}B\varpi, ST\omega, \tau), N_d(ST\omega, Q\omega, \tau)\}$$

Then, the mappings $\mathcal{A}B, ST, p$ *and* Q *have a unique common fixed point in X, and* $\mathcal{A}, B, p, Q, S$ *and* T *have a unique common fixed point in* Ξ.

Example 2 ([18]). *Let* $(\Xi, M_d, N_d, *, \Diamond)$ *be an intuitionistic fuzzy metric space, where the* $\Xi = [0,2]$ *t-norm is defined by* $a * b = min\{a, b\}$ *, t-conorm is defined by* $a \Diamond b = max\{a, b\}$ *for all* $a, b \in [0,\ 2]$*, and*

$$M_d(\varpi, \omega, \tau) = \left[\exp\left(\frac{|\varpi - \omega|}{\tau}\right)\right]^{-1}$$

and

$$N_d(\varpi, \omega, \tau) = \left[\exp\left(\frac{|\varpi - \omega|}{\tau}\right)\right]\left[\exp\left(\frac{|\varpi - \omega|}{\tau}\right)\right]^{-1} \text{for all } \varpi, \omega \in \Xi,\ \tau > 0$$

Clearly, $(\Xi, M_d, N_d, *, \Diamond)$ *is an intuionistic fuzzy metric space.*

Theorem 2 ([21]). *Let* $(\Xi, M_d, N_d, *, \Diamond)$ *be a complete intuionistic fuzzy metric space with continuous t-norm* $*$ *and continuous t-conorm* $\Diamond$ *defined by* $\tau * \tau \geq \tau$ *and* $(1-\tau) \Diamond (1-\tau) \leq (1-\tau)$ *for all* $\tau \in [0,\ 1]$.

Further, let $(\mathcal{A}, S)$ and (B, T) be pointwise R-weakly commuting pairs of self-mappings in a compatible pair $(\mathcal{A}, S)$ or (B, T) is continuous; then, $\mathcal{A}, B, S$ and T have a unique common fixed point.

Example 3. *Let* $\Xi = [0, \infty)$ *and let* M_d *and* N_d *be defined by*

$$M_d(\varpi, \omega, \tau) = \frac{\tau}{\tau + |\varpi - \omega|}$$

and

$$N_d(\varpi, \omega, \tau) = \frac{|\varpi - \omega|}{\tau + |\varpi - \omega|}$$

Then, $(\Xi, M_d, N_d, *, \Diamond)$ *is a complete intuitionistic fuzzy metric space. The t-norm is defined by* $a * b = min\{a, b\}$*, and the t-conorm is defined by* $a \Diamond b = max\{a, b\}$ *for all* $a, b \in [0,2]$.

Definition 5 ([15]). *A 6-tuple* $(\Xi, M, N, O *, \Diamond)$ *is said to be an NMS if* Ξ *is an arbitrary set,* $*$ *is a CTN,* $\Diamond$ *is a CTCN, and* M, N, O *are NS on* $\Xi^2 \times (0, \infty)$*, satisfying the following conditions:* $\forall\, \varpi, \omega, \sigma \in \Xi,\ \lambda, \tau > 0$,

(NMS1) $M(\varpi, \omega, \tau) + N(\varpi, \omega, \tau) + O(\varpi, \omega, \tau) \leq 3$;
(NMS2) $M(\varpi, \omega, 0) = 0$;
(NMS3) $M(\varpi, \omega, \tau) = 1\ \forall\, \tau > 0 \iff \varpi = \omega$;
(NMS4) $M(\varpi, \omega, \tau) = M(\omega, \varpi, \tau)$;
(NMS5) $M(\varpi, \omega, \tau) * M(\omega, \sigma, \lambda) \leq M(\varpi, \sigma, \tau + \lambda)\ \forall\, \varpi, \omega, \sigma \in \Xi, \lambda, \tau > 0$;
(NMS6) $M(\varpi, \omega,\ \cdot) : (0, \infty) \to [0, 1]$ *is left continuous;*
(NMS7) $\lim_{\tau \to \infty} M(\varpi, \omega, \sigma) = 1\ \ \forall\, \varpi, \omega \in \Xi$;
(NMS8) $N(\varpi, \omega, 0) = 1$;
(NMS9) $N(\varpi, \omega, \tau) = 0\ \forall\, \tau > 0 \iff \varpi = \omega$;
(NMS10) $N(\varpi, \omega, \tau) = N(\omega, \varpi, \tau)$;
(NMS11) $N(\varpi, \omega, \tau) \Diamond N(\omega, \sigma, \lambda) \geq N(\varpi, \sigma, \tau + \lambda)\ \forall\, \varpi, \omega, \sigma \in \Xi, \lambda, \tau > 0$;
(NMS12) $N(\varpi, \omega,\ \cdot) : (0, \infty) \to [0, 1]$ *is right continuous;*
(NMS13) $\lim_{\tau \to \infty} N(\varpi, \omega, \tau) = 0,\ \forall\, \varpi, \omega \in \Xi$;
(NMS14) $O(\varpi, \omega, 0) = 1$;
(NMS15) $O(\varpi, \omega, \tau) = 0\ \forall\, \tau > 0 \iff \varpi = \omega$;
(NMS16) $O(\varpi, \omega, \tau) = O(\omega, \varpi, \tau)$;
(NMS17) $O(\varpi, \omega, \tau) \Diamond O(\omega, \sigma, \lambda) \geq O(\varpi, \sigma, \tau + \lambda)\ \forall\, \varpi, \omega, \sigma \in \Xi, \lambda, \tau > 0$;
(NMS18) $O(\varpi, \omega,\ \cdot) : (0, \infty) \to [0, 1]$ *is right continuous;*
(NMS19) $\lim_{\tau \to \infty} O(\varpi, \omega, \tau) = 0,\ \forall\, \varpi, \omega \in \Xi$.

Then, (M, N, O) *is said to be a neutrosophic metric on* Ξ.

Definition 6 ([15])**.** *Let* $(\Xi, M, N, O, *, \ \Diamond\ )$ *be an NMS. Then:*

(a) *A sequence* $\{\varpi_n\}$ *in* Ξ *is said to a Cauchy sequence if for each* $\tau > 0$ *and* $p > 0$ *,*

$$\lim_{n\to\infty} M(\varpi_{n+p}, \varpi_n, \tau) = 1,$$

$$\lim_{n\to\infty} N(\varpi_{n+p}, \varpi_n, \tau) = 0,$$

$$\lim_{n\to\infty} O(\varpi_{n+p}, \varpi_n, \tau) = 0.$$

(b) *A NMS is only called complete if every Cauchy sequence is convergent.*

3. Main Results

In this section, we establish a common fp theorem for any even number of compatible mappings in a complete NMS.

Definition 7. *Let* $\mathcal{A}$ *and* B *represent mappings into an NMS* $(\Xi, M, N, O, *, \ \Diamond\ )$. *The maps* $\mathcal{A}$ *and* B *are said to be compatible with type* (α) *if* $\forall\ \tau > 0$,

$$\lim_{n\to\infty} M(\mathcal{A}B\varpi_n,\ BB\varpi_n, \tau) = 1, \lim_{n\to\infty} N(\mathcal{A}B\varpi_n,\ BB\varpi_n, \tau) = 0, \lim_{n\to\infty} O(\mathcal{A}B\varpi_n,\ BB\varpi_n, \tau) = 0,$$

and

$$\lim_{n\to\infty} M(\mathcal{A}B\varpi_n, \mathcal{A}\mathcal{A}\varpi_n, \tau) = 1,$$

$$\lim_{n\to\infty} N(\mathcal{A}B\varpi_n, \mathcal{A}\mathcal{A}\varpi_n, \tau) = 0,$$

$$\lim_{n\to\infty} O(\mathcal{A}B\varpi_n, \mathcal{A}\mathcal{A}\varpi_n, \tau) = 0,$$

whenever $\{\varpi_n\}$ *is a sequence in* Ξ *such that* $\lim_{n\to\infty} \mathcal{A}\varpi_n = \lim_{n\to\infty} B\varpi_n = \sigma$ *for some* $\sigma \in \Xi$.

Lemma 1. *Let* $(\Xi, M, N, O, *, \ \Diamond\ )$ *be an NMS and* ω_n *be a sequence in* Ξ. *If* $\exists$ *is a number* $\lambda \in (0, 1)$ *s.t*

$$M(\omega_{n+1}, \omega_{n+2}, \lambda\tau) \geq\ M(\omega_n, \omega_{n+1}, \tau),$$

$$N(\omega_{n+1}, \omega_{n+2}, \lambda\tau) \leq\ N(\omega_n, \omega_{n+1}, \tau),$$

$$O(\omega_{n+1}, \omega_{n+2}, \lambda\tau) \leq\ O(\omega_n, \omega_{n+1}, \tau),$$

$\forall\ \tau > 0$ *and* $n \in \mathbb{N}$, *then* $\{\omega_n\}$ *is a Cauchy sequence in* Ξ.

Lemma 2. *Let* $(\Xi, M, N, O, *, \ \Diamond\ )$ *be an NMS and be* $\forall\ \varpi, \omega \in \Xi, \tau > 0$. *If for a number* $\lambda \in (0, 1)$

$$M\ (\varpi, \omega, \lambda\tau) \geq M(\varpi, \omega, \tau),$$

$$N(\varpi, \omega, \lambda\tau) \leq N(\varpi, \omega, \tau),$$

$$O(\varpi, \omega, \lambda\tau) \leq O(\varpi, \omega, \tau).$$

Lemma 3. *Let* $(\Xi, M, N, O, *, \ \Diamond\ )$ *be a NMS and* $\forall\ \varpi, \omega \in \Xi,\ \tau > 0$. *If for a constant* $\lambda \in (0, 1)$

$$M(\varpi, \omega, \lambda\tau) \geq M(\varpi, \omega, \tau),$$

$$N(\varpi, \omega, \lambda\tau) \leq N(\varpi, \omega, \tau),$$

and

$$O(\varpi,\omega,\lambda\tau) \leq O(\varpi,\omega,\tau),$$

then $\varpi = \omega$.

Definition 8. *Let $\mathcal{A}$ and B be two mappings from Ξ into itself. If the maps A and B commute at their coincidence points, the maps are said to be weakly compatible, i.e., if $\mathcal{A}p = Bp$ are suitable for some $p \in \Xi$, then $\mathcal{A}Bp = B\mathcal{A}p$.*

Definition 9. *A pair (f,g) of self-mappings defined on an NMS $(\Xi, M, N, O, *, \Diamond)$ is said to satisfy the (CLRg) property if there is a sequence $\{\varpi_n\}$ of Ξ in which*

$$\lim_{n\to\infty} f\varpi_n = \lim_{n\to\infty} g\varpi_n = gu,$$

for some $u \in \Xi$.

Theorem 3. *Let $(\Xi, M, N, O, *, \Diamond)$ be a NMS with $\tau * \tau \geq \tau$ and*

$$(1-\tau) \Diamond (1-\tau) \leq (1-\tau), \forall \tau \in [0,1]$$

Further, let the pair (f,g) of self-mappings be weakly compatible, thus satisfying

$$\begin{aligned} M(f\varpi, f\omega, \lambda\tau) \geq\ & \{M(g\varpi, g\omega,\tau) * M(f\varpi, g\varpi,\tau) * M(f\omega, g\omega,\tau) * M(f\varpi, g\omega,\tau) \\ & *M(f\omega, g\varpi,\tau)\}, \end{aligned} \tag{1}$$

$$\begin{aligned} N(f\varpi, f\omega, \lambda\tau) \leq\ & \{N(g\varpi, g\omega,\tau) \Diamond N(f\varpi, g\varpi,\tau) \Diamond N(f\omega, g\omega,\tau) N \Diamond (f\varpi, g\omega,\tau) \\ & \Diamond N(f\omega, g\varpi,\tau)\}, \end{aligned} \tag{2}$$

and

$$\begin{aligned} O(f\varpi, f\omega, \lambda\tau) \leq\ & \{O(g\varpi, g\omega,\tau) \Diamond O(f\varpi, g\varpi,\tau) \Diamond O(f\omega, g\omega,\tau) \Diamond O(f\varpi, g\omega,\tau) \\ & \Diamond O(f\omega, g\varpi,\tau)\}. \end{aligned} \tag{3}$$

$\forall\ \varpi, \omega \in \Xi, \lambda \in (0,1)$ and $\tau > 0$. If f and g fulfill the (CLRg) property, then f and g have a unique common fixed point in Ξ.

Proof. Since the pair (f,g) fulfills the (CLRg) property, there is a sequence $\{\varpi_n\}$ in Ξ such that $\lim_{n\to\infty} f\varpi_n = \lim_{n\to\infty} g\varpi_n = gu$, for some $u \in \Xi$. Now, we emphasize that $fu = gu$. When utilizing Inequalities (1)–(3) with $\varpi = \varpi_n$, $\omega = u$, we find that

$$\begin{aligned} M(f\varpi_n, fu, \lambda\tau) \geq\ & \{M(g\varpi_n, gu,\tau) * M(f\varpi_n, g\varpi_n,\tau) * M(fu, gu,\tau) \\ & *M(f\varpi_n, gu,\tau) * M(fu, g\varpi_n,\tau)\}, \end{aligned}$$

$$\begin{aligned} N(f\varpi_n, fu, \lambda\tau) \leq\ & \{N(g\varpi_n, gu,\tau) \Diamond N(f\varpi_n, g\varpi_n,\tau) \Diamond N(fu, gu,\tau) \\ & \Diamond N(f\varpi_n, gu,\tau) \Diamond N(fu, g\varpi_n,\tau)\}, \end{aligned}$$

and

$$\begin{aligned} O(f\varpi_n, fu, \lambda\tau) \leq\ & \{O(g\varpi_n, gu,\tau) \Diamond O(f\varpi_n, g\varpi_n,\tau) \Diamond O(fu, gu,\tau) \\ & \Diamond O(f\varpi_n, gu,\tau) \Diamond O(fu, g\varpi_n,\tau)\}. \end{aligned}$$

It implies that

$$M(fu, gu, \lambda\tau) \geq \{1 * 1 * M(fu, gu,\tau) * 1 * M(fu, g\varpi_n,\tau)\} = M(fu, gu,\tau),$$

$$N(fu, gu, \lambda\tau) \leq \{0 \Diamond 0 \Diamond N(fu, gu,\tau) \Diamond 0 \Diamond N(fu, gu,\tau)\} = N(fu, gu,\tau),$$

and

$$O(fu, gu, \lambda\tau) \leq \{0 \lozenge 0 \lozenge O(fu, gu, \tau) \lozenge 0 \lozenge O(fu, gu, \tau)\} = O(fu, gu, \tau).$$

By applying Lemma 3, we deduce that $fu = gu$. Now, consider $z = fu = gu$. Therefore, the pair (f, g) is weakly compatible, and we obtain $fz = fgu = gfu = gz$. Now, we examine that z is a common fixed point of the mappings f and g. Now, utilizing inequalities (1), (2), and (3) with $\varpi = z, \omega = u$, we deduce that

$$M(fz, fu, \lambda\tau) \geq \{M(gz, gu, \tau) * M(fz, gz, \tau) * M(fu, gu, \tau) * M(fz, gu, \tau) * M(fu, gz, \tau)\},$$

$$N(fz, fu, \lambda\tau) \leq \{N(gz, gu, \tau) \lozenge N(fz, gz, \tau) \lozenge N(fu, gu, \tau) \lozenge N(fz, gu, \tau) \lozenge N(fu, gz, \tau)\},$$

and

$$O(fz, fu, \lambda\tau) \leq \{O(gz, gu, \tau) \lozenge O(fz, gz, \tau) \lozenge O(fu, gu, \tau) \lozenge O(fz, gu, \tau) \lozenge O(fu, gz, \tau)\}.$$

It implies

$$M(fz, z, \lambda\tau) \geq \{M(fz, z, \tau) * 1 * 1 * M(fz, z, \tau) * M(z, fz, \tau)\} = M(fz, z, \tau),$$

$$N(fz, z, \lambda\tau) \leq \{N(fz, z, \tau) \lozenge 0 \lozenge 0 \lozenge N(fz, z, \tau) \lozenge N(z, fz, \tau)\} = N(fz, z, \tau),$$

and

$$O(fz, z, \lambda\tau) \leq \{O(fz, z, \tau) \lozenge 0 \lozenge 0 \lozenge O(fz, z, \tau) \lozenge O(z, fz, \tau)\} = O(fz, z, \tau).$$

By utilizing Lemma 3, we find that $z = fz = gz$, which shows that z is a common fixed point of the mappings f and g. To show the uniqueness, we suppose that w will be another common fixed point of the mappings f and g. When using inequalities (1), (2), and (3) with $\varpi = z,\ \omega = w$, we have

$$M(fz, fw, \lambda\tau) \geq \{M(gz, gw, \tau) * M(fz, gz, \tau) * M(fw, gw, \tau) * M(fz, gw, \tau) * M(fw, gz, \tau)\},$$

$$N(fz, fw, \lambda\tau) \leq \{N(gz, gw, \tau) \lozenge N(fz, gz, \tau) \lozenge N(fw, gw, \tau) \lozenge N(fz, gw, \tau) \lozenge N(fw, gz, \tau)\},$$

and

$$O(fz, fw, \lambda\tau) \leq \{O(gz, gw, \tau) \lozenge O(fz, gz, \tau) \lozenge O(fw, gw, \tau) \lozenge O(fz, gw, \tau) \lozenge O(fw, gz, \tau)\},$$

Or, equivalently,

$$M(z, w, \lambda\tau) \geq \{M(z, w, \tau) * M(z, z, \tau) * M(w, w, \tau) * M(z, w, \tau) * M(w, z, \tau)\} = M(z, w, \tau),$$

$$N(z, w, \lambda\tau) \leq \{N(z, w, \tau) \lozenge N(z, z, \tau) \lozenge N(w, w, \tau) \lozenge N(z, w, \tau) \lozenge N(w, z, \tau)\} = N(z, w, \tau),$$

and

$$O(z, w, \lambda\tau) \leq \{O(z, w, \tau) \lozenge O(z, z, \tau) \lozenge O(w, w, \tau) \lozenge O(z, w, \tau) \lozenge O(w, z, \tau)\} = O(z, w, \tau).$$

Appealing to Lemma 3, we have $z = w$. Therefore, the mappings f and g have a unique common fixed point in Ξ. □

Example 4. *Let $\Xi = [1, 15)$ with metric d be defined by $d(\varpi, \omega) = |\varpi - \omega|$ and, thus, define*

$$M(\varpi, \omega, \tau) = \begin{cases} \frac{\tau}{\tau + |\varpi - \omega|}, \text{ if } \tau > 0, \\ 0, \text{ if } \tau = 0, \end{cases}$$

$$N(\varpi,\omega,\tau)=\begin{cases}\frac{|\varpi-\omega|}{\tau+|\varpi-\omega|}, \text{ if } \tau>0,\\ 1, \text{ if } \tau=0,\end{cases}$$

$$O(\varpi,\omega,\tau)=\begin{cases}\frac{|\varpi-\omega|}{\tau}, \text{ if } \tau>0,\\ 1, \text{ if } \tau=0,\end{cases}$$

$\forall\varpi,\omega\in\Xi$. Then, $(\Xi, M, N, O, *, \Diamond)$ is an IFM-space where, $*$ and $\Diamond$ are the continuous t-norm and continuous t-co-norm defined by $a*b=min\{a,b\}$ and $a\Diamond b=max\{a,b\},\forall a,b\in[0,1]$. Now, we define the self-mappings f and g on Ξ by

$$f(\varpi)=\begin{cases}1, \text{ if } \varpi\in\{1\}\bigcup(3,15),\\ 8, \text{ if } \varpi\in(1,3].\end{cases}$$

$$g(\varpi)=\begin{cases}1, \text{ if } \varpi=1;\\ 7, \quad \text{if } \varpi\in(1,3];\\ \frac{\varpi+1}{4}, \text{ if } \varpi\in(3,15)\end{cases}$$

Consider a sequence $\{\varpi_n\}=\left\{3+\frac{1}{n}\right\}_{n\in\mathbb{N}}$ or $\{\varpi_n\}=1$.

Then, we have

$$\lim_{n\to\infty}f\varpi_n=\lim_{n\to\infty}g\varpi_n=1=g(1)\in\Xi.$$

Hence, the pair (f,g) fulfill the (CLRs) property. It is clear that $f(\varpi)=\{1,8\}\not\subset[1,4)\cup\{7\}=g(\Xi)$. Here, $g(\Xi)$ is not a closed subsequence of Ξ. That is, all the conditions of Theorem 3 are fulfilled for some $\lambda\in(0,1)$, and 1 is a unique common fixed point of the mappings f and g. Additionally, at their unique common fixed point, every mapping that is involved is discontinuous.

Proposition 1. *Let $(\Xi, M, N, O, *, \Diamond)$ be an NMS, and if self-mappings $\mathcal{A}$ and B are compatible, then they are weakly compatible.*

Proof. We suppose $\mathcal{A}\varpi=\mathrm{B}\varpi$ for some p in Ξ. Consider the constant sequence $\{\varpi_n\}=\varpi$. Now, $\mathcal{A}\varpi_n\to\mathcal{A}p$ and $\mathrm{B}\varpi_n\to\mathrm{B}\varpi(=\mathcal{A}\varpi)$. As $\mathcal{A}$ and B are compatible, we have

$$\lim_{n\to\infty}M(\mathcal{A}B\varpi_n,\ B\mathcal{A}\varpi_n,\tau)=1,$$

$$\lim_{n\to\infty}N(\mathcal{A}B\varpi_n,\ B\mathcal{A}\varpi_n,\tau)=0,$$

$$\lim_{n\to\infty}O(\mathcal{A}B\varpi_n,\ B\mathcal{A}\varpi_n,\tau)=0,$$

For all $\tau>0$. Thus, $\mathcal{A}\mathrm{B}\varpi=\mathrm{B}\mathcal{A}\varpi$, and $(\mathcal{A},\mathrm{B})$ is weakly compatible. □

Proposition 2. *Let $(\Xi, M, N, O, *, \Diamond)$ be a complete NMS with $\tau*\tau\geq\tau$ and $(1-\tau)\Diamond(1-\tau)\leq 1-\tau$ for all $\tau\in[0,1]$, and let $\mathcal{A}$ and B be continuous mappings from Ξ into themselves. If $\mathcal{A}$ and B are compatible mappings of type α, then they are compatible.*

Theorem 4. *Let $(\Xi, M, N, O, *, \Diamond)$ be a complete NMS with $\tau*\tau\geq\tau$ and $(1-\tau)\Diamond(1-\tau)\leq1-\tau$ for all $\tau\in[0,1]$. Let $\zeta_1,\zeta_2,\ldots,\zeta_{2n},Q_0$ and Q_1 be mappings from Ξ into themselves that satisfy the following conditions:*

(1) $Q_0(\Xi)\subset\zeta_1\zeta_3\ldots\zeta_{2n-1}(\Xi)$, $Q_1(\Xi)\subset\zeta_2\zeta_4\ldots\zeta_{2n}(\Xi)$;

(2) $\zeta_2(\zeta_4\ldots\zeta_{2n})=(\zeta_4\ldots\zeta_{2n})\zeta_2$

$$\begin{aligned}
\zeta_2\zeta_4(\zeta_6\ldots\zeta_{2n}) &= (\zeta_6\ldots\zeta_{2n})\zeta_2\zeta_4\\
&\vdots\\
\zeta_2\ldots\zeta_{2n-2}(\zeta_{2n}) &= (\zeta_{2n})\zeta_2\ldots\zeta_{2n-2},\\
Q_0(\zeta_4\ldots\zeta_{2n}) &= (\zeta_4\ldots\zeta_{2n})Q_0,\\
Q_0(\zeta_6\ldots\zeta_{2n}) &= (\zeta_6\ldots\zeta_{2n})Q_0\\
&\vdots\\
Q_0\zeta_{2n} &= \zeta_{2n}Q_0,\\
\zeta_1(\zeta_3\ldots\zeta_{2n-1}) &= (\zeta_3\ldots\zeta_{2n-1})\zeta_1,\\
\zeta_1\zeta_3(\zeta_5\ldots\zeta_{2n-1}) &= (\zeta_5\ldots\zeta_{2n-1})\zeta_1\zeta_3,\\
&\vdots\\
\zeta_1\ldots\zeta_{2n-3}\,(\zeta_{2n-1}) &= (\zeta_{2n-1})\zeta_1\ldots\zeta_{2n-3},\\
Q_1(\zeta_3\ldots\zeta_{2n-1}) &= (\zeta_3\ldots\zeta_{2n-1})Q_1,\\
Q_1(\zeta_5\ldots\zeta_{2n-1}) &= (\zeta_5\ldots\zeta_{2n-1})Q_1,\\
&\vdots\\
Q_1\zeta_{2n-1} &= \zeta_{2n-1}Q_1;
\end{aligned}$$

(3) *either $\zeta_2\ldots\zeta_{2n}$ or Q_0 is continuous;*

(4) *$(Q_0,\ \zeta_2\ldots\zeta_{2n})$ is compatible, and $(Q_1,\ \zeta_1\ldots\zeta_{2n-1})$ is weakly compatible;*

(5) *$\exists\ \varrho\lambda \in (0,1)$ such that*

$$\begin{gathered}
M(Q_0\varpi, Q_1\omega,\ \lambda\tau) \geq M(\zeta_2\zeta_4\ldots\zeta_{2n}\varpi, Q_0\varpi, \tau) * M(\zeta_1\zeta_3\ldots\zeta_{2n-1}\omega, Q_1\omega, \tau)\\
*M(\zeta_1\zeta_3\ldots\zeta_{2n-1}\omega,\ Q_0\varpi,\ \beta\tau);\\
*M(\zeta_2\zeta_4\ldots\zeta_{2n}\varpi,\ Q_1\omega, (2-\beta)\tau) * M(\zeta_2\zeta_4\ldots\zeta_{2n}\varpi,\ \zeta_1\zeta_3\ldots\zeta_{2n-1}\omega, \tau);\\
N(Q_0\varpi, Q_1\omega, \lambda\tau) \leq N(\zeta_2\zeta_4\ldots\zeta_{2n}\varpi,\ Q_0\varpi, \tau) \lozenge N(\zeta_1\zeta_3\ldots\zeta_{2n-1}\omega, Q_1\omega, \tau)\\
\lozenge N(\zeta_1\zeta_3\ldots\zeta_{2n-1}\omega,\ Q_0\varpi,\ \beta\tau);\\
\lozenge N(\zeta_2\zeta_4\ldots\zeta_{2n}\varpi,\ Q_1\omega, (2-\beta)\tau)\ \lozenge N(\zeta_2\zeta_4\ldots\zeta_{2n}\varpi,\ \zeta_1\zeta_3\ldots\zeta_{2n-1}\omega, \tau).
\end{gathered}$$

and

$$\begin{gathered}
O(Q_0\varpi, Q_1\omega, \lambda\tau) \leq O(\zeta_2\zeta_4\ldots\zeta_{2n}\varpi,\ Q_0\varpi, \tau) \lozenge O(\zeta_1\zeta_3\ldots\zeta_{2n-1}\omega, Q_1\omega, \tau)\\
\lozenge O(\zeta_1\zeta_3\ldots\zeta_{2n-1}\omega,\ Q_0\varpi,\ \beta\tau)\\
\lozenge O(\zeta_2\zeta_4\ldots\zeta_{2n}\varpi,\ Q_1\omega, (2-\beta)\tau)\ \lozenge O(\zeta_2\zeta_4\ldots\zeta_{2n}\varpi,\ \zeta_1\zeta_3\ldots\zeta_{2n-1}\omega, \tau);
\end{gathered}$$

for all $\varpi, \omega \in \Xi, \beta \in (0,2)$ and $\tau > 0$. Then, $\zeta_1, \zeta_2\ldots\zeta_{2n}, Q_0$ and Q_1 have a unique CFP in Ξ.

Proof. Let ϖ_0 be a random point in Ξ from the condition (1) $\exists\ \varpi_1,\ \varpi_2 \in \Xi$, s.t

$$\begin{gathered}
Q_0\varpi_0 = \zeta_1\zeta_3\ldots\zeta_{2n-1}\varpi_1 = \omega_0\\
Q_1\varpi_1 = \zeta_2\zeta_4\ldots\zeta_{2n}\varpi_2 = \omega_1.
\end{gathered}$$

Using induction, we find a sequence ϖ_n and ω_n in Ξ

$$\begin{gathered}
Q_0\varpi_{2\lambda} = \zeta_1\zeta_3\ldots\zeta_{2n-1}\varpi_{2\lambda+1} = \omega_{2\lambda}\\
Q_1\varpi_{2\lambda+1} = \zeta_2\zeta_4\ldots\zeta_{2n}\varpi_{2\lambda+2} = \omega_{2\lambda+1}
\end{gathered}$$

for $\lambda = 0, 1, \ldots$, etc., based on the condition (5) for all $\tau > 0$ and $\beta = 1-q$ with $q \in (0,1)$, we have

$$\begin{gathered}
M(\omega_{2\lambda},\ \omega_{2\lambda+1},\ \lambda\tau) = M(Q_0\varpi_{2\lambda},\ Q_1\varpi_{2\lambda+1}, \lambda\tau) \geq M(\zeta_2\zeta_4\ldots\zeta_{2n}\varpi_{2\lambda},\ Q_0\varpi_{2\lambda}, \tau);\\
*M(\zeta_1\zeta_3\ldots\zeta_{2n-1}\varpi_{2\lambda+1},\ Q_1\varpi_{2\lambda}, \tau) * M(\zeta_1\zeta_3\ldots\zeta_{2n-1}\varpi_{2\lambda+1},\ Q_0\varpi_{2\lambda}, \beta\tau);\\
*M(\zeta_2\zeta_4\ldots\zeta_{2n}\varpi_{2\lambda},\ Q_1\varpi_{2\lambda+1}, (2-\beta)\tau) * M(\zeta_2\zeta_4\ldots\zeta_{2n}\varpi_{2\lambda},\ \zeta_1, \zeta_3\ldots\zeta_{2n-1}\,\varpi_{2\lambda+1}, \tau);\\
= M(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau) * M(\omega_{2\lambda},\ \omega_{2\lambda+1}, \tau) * M(\omega_{2\lambda},\ \omega_{2\lambda},\ (1-q)\tau);\\
*M(\omega_{2\lambda-1},\ \omega_{2\lambda+1}, (1+q)\tau) * M(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau);\\
\geq M(\omega_{2\lambda-1},\ \omega_{2\lambda}, \tau) * M(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau) * 1 * M(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau);\\
*M(\omega_{2\lambda},\ \omega_{2\lambda+1}, q\tau) * M(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau);\\
\geq M(\omega_{2\lambda-1}, \omega_{2\lambda},\ \tau) * M(\omega_{2\lambda},\ \omega_{2\lambda+1}, \tau) * M(\omega_{2\lambda},\ \omega_{2\lambda+1}, q\tau);
\end{gathered}$$

$$
\begin{aligned}
N(\omega_{2\lambda}, \omega_{2\lambda+1}, \lambda\tau) &= N(Q_0\varpi_{2\lambda}, Q_1\varpi_{2\lambda+1}, \lambda\tau) \leq N(\zeta_2\zeta_4 \ldots \zeta_{2n}\varpi_{2\lambda}, Q_0\varpi_{2\lambda}, \tau);\\
&\lozenge\, N(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\varpi_{2\lambda+1}, Q_1\varpi_{2\lambda+1}, \tau);\\
&\lozenge\, N(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\varpi_{2\lambda+1}, Q_0\varpi_{2\lambda}, \beta\tau);\\
&\lozenge\, N(\zeta_2\zeta_4 \ldots \zeta_{2n}\varpi_{2\lambda}, Q_1\varpi_{2\lambda+1}, (2-\beta)\tau);\\
&\lozenge\, N(\zeta_2\zeta_4 \ldots p_{2n}\varpi_{2\lambda}, \zeta_1, \zeta_3 \ldots \zeta_{2n-1}\varpi_{2\lambda+1}, \tau);\\
&= N(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau) \lozenge N(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau) \lozenge N(\omega_{2\lambda}, \omega_{2\lambda}, (1-q)\tau);\\
&\lozenge\, N(\omega_{2\lambda-1}, \omega_{2\lambda+1}, (1+q)\tau) \lozenge N(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau);\\
&\leq N(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau) \lozenge N(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau) \lozenge 0 \lozenge N(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau);\\
&\lozenge\, N(\omega_{2\lambda}, \omega_{2\lambda+1}, q\tau) \lozenge N(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau);\\
&\leq N(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau) \lozenge N(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau) \lozenge N(\omega_{2\lambda}, \omega_{2\lambda+1}, q\tau).
\end{aligned}
$$

and

$$
\begin{aligned}
&O(\omega_{2\lambda}, \omega_{2\lambda+1}, \lambda\tau) = O(Q_0\varpi_{2\lambda}, Q_1\varpi_{2\lambda+1}, \lambda\tau) \leq O(\zeta_2\zeta_4 \ldots \zeta_{2n}\varpi_{2\lambda}, Q_0\varpi_{2\lambda}, \tau);\\
&\lozenge\, O(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\varpi_{2\lambda+1}, Q_1\varpi_{2\lambda+1}, \tau) \lozenge O(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\varpi_{2\lambda+1}, Q_0\varpi_{2\lambda}, \beta\tau);\\
&\lozenge\, O(\zeta_2\zeta_4 \ldots \zeta_{2n}\varpi_{2\lambda}, Q_1\varpi_{2\lambda+1}, (2-\beta)\tau) \lozenge O(\zeta_2\zeta_4 \ldots p_{2n}\varpi_{2\lambda}, \zeta_1, \zeta_3 \ldots \zeta_{2n-1}\varpi_{2\lambda+1}, \tau);\\
&\qquad\qquad = O(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau) \lozenge O(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau) \lozenge O(\omega_{2\lambda}, \omega_{2\lambda}, (1-q)\tau)\\
&\lozenge\, O(\omega_{2\lambda-1}, \omega_{2\lambda+1}, (1+q)\tau) \lozenge O(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau)\\
&\qquad\qquad \leq O(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau) \lozenge O(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau) \lozenge 0 \lozenge O(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau);\\
&\qquad\qquad \lozenge\, O(\omega_{2\lambda}, \omega_{2\lambda+1}, q\tau) \lozenge O(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau);\\
&\qquad\qquad \leq O(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau) \lozenge O(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau) \lozenge O(\omega_{2\lambda}, \omega_{2\lambda+1}, q\tau).
\end{aligned}
$$

Thus, it follows that

$$
\begin{aligned}
&M(\omega_{2\lambda}, \omega_{2\lambda+1}, \lambda\tau)\\
&\qquad \geq M(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau) * M(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau);\\
&\qquad * M(\omega_{2\lambda}, \omega_{2\lambda+1}, q\tau), N(\omega_{2\lambda}, \omega_{2\lambda+1}, \lambda\tau);\\
&\qquad \leq N(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau) \lozenge N(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau);\\
&\qquad \lozenge\, N(\omega_{2\lambda}, \omega_{2\lambda+1}, q\tau), O(\omega_{2\lambda}, \omega_{2\lambda+1}, \lambda\tau);\\
&\qquad \leq O(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau) \lozenge O(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau) \lozenge O(\omega_{2\lambda}, \omega_{2\lambda+1}, q\tau).
\end{aligned}
$$

For CTN $*$, CTCN $\lozenge$, $M(\varpi, \omega, \cdot)$, and $N(\varpi, \omega, \cdot)$ both the left and the right are continuous. Given $q \to 1$, we have

$$
\begin{aligned}
M(\omega_{2\lambda}, \omega_{2\lambda+1}, \lambda\tau) \quad &\geq M(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau) * M(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau), N(\omega_{2\lambda}, \omega_{2\lambda+1}, \lambda\tau);\\
&\leq N(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau) \lozenge N(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau), O(\omega_{2\lambda}, \omega_{2\lambda+1}, \lambda\tau);\\
&\leq O(\omega_{2\lambda-1}, \omega_{2\lambda}, \tau) \lozenge O(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&M(\omega_{2\lambda+1}, \omega_{2\lambda+2}, \lambda\tau)\\
&\qquad \geq M(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau) * M(\omega_{2\lambda+1}, \omega_{2\lambda+2}, \tau), N(\omega_{2\lambda+1}, \omega_{2\lambda+2}, \lambda\tau);\\
&\qquad \leq N(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau) \lozenge N(\omega_{2\lambda+1}, \omega_{2\lambda+2}, \tau), O(\omega_{2\lambda+1}, \omega_{2\lambda+2}, \lambda\tau);\\
&\qquad \leq O(\omega_{2\lambda}, \omega_{2\lambda+1}, \tau) \lozenge O(\omega_{2\lambda+1}, \omega_{2\lambda+2}, \tau).
\end{aligned}
$$

In general, for $m = 1, 2, \ldots,$ we have

$$
\begin{aligned}
M(\omega_{m+1}, \omega_{m+2}, \lambda\tau) &\geq M(\omega_m, \omega_{m+1}, \tau) * M(\omega_{m+1}, \omega_{m+2}, \tau);\\
N(\omega_{m+1}, \omega_{m+2}, \lambda\tau) &\leq N(\omega_m, \omega_{m+1}, \tau) \lozenge N(\omega_{m+1}, \omega_{m+2}, \tau);\\
O(\omega_{m+1}, \omega_{m+2}, \lambda\tau) &\leq O(\omega_m, \omega_{m+1}, \tau) \lozenge O(\omega_{m+1}, \omega_{m+2}, \tau).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&M(\omega_{m+1}, \omega_{m+2}, \lambda\tau) \geq M(\omega_m, \omega_{m+1}, \tau) * M(\omega_{m+1}, \omega_{m+2}, \tau/\lambda^p);\\
&N(\omega_{m+1}, \omega_{m+2}, \lambda\tau) \leq N(\omega_m, \omega_{m+1}, \tau) * N(\omega_{m+1}, \omega_{m+2}, \tau/\lambda^p), O(\omega_{m+1}, \omega_{m+2}, \lambda\tau);\\
&\qquad\qquad \leq O(\omega_m, \omega_{m+1}, \tau) * O(\omega_{m+1}, \omega_{m+2}, \tau/\lambda^p).
\end{aligned}
$$

Noting that $M(\omega_{m+1}, \omega_{m+2}, \tau/\lambda^p) \to 1$, $N(\omega_{m+1}, \omega_{m+2}, \tau/\lambda^p) \to 0$, and $O(\omega_{m+1}, \omega_{m+2}, \tau/\lambda^p) \to 0$ as $p \to \infty$, we have, for $m = 1, 2, \ldots,$

$$
\begin{aligned}
&M(\omega_{m+1}, \omega_{m+2}, \lambda\tau) \geq M(\omega_m, \omega_{m+1}, \tau), N(\omega_{m+1}, \omega_{m+2}, \lambda\tau)\\
&\qquad\qquad \leq N(\omega_m, \omega_{m+1}, \tau), O(\omega_{m+1}, \omega_{m+2}, \lambda\tau) \leq O(\omega_m, \omega_{m+1}, \tau).
\end{aligned}
$$

Based on Lemma 1, $\{\omega_m\}$ is a Cauchy sequence in Ξ. Since Ξ is complete, $\{\omega_m\}$ converges to the point $\sigma \in \Xi$. Also, we have its consequences as follows:

$$Q_1\varpi_{2\lambda+1} \to \sigma \text{ and } \zeta_1\zeta_3\ldots\zeta_{2n-1}\varpi_{2\lambda+1} \to \sigma;\ Q_0\varpi_{2\lambda} \to \sigma\text{and } \zeta_2\zeta_4\ldots\zeta_{2n}\varpi_{2\lambda} \to \sigma.$$

□

Case 1. *$\zeta_2\zeta_4\ldots\ \zeta_{2n}$ is continuous. Define $\zeta_1' = \zeta_2\zeta_4\ldots\ \zeta_{2n}$. ζ_1' is continuous, $\zeta_1'{}^\circ\zeta_1'\varpi_{2\lambda} \to \zeta_1'\sigma$ and $\zeta_1'Q_0\varpi_{2\lambda} \to \zeta_1'\sigma$. Also, as (Q_0, ζ_1') is compatible, this implies that $Q_0\zeta_1'\varpi_{2\lambda} \to \zeta'_1\sigma$.*

(a) *Putting $\varpi = \zeta_2\zeta_4\ldots\ \zeta_{2n}\varpi_{2\lambda} = \zeta_1'\varpi_{2\lambda}$, $\omega = \varpi_{2\lambda+1}$, and $\zeta_2' = \zeta_1\zeta_3\ldots\zeta_{2n-1}$ with $\beta = 1$ in condition (5), we have*

$$\begin{aligned} M\big(Q_0\zeta_1'\varpi_{2\lambda}, Q_1\varpi_{2\lambda+1}, \lambda\tau\big) & \\ & \geq M\big(\zeta_1'\zeta_1'\varpi_{2\lambda}, Q_0\zeta_1'\varpi_{2\lambda}, \tau\big) * M\big(\zeta_2'\varpi_{2\lambda+1}, Q_1\varpi_{2\lambda+1}, \tau\big) \\ & *M\big(\zeta_2'\varpi_{2\lambda+1}, Q_0\zeta_1'\varpi_{2\lambda}, \tau\big) * M\big(\zeta_1'\zeta_1'\varpi_{2\lambda}, Q_1\varpi_{2\lambda+1}, \tau\big) \\ & *M\big(\zeta_1'\zeta_1'\varpi_{2\lambda}, \zeta_2'\varpi_{2\lambda+1}, \tau\big), N\big(Q_0\zeta_1'\varpi_{2\lambda}, Q_1\varpi_{2\lambda+1}, \lambda\tau\big) \\ & \leq N\big(\zeta_1'\zeta_1'\varpi_{2\lambda}, Q_0\zeta_1'\varpi_{2\lambda}, \tau\big) \lozenge N\big(\zeta_2'\varpi_{2\lambda+1}, Q_1\varpi_{2\lambda+1}, \tau\big) \\ & \lozenge N\big(\zeta_2'\varpi_{2\lambda+1},\ Q_0\zeta_1'\varpi_{2\lambda}, \tau\big) \lozenge N\big(\zeta_1'\zeta_1'\varpi_{2\lambda},\ Q_1\varpi_{2\lambda+1}, \tau\big) \\ & \lozenge N\big(\zeta_1'\zeta_1'\varpi_{2\lambda}, \zeta_2'\varpi_{2\lambda+1}, \tau\big), O\big(Q_0\zeta_1'\varpi_{2\lambda}, Q_1\varpi_{2\lambda+1}, \lambda\tau\big) \\ & \leq O\big(\zeta_1'\zeta_1'\varpi_{2\lambda}, Q_0\zeta_1'\varpi_{2\lambda}, \tau\big) \lozenge O\big(\zeta_2'\varpi_{2\lambda+1}, Q_1\varpi_{2\lambda+1}, \tau\big) \\ & \lozenge O\big(\zeta_2'\varpi_{2\lambda+1},\ Q_0\zeta_1'\varpi_{2\lambda}, \tau\big) \lozenge O\big(\zeta_1'\zeta_1'\varpi_{2\lambda},\ Q_1\varpi_{2\lambda+1}, \tau\big) \\ & \lozenge O\big(\zeta_1'\zeta_1'\varpi_{2\lambda}, \zeta_2'\varpi_{2\lambda+1}, \tau\big). \end{aligned}$$

Which implies that as $\lambda \to \infty$

$$\begin{aligned} M\big(\zeta_1'\sigma, \sigma, \lambda\tau\big) \geq\ & 1 * 1 * M\big(\sigma, \zeta_1'\sigma, \tau\big) * M\big(\zeta_1'\sigma, \sigma, \tau\big) * M\big(\zeta_1'\sigma, \sigma, \tau\big); \\ & \geq M\big(\zeta_1'\sigma, \sigma, \tau\big), N\big(\zeta_1'\sigma, \sigma, \lambda\tau\big); \\ & \leq 0 * 0 * N\big(\sigma, \zeta_1'\sigma, \tau\big) * N\big(\zeta_1'\sigma, \sigma, \tau\big) * N\big(\zeta_1'\sigma, \sigma, \tau\big); \\ & \leq N\big(\zeta_1'\sigma, \sigma, \tau\big), O\big(\zeta_1'\sigma, \sigma, \lambda\tau\big); \\ & \leq 0 * 0 * O\big(\sigma, \zeta_1'\sigma, \tau\big) * O\big(\zeta_1'\sigma, \sigma, \tau\big) * O\big(\zeta_1'\sigma, \sigma, \tau\big); \\ & \leq O\big(\zeta_1'\sigma, \sigma, \tau\big). \end{aligned}$$

Therefore, based on Lemma 2, we have $\zeta_1'\sigma = \sigma$, i.e., $\zeta_2\zeta_4\ldots\zeta_{2n}\sigma = \sigma$.

(b) *If $\varpi = \sigma, \omega = \varpi_{2\lambda+1}$, $\zeta_1' = \zeta_2\zeta_4\ldots\zeta_{2n}$, and $\zeta_2' = \zeta_1\zeta_3\ldots\zeta_{2n-1}$. With $\beta = 1$ in condition (5), we have*

$$\begin{aligned} M(Q_0Z, Q_1\varpi_{2\lambda+1}, \lambda\tau) &\geq M\big(\zeta_1'\sigma, Q_0\sigma, \tau\big) * M\big(\zeta_2'\varpi_{2\lambda+1}, Q_1\varpi_{2\lambda+1}, \tau\big) * M\big(\zeta_2'\varpi_{2\lambda+1}, Q_0\sigma, \tau\big) \\ & *M\big(\zeta_1'\sigma, Q_1\varpi_{2\lambda+1}, \tau\big) * M\big(\zeta_1'\sigma, \zeta_2'\varpi_{2\lambda+1}, \tau\big); \\ N(Q_0\sigma, Q_1\varpi_{2\lambda+1}, \lambda\tau) &\leq\ N\big(\zeta_1'\sigma, Q_0\sigma, \tau\big) \lozenge N\big(\zeta_2'\varpi_{2\lambda+1}, Q_1\varpi_{2\lambda+1}, \tau\big) \lozenge N\big(\zeta_2'\varpi_{2\lambda+1}, Q_0\sigma, \tau\big) \\ & \lozenge N\big(\zeta_1'\sigma, Q_1\varpi_{2\lambda+1}, \tau\big) \lozenge N\big(\zeta_1'\sigma, \zeta_2'\varpi_{2\lambda+1}, \tau\big); \\ O(Q_0\sigma, Q_1\varpi_{2\lambda+1}, \lambda\tau) &\leq\ O\big(\zeta_1'\sigma, Q_0\sigma, \tau\big) \lozenge O\big(\zeta_2'\varpi_{2\lambda+1}, Q_1\varpi_{2\lambda+1}, \tau\big) \lozenge O\big(\zeta_2'\varpi_{2\lambda+1}, Q_0\sigma, \tau\big) \\ & \lozenge O\big(\zeta_1'\sigma, Q_1\varpi_{2\lambda+1}, \tau\big) \lozenge O\big(\zeta_1'\sigma, \zeta_2'\varpi_{2\lambda+1}, \tau\big). \end{aligned}$$

Which implies that as $\lambda \to \infty$

$$\begin{aligned} M(Q_0\sigma, \sigma, \lambda\tau) \geq\ & M(\sigma, Q_0\ \sigma, \tau) * 1 * M(\sigma, Q_0\sigma, \tau) * 1 * 1 \geq M(Q_0\sigma, \sigma, \tau), N(Q_0\sigma, \sigma, \lambda\tau); \\ & \leq N(\sigma, Q_0\ \sigma, \tau) \lozenge 1 \lozenge N(\sigma, Q_0\sigma, \tau) \lozenge 1 \lozenge 1 \leq N(Q_0\sigma, \sigma, \tau), O(Q_0\sigma, \sigma, \lambda\tau); \\ & \leq O(\sigma, Q_0\ \sigma, \tau) \lozenge 1 \lozenge O(\sigma, Q_0\sigma, \tau) \lozenge 1 \lozenge 1 \leq O(Q_0\sigma, \sigma, \tau). \end{aligned}$$

Therefore, based on Lemma 2, we have $Q_0\sigma = \sigma$. Hence, $Q_0\sigma = \zeta_2\zeta_4\ldots\zeta_{2n}\sigma = \sigma$.

(c) *If $\varpi = \zeta_4\zeta_6\ldots\zeta_{2n}\sigma$, $\omega = \varpi_{2\lambda+1}$, $\zeta_1' = \zeta_2\zeta_4\ldots\zeta_{2n}$, and $\zeta_2' = \zeta_1\zeta_3\ldots\zeta_{2n-1}$, with $\beta = 1$ in condition (5). Using the conditions $\zeta_2(\zeta_4\ldots\zeta_{2n}) = (\zeta_4\ldots\zeta_{2n})\zeta_2$, and $Q_0(\zeta_4\ldots\ \zeta_{2n}) = (\zeta_4\ldots\zeta_{2n})Q_0$ in condition (2), we have*

$$
\begin{aligned}
&M(Q_0\zeta_4\zeta_6\ldots\zeta_{2n}\sigma, Q_1\varpi_{2\lambda+1}, \lambda\tau)\\
&\qquad \geq M\big(\zeta_1'\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\ Q_0\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\tau\big) * M\big(\zeta_2'\varpi_{2\lambda+1}, Q_1\varpi_{2\lambda+1},\tau\big)\\
&\qquad *M\big(\zeta_2'\varpi_{2\lambda+1}, Q_0\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\tau\big) * M\big(\zeta_1'\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\ Q_1\varpi_{2\lambda+1},\tau\big)\\
&\qquad *M\big(\zeta_1'\zeta_4\zeta_6\ldots\zeta_{2n}\sigma, \zeta_2'\varpi_{2\lambda+1},\tau\big), N(Q_0\zeta_4\zeta_6\ldots\zeta_{2n}\sigma, Q_1\varpi_{2\lambda+1},\lambda\tau);\\
&\qquad \leq N\big(\zeta_1'\zeta_4\zeta_6\ldots\zeta_{2n}\sigma, Q_0\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\tau\big) \lozenge N\big(\zeta_2'\varpi_{2\lambda+1}, Q_1\varpi_{2\lambda+1},\tau\big)\\
&\qquad \lozenge N\big(\zeta_2'\varpi_{2\lambda+1}, Q_0\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\tau\big) \lozenge N\big(\zeta_1'\zeta_4\zeta_6\ldots\zeta_{2n}\sigma, Q_1\varpi_{2\lambda+1},\tau\big)\\
&\qquad \lozenge N\big(\zeta_1'\zeta_4\zeta_6\ldots\zeta_{2n}\sigma, \zeta_2'\varpi_{2\lambda+1},\tau\big), O(Q_0\zeta_4\zeta_6\ldots\zeta_{2n}\sigma, Q_1\varpi_{2\lambda+1},\lambda\tau);\\
&\qquad \leq O\big(\zeta_1'\zeta_4\zeta_6\ldots\zeta_{2n}\sigma, Q_0\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\tau\big) \lozenge O\big(\zeta_2'\varpi_{2\lambda+1}, Q_1\varpi_{2\lambda+1},\tau\big)\\
&\qquad \lozenge O\big(\zeta_2'\varpi_{2\lambda+1}, Q_0\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\tau\big) \lozenge O\big(\zeta_1'\zeta_4\zeta_6\ldots\zeta_{2n}\sigma, Q_1\varpi_{2\lambda+1},\tau\big)\\
&\qquad \lozenge O\big(\zeta_1'\zeta_4\zeta_6\ldots\zeta_{2n}\sigma, \zeta_2'\varpi_{2\lambda+1},\tau\big).
\end{aligned}
$$

Which implies that as $\lambda \to \infty$

$$
\begin{aligned}
&M(\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\sigma,\lambda\tau)\\
&\qquad \geq 1*1*M(\sigma,\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\tau)*M(\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\sigma,\tau)\\
&\qquad *M(\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\sigma,\tau);\\
&\qquad \geq M(\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\sigma,\tau), N(\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\sigma,\lambda\tau)\\
&\qquad \leq 0\lozenge 0\lozenge N(\sigma,\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\tau)\lozenge N(\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\sigma,\tau)\\
&\qquad \lozenge N(\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\sigma,\tau);\\
&\qquad \leq N(\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\sigma,\tau), O(\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\sigma,\lambda\tau)\\
&\qquad \leq 0\lozenge 0\lozenge O(\sigma,\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\tau)\lozenge O(\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\sigma,\tau)\\
&\qquad \lozenge O(\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\sigma,\tau)\leq O(\zeta_4\zeta_6\ldots\zeta_{2n}\sigma,\sigma,\tau).
\end{aligned}
$$

Therefore, based on Lemma 2, we have $\zeta_4\zeta_6\ldots\zeta_{2n}\sigma = \sigma$. *Then,* $\zeta_2(\zeta_4\zeta_6\ldots\zeta_{2n}\sigma)$, $\sigma = \zeta_2\sigma$, *and* $\zeta_2\sigma = \zeta_2\zeta_4\zeta_6\ldots\zeta_{2n}\sigma = \sigma$. *Continuing this procedure, we obtain*

$$Q_0\sigma = \zeta_2\sigma = \zeta_4\sigma = \ldots = \zeta_{2n}\sigma = \sigma.$$

(d) *As* $Q_0(\Xi) \subset \zeta_1\zeta_3\ldots\zeta_{2n-1}(\Xi)$, *there is* $\omega \in \Xi$ *such that* $\sigma = Q_0\sigma = \zeta_1\zeta_3\ldots\zeta_{2n-1}\omega$. *If* $\varpi = \varpi_{2\lambda}, \zeta_1' = \zeta_2\zeta_4\ldots\zeta_{2n}$, *and* $\zeta_2' = \zeta_1\zeta_3\ldots\zeta_{2n-1}$, *with the* $\beta = 1$ *in condition (5), we have*

$$
\begin{aligned}
M(Q_0\varpi_{2\lambda}, Q_1\omega,\lambda\tau) &\geq M\big(\zeta_1'\varpi_{2\lambda}, Q_0\varpi_{2\lambda},\tau\big)*M\big(\zeta_2'\omega, Q_1\omega,\tau\big)*M\big(\zeta_2'\omega, Q_0\varpi_{2\lambda},\tau\big)\\
&\quad *M\big(\zeta_1'\varpi_{2\lambda}, Q_1\omega,\tau\big)*M\big(\zeta_1'\varpi_{2\lambda},\zeta_2'\omega,\tau\big);\\
N(Q_0\varpi_{2\lambda}, Q_1\omega,\lambda\tau) &\leq N\big(\zeta_1'\varpi_{2\lambda}, Q_0\varpi_{2\lambda},\tau\big)\lozenge N\big(\zeta_2'\omega, Q_1\omega,\tau\big)\lozenge N\big(\zeta_2'\omega, Q_0\varpi_{2\lambda},\tau\big)\\
&\quad \lozenge N\big(\zeta_1'\varpi_{2\lambda}, Q_1\omega,\tau\big)\lozenge N\big(\zeta_1'\varpi_{2\lambda},\zeta_2'\omega,\tau\big);\\
O(Q_0\varpi_{2\lambda}, Q_1\omega,\lambda\tau) &\leq O\big(\zeta_1'\varpi_{2\lambda}, Q_0\varpi_{2\lambda},\tau\big)\lozenge O\big(\zeta_2'\omega, Q_1\omega,\tau\big)\lozenge O\big(\zeta_2'\omega, Q_0\varpi_{2\lambda},\tau\big)\\
&\quad \lozenge O\big(\zeta_1'\varpi_{2\lambda}, Q_1\omega,\tau\big)\lozenge O\big(\zeta_1'\varpi_{2\lambda},\zeta_2'\omega,\tau\big).
\end{aligned}
$$

Which implies that as $\lambda \to \infty$,

$$
\begin{aligned}
M(\sigma, Q_1\omega,\ \lambda\tau) \geq\ & 1*M(\sigma,Q_1\omega,\tau)*1*M(\sigma,Q_1\omega,\tau)*1;\\
& \geq M(\sigma,Q_1\omega,\tau), N(\sigma,Q_1\omega,\ \lambda\tau);\\
& \leq 0\lozenge N(\sigma,Q_1\omega,\tau)\lozenge 0\lozenge N(\sigma,Q_1\omega,\tau)\lozenge 0;\\
& \leq N(\sigma,Q_1\omega,\tau), O(\sigma,Q_1\omega,\ \lambda\tau);\\
& \leq 0\lozenge O(\sigma,Q_1\omega,\tau)\lozenge 0\lozenge O(\sigma,Q_1\omega,\tau)\lozenge 0\leq O(\sigma,Q_1\omega,\tau).
\end{aligned}
$$

Therefore, based on Lemma 2, we have $Q_1\omega = \sigma$. *Hence,* $\zeta_1\zeta_3\ldots\zeta_{2n-1}\omega = Q_1\omega = \sigma$. *As* $(Q_1, \zeta_1\zeta_3\ldots\zeta_{2n-1})$ *is weakly compatible, we have* $\zeta_1\zeta_3\ldots\zeta_{2n-1}Q_1\omega = Q_1\zeta_1\zeta_3\ldots\zeta_{2n-1}\omega$. *Thus,* $\zeta_1\zeta_3\ldots\zeta_{2n-1}\sigma = Q_1\sigma$.

(e) *If* $\varpi = \varpi_2\lambda$, $\omega = \sigma$, $\zeta_1' = \zeta_2\zeta_4\ldots\zeta_{2n}$, *and* $\zeta_2' = \zeta_1\zeta_3\ldots\zeta_{2n-1}$ *are with* $\beta = 1$ *in condition (5), we have*

$$
\begin{aligned}
M(Q_0\varpi_{2\lambda}, Q_1\sigma,\lambda\tau) &\geq M\big(\zeta_1'\varpi_{2\lambda}, Q_0\varpi_{2\lambda},\tau\big)*M\big(\zeta_2'\sigma, Q_1\sigma,\tau\big)*M\big(\zeta_2'\sigma, Q_0\varpi_{2\lambda},\tau\big)\\
&\quad *M\big(\zeta_1'\varpi_{2\lambda}, Q_1\sigma,\tau\big)*M\big(\zeta_1'\varpi_{2\lambda},\zeta_2'\sigma,\tau\big);\\
N(Q_0\varpi_{2\lambda}, Q_1\sigma,\lambda\tau) &\leq N\big(\zeta_1'\varpi_{2\lambda}, Q_0\varpi_{2\lambda},\tau\big)\lozenge N\big(\zeta_2'\sigma, Q_1\sigma,\tau\big)\lozenge N\big(\zeta_2'\sigma, Q_0\varpi_{2\lambda},\tau\big)\\
&\quad \lozenge N\big(\zeta_1'\varpi_{2\lambda}, Q_1\sigma,\tau\big)\lozenge N\big(\zeta_1'\varpi_{2\lambda},\zeta_2'\sigma,\tau\big);\\
O(Q_0\varpi_{2\lambda}, Q_1\sigma,\lambda\tau) &\leq O\big(\zeta_1'\varpi_{2\lambda}, Q_0\varpi_{2\lambda},\tau\big)\lozenge O\big(\zeta_2'\sigma, Q_1\sigma,\tau\big)\lozenge O\big(\zeta_2'\sigma, Q_0\varpi_{2\lambda},\tau\big)\\
&\quad \lozenge O\big(\zeta_1'\varpi_{2\lambda}, Q_1\sigma,\tau\big)\lozenge O\big(\zeta_1'\varpi_{2\lambda},\zeta_2'\sigma,\tau\big).
\end{aligned}
$$

Which implies that as $\lambda \to \infty$

$$\begin{aligned} M(\sigma, Q_1\sigma, \lambda\tau) \geq\ & 1 * M(\sigma, Q_1\sigma, \tau) * 1 * M(\sigma, Q_1\sigma, \tau) * 1; \\ & \geq M(\sigma, Q_1\sigma, \tau), N(\sigma, Q_1\sigma, \lambda\tau); \\ & \leq 0 \lozenge N(\sigma, Q_1\sigma, \tau) \lozenge 0 \lozenge N(\sigma, Q_1\sigma, \tau) \lozenge 0; \\ & \leq N(\sigma, Q_1\sigma, \tau), O(\sigma, Q_1\sigma, \lambda\tau); \\ & \leq 0 \lozenge O(\sigma, Q_1\sigma, \tau) \lozenge 0 \lozenge O(\sigma, Q_1\sigma, \tau) \lozenge 0 \leq O(\sigma, Q_1\sigma, \tau). \end{aligned}$$

Therefore, based on Lemma 2, we have $Q_1\sigma = \sigma$. *Hence,* $\zeta_1\zeta_3 \ldots \zeta_{2n-1}\sigma = Q_1\sigma = \sigma$.

(f) *If* $\varpi = \varpi_{2\lambda}, \omega = \zeta_3 \ldots \zeta_{2n-1}\sigma$, $\zeta_1' = \zeta_2\zeta_4 \ldots \zeta_{2n}$, *and* $\zeta_2' = \zeta_1\zeta_3 \ldots$ *are With* $\beta = 1$ *in condition (5), we have*

$$\begin{aligned} & M(Q_0\varpi_{2\lambda}, Q_1\zeta_3 \ldots \zeta_{2n-1}\sigma, \lambda\tau) \geq M\big(\zeta_1'\varpi_{2\lambda}, Q_0\varpi_{2\lambda}, \tau\big) * M\big(\zeta_2'\zeta_3 \ldots \zeta_{2n-1}\sigma, Q_1\zeta_3 \ldots \zeta_{2n-1}\sigma, \tau\big) \\ & * M\big(\zeta_2'\zeta_3 \ldots \zeta_{2n-1}\sigma, Q_0\varpi_{2\lambda}, \tau\big) * M\big(\zeta_1'\varpi_{2\lambda}, Q_1\zeta_3 \ldots \zeta_{2n-1}\sigma, \tau\big) * M\big(\zeta_1'\varpi_{2\lambda}, \zeta_2'\zeta_3 \ldots \zeta_{2n-1}\sigma, \tau\big); \\ & N(Q_0\varpi_{2\lambda}, Q_1\zeta_3 \ldots \zeta_{2n-1}\sigma, \lambda\tau) \leq N\big(\zeta_1'\varpi_{2\lambda}, Q_0\varpi_{2\lambda}, \tau\big) \lozenge N\big(\zeta_2'\zeta_3 \ldots \zeta_{2n-1}\sigma, Q_1\zeta_3 \ldots \zeta_{2n-1}\sigma, \tau\big) \\ & \lozenge N\big(\zeta_2'\zeta_3 \ldots \zeta_{2n-1}\sigma, Q_0\varpi_{2\lambda}, \tau\big) \lozenge N\big(\zeta_1'\varpi_{2\lambda}, Q_1\zeta_3 \ldots \zeta_{2n-1}\sigma, \tau\big) \lozenge N\big(\zeta_1'\varpi_{2\lambda}, \zeta_2'\zeta_3 \ldots \zeta_{2n-1}\sigma, \tau\big); \\ & O(Q_0\varpi_{2\lambda}, Q_1\zeta_3 \ldots \zeta_{2n-1}\sigma, \lambda\tau) \leq O\big(\zeta_1'\varpi_{2\lambda}, Q_0\varpi_{2\lambda}, \tau\big) \lozenge O\big(\zeta_2'\zeta_3 \ldots \zeta_{2n-1}\sigma, Q_1\zeta_3 \ldots \zeta_{2n-1}\sigma, \tau\big) \\ & \lozenge O\big(\zeta_2'\zeta_3 \ldots \zeta_{2n-1}\sigma, Q_0\varpi_{2\lambda}, \tau\big) \lozenge O\big(\zeta_1'\varpi_{2\lambda}, Q_1\zeta_3 \ldots \zeta_{2n-1}\sigma, \tau\big) \lozenge O\big(\zeta_1'\varpi_{2\lambda}, \zeta_2'\zeta_3 \ldots \zeta_{2n-1}\sigma, \tau\big). \end{aligned}$$

Which implies that as $\lambda \to \infty$

$$\begin{aligned} & M(\sigma, \zeta_3 \ldots \zeta_{2n-1}\sigma, \lambda\tau) \\ & \qquad \geq 1 * 1 * M(\zeta_3 \ldots \zeta_{2n-1}\sigma, \sigma, \tau) * M(\sigma, \zeta_3 \ldots \zeta_{2n-1}\sigma, \tau) \\ & \qquad * M(\sigma, \zeta_3 \ldots \zeta_{2n-1}\sigma, \tau) \\ & \qquad \geq M(\sigma, \zeta_3 \ldots \zeta_{2n-1}\sigma, \tau); \\ & N(\sigma, \zeta_3 \ldots \zeta_{2n-1}\sigma, \lambda\tau) \\ & \qquad \leq 0 \lozenge 0 \lozenge N(\zeta_3 \ldots \zeta_{2n-1}\sigma, \sigma, \tau) \lozenge N(\sigma, \zeta_3 \ldots \zeta_{2n-1}\sigma, \tau) \\ & \qquad \lozenge N(\sigma, \zeta_3 \ldots \zeta_{2n-1}\sigma, \tau) \\ & \qquad \leq N(\sigma, \zeta_3 \ldots \zeta_{2n-1}\sigma, \tau); \\ & O(\sigma, \zeta_3 \ldots \zeta_{2n-1}\sigma, \lambda\tau) \\ & \qquad \leq 0 \lozenge 0 \lozenge O(\zeta_3 \ldots \zeta_{2n-1}\sigma, \sigma, \tau) \lozenge O(\sigma, \zeta_3 \ldots \zeta_{2n-1}\sigma, \tau) \\ & \qquad \lozenge O(\sigma, \zeta_3 \ldots \zeta_{2n-1}\sigma, \tau) \\ & \qquad \leq O(\sigma, \zeta_3 \ldots \zeta_{2n-1}\sigma, \tau). \end{aligned}$$

Therefore, based on Lemma 2, we have $\zeta_3 \ldots \zeta_{2n-1}\sigma = \sigma$. *Hence,* $\zeta_1\sigma = \sigma$. *Continuing this procedure, we have*

$$Q_1\sigma = \zeta_1\sigma = \zeta_3\sigma = \ldots = \zeta_{2n-1}\sigma.$$

Thus, we have

$$Q_0\sigma = Q_1\sigma = \zeta_1\sigma = \zeta_2\sigma = \zeta_3\sigma = \ldots = \zeta_{2n-1}\sigma = \zeta_{2n}\sigma = \sigma.$$

Case 2. Q_0 *is continuous. Since* Q_0 *is continuous,* $Q_0^2\varpi_{2\lambda} \to Q_0\sigma$ *and* $Q_0(\zeta_2\zeta_4 \ldots \zeta_{2n})\varpi_{2\lambda} \to Q_0\sigma$. *As* $(Q_0,\ \zeta_2\zeta_4 \ldots \zeta_{2n})$ *is compatible, we have* $(\zeta_2\zeta_4 \ldots \zeta_{2n})Q_0\varpi_{2\lambda} \to Q_0\sigma$.

(g) *If* $\varpi = Q_0\varpi_{2\lambda}$, $\omega = \varpi_{2\lambda+1}$, $\zeta_1' = \zeta_2\zeta_4 \ldots\ \zeta_{2n}$, *and* $\zeta_2' = \zeta_1\zeta_3 \ldots\ \zeta_{2n-1}$ *are with* $\beta = 1$ *in condition (5), we have*

$$\begin{aligned} & M(Q_0Q_0\varpi_{2\lambda},\ Q_1\varpi_{2\lambda+1}, \lambda\tau) \\ & \qquad \geq M\big(\zeta_1'Q_0\varpi_{2\lambda},\ Q_0Q_0\varpi_{2\lambda}, \tau\big) * \big(\zeta_2'\varpi_{2\lambda+1},\ Q_1\varpi_{2\lambda+1}, \tau\big) \\ & \qquad * M\big(\zeta_2'\varpi_{2\lambda+1}, Q_0Q_0\varpi_{2\lambda}, \tau\big) * M\big(\zeta_1'Q_0\varpi_{2\lambda}, Q_1\varpi_{2\lambda+1}, \tau\big) \\ & \qquad * M\big(\zeta_1'Q_0\varpi_{2\lambda}, \zeta_2'\varpi_{2\lambda+1}, \tau\big). \\ & N(Q_0Q_0\varpi_{2\lambda},\ Q_1\varpi_{2\lambda+1}, \lambda\tau) \\ & \qquad \leq N\big(\zeta_1'Q_0\varpi_{2\lambda},\ Q_0Q_0\varpi_{2\lambda}, \tau\big) \lozenge N\big(\zeta_2'\varpi_{2\lambda+1},\ Q_1\varpi_{2\lambda+1}, \tau\big) \\ & \qquad \lozenge N\big(\zeta_2'\varpi_{2\lambda+1}, Q_0Q_0\varpi_{2\lambda}, \tau\big) \lozenge N\big(\zeta_1'Q_0\varpi_{2\lambda}, Q_1\varpi_{2\lambda+1}, \tau\big) \\ & \qquad \lozenge N\big(\zeta_1'Q_0\varpi_{2\lambda}, \zeta_2'\varpi_{2\lambda+1}, \tau\big), O(Q_0Q_0\varpi_{2\lambda},\ Q_1\varpi_{2\lambda+1}, \lambda\tau); \\ & \qquad \leq O\big(\zeta_1'Q_0\varpi_{2\lambda},\ Q_0Q_0\varpi_{2\lambda}, \tau\big) \lozenge O\big(\zeta_2'\varpi_{2\lambda+1},\ Q_1\varpi_{2\lambda+1}, \tau\big) \\ & \qquad \lozenge O\big(\zeta_2'\varpi_{2\lambda+1}, Q_0Q_0\varpi_{2\lambda}, \tau\big) \lozenge O\big(\zeta_1'Q_0\varpi_{2\lambda}, Q_1\varpi_{2\lambda+1}, \tau\big) \\ & \qquad \lozenge O\big(\zeta_1'Q_0\varpi_{2\lambda}, \zeta_2'\varpi_{2\lambda+1}, \tau\big). \end{aligned}$$

Which implies that as $\lambda \to \infty$

$$\begin{aligned} M(Q_0\sigma, \sigma, \lambda\tau) \geq\ & 1 * 1 * M(\sigma, Q_0\sigma, \tau) * M(Q_0\sigma, \sigma, \tau) * M(Q_0\sigma, \sigma, \tau) \\ & \geq M(Q_0\sigma, \sigma, \tau), N(Q_0\sigma, \sigma, \lambda\tau); \\ & \leq 0 \lozenge 0 \lozenge N(\sigma, Q_0\sigma, \tau) \lozenge N(Q_0\sigma, \sigma, \tau) \lozenge N(Q_0\sigma, \sigma, \tau) \\ & \leq N(Q_0\sigma, \sigma, \tau), O(Q_0\sigma, \sigma, \lambda\tau); \\ & \leq 0 \lozenge 0 \lozenge O(\sigma, Q_0\sigma, \tau) \lozenge O(Q_0\sigma, \sigma, \tau) \lozenge O(Q_0\sigma, \sigma, \tau) \\ & \leq O(Q_0\sigma, \sigma, \tau). \end{aligned}$$

Therefore, based on Lemma 2, we have $Q_0\sigma = \sigma$. *As a result, using steps (d), (e), and (f) while continuing with step (f) provides us with the following information: (f) using steps (d), (e), and (f) now, carry on to step (f)*

$$Q_1\sigma = \zeta_1\sigma = \zeta_3\sigma = \ldots = \zeta_{2n-1}\sigma = \sigma.$$

(h) *As* $Q_1(\Xi) \subset \zeta_2\zeta_4 \ldots \zeta_{2n}(\Xi)$, *there is* $\omega \in \Xi$ *such that* $\sigma = Q_1\sigma = \zeta_2\zeta_4 \ldots \zeta_{2n}\omega$. *If* $\varpi = \omega, \omega = \varpi_{2\lambda+1}$, $\zeta_1' = \zeta_2\zeta_4 \ldots \zeta_{2n}$, *and* $\zeta_2' = \zeta_1\zeta_3 \ldots \zeta_{2n-1}$ *are with* $\beta = 1$ *in condition (5), we have*

$$\begin{aligned} M(Q_0\omega, Q_1\varpi_{2\lambda+1}, \lambda\tau) \quad & \geq M(\zeta_1'\omega,\ Q_0\omega, \tau) * M(\zeta_2'\varpi_{2\lambda+1}, Q_1\varpi_{2\lambda+1}, \tau) * M(\zeta_2'\varpi_{2\lambda+1}, Q_0\omega, \tau) \\ & * M(\zeta_1'\omega, Q_1\varpi_{2\lambda+1}, \tau) * M(\zeta_1'\omega, \zeta_2'\varpi_{2\lambda+1}, \tau); \\ N(Q_0\omega, Q_1\varpi_{2\lambda+1}, \lambda\tau) \quad & \leq N(\zeta_1'\omega,\ Q_0\omega, \tau) \lozenge N(\zeta_2'\varpi_{2\lambda+1}, Q_1\varpi_{2\lambda+1}, \tau) \lozenge N(\zeta_2'\varpi_{2\lambda+1}, Q_0\omega, \tau) \\ \lozenge N(\zeta_1'\omega, Q_1\varpi_{2\lambda+1}, \tau) \quad & \lozenge N(\zeta_1'\omega, \zeta_2'\varpi_{2\lambda+1}, \tau); \\ O(Q_0\omega, Q_1\varpi_{2\lambda+1}, \lambda\tau) \quad & \leq O(\zeta_1'\omega,\ Q_0\omega, \tau) \lozenge O(\zeta_2'\varpi_{2\lambda+1}, Q_1\varpi_{2\lambda+1}, \tau) \lozenge O(\zeta_2'\varpi_{2\lambda+1}, Q_0\omega, \tau) \\ & \lozenge O(\zeta_1'\omega, Q_1\varpi_{2\lambda+1}, \tau) \lozenge O(\zeta_1'\omega, \zeta_2'\varpi_{2\lambda+1}, \tau). \end{aligned}$$

Which implies that as $\lambda \to \infty$

$$\begin{aligned} M(Q_0\omega, \sigma, \lambda\tau) \geq\ & M(\sigma, Q_0\omega, \tau) * 1 * M(\sigma, Q_0\omega, \tau) * 1 * 1 \\ & \geq M(\sigma, Q_0\omega, \tau), N(Q_0\omega, \sigma, \lambda\tau); \\ & \leq N(\sigma, Q_0\omega, \tau) \lozenge 0 \lozenge N(\sigma, Q_0\omega, \tau) \lozenge 0 \lozenge 0 \\ & \leq N(\sigma, Q_0\omega, \tau), O(Q_0\omega, \sigma, \lambda\tau); \\ & \leq O(\sigma, Q_0\omega, \tau) \lozenge 0 \lozenge O(\sigma, Q_0\omega, \tau) \lozenge 0 \lozenge 0 \leq O(\sigma, Q_0\omega, \tau). \end{aligned}$$

Therefore, based on Lemma 2, we have $Q_0\ \omega = \sigma$. *Hence,* $Q_0\omega = \sigma = \zeta_2\zeta_4 \ldots \zeta_{2n}\omega$. *As* $(Q_0\zeta_2\zeta_4 \ldots . \zeta_{2n})$ *is weakly compatible, we have*

$$\zeta_2\zeta_4 \ldots \zeta_{2n}Q_0\omega = Q_0\zeta_2\zeta_4 \ldots \zeta_{2n}\omega$$

Thus, $\zeta_2\zeta_4 l \ldots \zeta_{2n}\sigma = Q_0\sigma = \sigma$. *Similarly, in step (c), it is shown that* $\zeta_2\sigma = \zeta_4\sigma \ldots = \zeta_{2n}\sigma = Q_0\sigma = \sigma$. *Thus, we have proved that*

$$Q_0\sigma = Q_1\sigma = \zeta_1\sigma = \zeta_2\sigma = \zeta_3\sigma = \ldots = \zeta_{2n}\sigma = \sigma.$$

Proof of uniqueness: Let σ'*be another common fixed point (CFP) of the above-mentioned mappings; then,* $Q_0\sigma' = Q_1\sigma' = \zeta_1\sigma' = \zeta_2\sigma' = \zeta_3\sigma' = \ldots = \zeta_{2n}\sigma' = \sigma'$.

If $\varpi = \sigma,\ \omega = \sigma'$, $\zeta_1' = \zeta_2\zeta_4 \ldots \zeta_{2n}$ *and* $\zeta_2' = \zeta_1\zeta_3 \ldots \zeta_{2n-1}$ *are with* $\beta = 1$ *in condition (5), we have*

$$\begin{aligned} M(Q_0\sigma,\ Q_1\sigma',\ \lambda\tau) \quad & \geq M(\zeta_1'\sigma, Q_0\sigma, \tau) * M(\zeta_2'\sigma', Q_1\sigma', \tau) * M(\zeta_2'\sigma', Q_0\sigma', \tau) \\ & * M(\zeta_1'\sigma, Q_1\sigma, \tau) * M(\zeta_1'\sigma, \zeta_1'\sigma', \tau); \\ N(Q_0\sigma,\ Q_1\sigma',\ \lambda\tau) \quad & \leq N(\zeta_1'\sigma, Q_0\sigma, \tau) \lozenge N(\zeta_2'\sigma', Q_1\sigma', \tau) \lozenge N(\zeta_2'\sigma', Q_0\sigma', \tau) \\ & \lozenge N(\zeta_1'\sigma, Q_1\sigma, \tau) \lozenge N(\zeta_1'\sigma, \zeta_1'\sigma', \tau); \\ O(Q_0\sigma,\ Q_1\sigma',\ \lambda\tau) \quad & \leq O(\zeta_1'\sigma, Q_0\sigma, \tau) \lozenge O(\zeta_2'\sigma', Q_1\sigma', \tau) \lozenge O(\zeta_2'\sigma', Q_0\sigma', \tau) \\ & \lozenge O(\zeta_1'\sigma, Q_1\sigma, \tau) \lozenge O(\zeta_1'\sigma, \zeta_1'\sigma', \tau). \end{aligned}$$

Which implies that as $\lambda \to \infty$

$$\begin{aligned} M(\sigma, \sigma', \lambda\tau) \geq\ & 1 * 1 * M(\sigma', \sigma, , \tau) * M(\sigma, \sigma', \tau) * M(\sigma, \sigma', \tau) \\ & \geq M(\sigma, \sigma', \tau), N(\sigma, \sigma', \lambda\tau); \\ & \leq 0 \lozenge 0 \lozenge N(\sigma', \sigma, , \tau) \lozenge N(\sigma, \sigma', \tau) \lozenge N(\sigma, \sigma', \tau) \\ & \leq N(\sigma, \sigma', \tau), O(\sigma, \sigma', \lambda\tau); \\ & \leq 0 \lozenge 0 \lozenge O(\sigma', \sigma, , \tau) \lozenge O(\sigma, \sigma', \tau) \lozenge O(\sigma, \sigma', \tau) \leq O(\sigma, \sigma', \tau). \end{aligned}$$

Therefore, based on Lemma 2, we have $\sigma = \sigma'$, *and this shows that* σ *is a unique common fixed point of mappings. Now that Theorem 4 has been slightly generalized, we will prove a common fixed-point theorem.*

Theorem 5. *Let $(\Xi, M, N, O, *, \Diamond)$ be a complete NMS with $\tau * \tau \geq \tau$ and $(1-\tau) \Diamond (1-\tau) \leq 1-\tau$ for all $\tau \in [0,1]$. Let $\{T_\mu\}_{\mu \in J}$ and $\{\zeta_i\}_{i=1}^{2n}$ be two families of self-mappings of Ξ. We suppose that there exists a fixed $v \in J$ such that the following conditions exist:*

(1) $T_\mu(\Xi) \subset \zeta_2\zeta_4 \ldots \zeta_{2n}(\Xi)$, *for each $\mu \in J$ and $T_v(\Xi) \subset \zeta_1\zeta_3 \ldots \zeta_{2n-1}(\Xi)$ for some $v \in J$;*

(2) $\zeta_2(\zeta_4 \ldots \zeta_{2n}) = (\zeta_4 \ldots \zeta_{2n})\zeta_2$,

$$\begin{gathered}
\zeta_2\zeta_4(\zeta_6 \ldots \zeta_{2n}) = (\zeta_6 \ldots \zeta_{2n})\zeta_2 \\
\vdots \\
\zeta_2 \ldots \zeta_{2n-2}(\zeta_{2n}) = (\zeta_{2n})\zeta_2 \ldots \zeta_{2n-2}, \\
T_v(\zeta_4 \ldots \zeta_{2n}) = (\zeta_4 \ldots \zeta_{2n})T_v, \\
T_v(\zeta_6 \ldots \zeta_{2n}) = (\zeta_6 \ldots \zeta_{2n})T_v, \\
\vdots \\
T_v\zeta_{2n} = \zeta_{2n}T_v, \zeta_1(\zeta_3 \ldots \zeta_{2n-1}) = (\zeta_3 \ldots \zeta_{2n-1})\zeta_1, \\
\zeta_1\zeta_3(\zeta_5 \ldots \zeta_{2n-1}) = (\zeta_5 \ldots \zeta_{2n-1})\zeta_1\zeta_3, \\
\vdots \\
\zeta_1 \ldots \zeta_{2n-3}(\zeta_{2n-1}) = (\zeta_{2n-1})\zeta_1 \ldots \zeta_{2n-3}, \\
T_\mu(\zeta_3 \ldots \zeta_{2n-1}) = (\zeta_3 \ldots \zeta_{2n-1})T_\mu, \\
T_\mu(\zeta_5 \ldots \zeta_{2n-1}) = (\zeta_5 \ldots \zeta_{2n-1})T_\mu, \\
\vdots \\
T_\mu\zeta_{2n-1} = \zeta_{2n-1}T_\mu;
\end{gathered}$$

(3) *either $\zeta_2 \ldots \zeta_{2n}$ or T_v is continuous;*

(4) *$(T_v, \zeta_2 \ldots \zeta_{2n})$ is compatible and $(T_\mu, \zeta_1 \ldots \zeta_{2n-1})$ is weakly compatible;*

(5) *there exist $\varrho\lambda \in (0,1)$ such that*

$$\begin{aligned}
M(T_v\varpi, T_\mu\omega, \lambda\tau) &\geq M(\zeta_2\zeta_4 \ldots \zeta_{2n}\varpi, T_v\varpi, \tau) * M(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\omega, T_\mu\omega, \tau) * M(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\omega, T_v\varpi, \beta\tau) \\
&\quad * M(\zeta_2\zeta_4 \ldots \zeta_{2n}\varpi, T_\mu\omega, (2-\beta)\tau) * M(\zeta_2\zeta_4 \ldots \zeta_{2n}\varpi, \zeta_1\zeta_3 \ldots \zeta_{2n-1}\omega, \tau); \\
N(T_v\varpi, T_\mu\omega, \lambda\tau) &\leq N(\zeta_2\zeta_4 \ldots \zeta_{2n}\varpi, T_v\varpi, \tau) \Diamond N(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\omega, T_\mu\omega, \tau) \Diamond N(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\omega, T_v\varpi, \beta\tau) \\
&\quad \Diamond N(\zeta_2\zeta_4 \ldots \zeta_{2n}\varpi, T_\mu\omega, (2-\beta)\tau) \Diamond N(\zeta_2\zeta_4 \ldots \zeta_{2n}\varpi, \zeta_1\zeta_3 \ldots \zeta_{2n-1}\omega, \tau); \\
O(T_v\varpi, T_\mu\omega, \lambda\tau) &\leq O(\zeta_2\zeta_4 \ldots \zeta_{2n}\varpi, T_v\varpi, \tau) \Diamond O(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\omega, T_\mu\omega, \tau) \Diamond O(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\omega, T_v\varpi, \beta\tau) \\
&\quad \Diamond O(\zeta_2\zeta_4 \ldots \zeta_{2n}\varpi, T_\mu\omega, (2-\beta)\tau) \Diamond O(\zeta_2\zeta_4 \ldots \zeta_{2n}\varpi, \zeta_1\zeta_3 \ldots \zeta_{2n-1}\omega, \tau).
\end{aligned}$$

for all $\varpi, \omega \in \Xi$, $\beta \in (0,2)$, and $\tau > 0$. Then, $\{\zeta_i\}_{i=1}^{2n}$ and T_μ have a UCFP in Ξ.

Proof. Let T_{μ_0} be a fixed element in $\{T_\mu\}_{\mu \in J}$. Using Theorem 4 with $Q_0 = T_v$ and $Q_1 = T_{\mu_0}$, it follows that there is some $\sigma \in \Xi$ such that

$$T_v\sigma = T_{\mu_0}\sigma = \zeta_2\zeta_4 \ldots \zeta_{2n}\sigma = \zeta_1\zeta_3 \ldots \zeta_{2n-1}\sigma = \sigma,$$

Let $\mu \in J$ be arbitrary and $\beta = 1$. Then, from condition (5),

$$\begin{aligned}
M(T_v\sigma, T_\mu\sigma, \lambda\tau) \quad &\geq M(\zeta_2\zeta_4 \ldots \zeta_{2n}\sigma, T_v\sigma, \tau) * M(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\sigma, T_\mu\sigma, \tau) \\
&* M(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\sigma, T_v\sigma, \tau) * M(\zeta_2\zeta_4 \ldots \zeta_{2n}\sigma, T_\mu\sigma, \tau) \\
&* M(\zeta_2\zeta_4 \ldots \zeta_{2n}\sigma, \zeta_1\zeta_3 \ldots \zeta_{2n-1}\sigma, \tau), N(T_v\sigma, T_\mu\sigma, \lambda\tau); \\
&\leq N(\zeta_2\zeta_4 \ldots \zeta_{2n}\sigma, T_v\sigma, \tau) \Diamond N(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\sigma, T_\mu\sigma, \tau) \\
&\Diamond N(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\sigma, T_v\sigma, \tau) \Diamond N(\zeta_2\zeta_4 \ldots \zeta_{2n}\sigma, T_\mu\sigma, \tau) \\
&\Diamond N(\zeta_2\zeta_4 \ldots \zeta_{2n}\sigma, \zeta_1\zeta_3 \ldots \zeta_{2n-1}\sigma, \tau), O(T_v\sigma, T_\mu\sigma, \lambda\tau); \\
&\leq O(\zeta_2\zeta_4 \ldots \zeta_{2n}\sigma, T_v\sigma, \tau) \Diamond O(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\sigma, T_\mu\sigma, \tau) \\
&\Diamond O(\zeta_1\zeta_3 \ldots \zeta_{2n-1}\sigma, T_v\sigma, \tau) \Diamond O(\zeta_2\zeta_4 \ldots \zeta_{2n}\sigma, T_\mu\sigma, \tau) \\
&\Diamond O(\zeta_2\zeta_4 \ldots \zeta_{2n}\sigma, \zeta_1\zeta_3 \ldots \zeta_{2n-1}\sigma, \tau).
\end{aligned}$$

and hence

$$\begin{aligned}
M(\sigma, T_\mu\sigma, \lambda\tau) \geq \quad & 1 * M(\sigma, T_\mu\sigma, \tau) * 1 * M(\sigma, T_\mu\sigma, \tau) * 1 \\
&\geq M(\sigma, T_\mu\sigma, \tau), N(\sigma, T_\mu\sigma, \lambda\tau); \\
&\leq 0 \Diamond N(\sigma, T_\mu\sigma, \tau) \Diamond 0 \Diamond N(\sigma, T_\mu\sigma, \tau) \Diamond 0 \\
&\leq N(\sigma, T_\mu\sigma, \tau), O(\sigma, T_\mu\sigma, \lambda\tau); \\
&\leq 0 \Diamond O(\sigma, T_\mu\sigma, \tau) \Diamond 0 \Diamond O(\sigma, T_\mu\sigma, \tau) \Diamond 0 \leq O(\sigma, T_\mu\sigma, \tau).
\end{aligned}$$

Therefore, based on Lemma 2, we have $T_\mu \sigma = \sigma$ for each $\mu \in J$, since condition (5) implies the uniqueness of the common fp. □

Corollary 1. *Let $\mathcal{A}, S, B$, and T be self-mappings on a complete NMS $(\Xi, M, N, O, *, \Diamond)$ with $\tau * \tau \geq \tau$ and $(1-\tau) \Diamond (1-\tau) \leq 1-\tau$ for all $\tau \in [0,1]$, satisfying the following conditions:*

(1) $\mathcal{A}(\Xi) \subseteq T(\Xi), B(\Xi) \subseteq S(\Xi)$;
(2) *either $\mathcal{A}$ or S is continuous;*
(3) *(B, T) is weakly compatible, and $(\mathcal{A}, S)$ is compatible;*
(4) $\exists\, \lambda \in (0,1)$ *s.t.*

$$\begin{aligned}
M(\mathcal{A}\varpi, B\omega, \lambda\tau) &\geq M(S\varpi, \mathcal{A}\varpi, \tau) * M(T\omega, B\omega, \tau) * M(T\omega, \mathcal{A}\varpi, \beta\tau) * M(S\varpi, B\omega, (2-\beta)\tau) * M(S\varpi, T\omega, \tau);\\
N(\mathcal{A}\varpi, B\omega, \lambda\tau) &\leq N(S\varpi, \mathcal{A}\varpi, \tau) \Diamond N(T\omega, B\omega, \tau) \Diamond N(T\omega, \mathcal{A}\varpi, \beta\tau) \Diamond N(S\varpi, B\omega, (2-\beta)\tau)\\
&\Diamond N(S\varpi, T\omega, \tau), O(\mathcal{A}\varpi, B\omega, \lambda\tau);\\
&\leq O(S\varpi, \mathcal{A}\varpi, \tau) \Diamond O(T\omega, B\omega, \tau) \Diamond O(T\omega, \mathcal{A}\varpi, \beta\tau) \Diamond O(S\varpi, B\omega, (2-\beta)\tau) \Diamond O(S\varpi, T\omega, \tau).
\end{aligned}$$

$\forall\, \varpi, \omega \in \Xi, \beta \in (0,2)$ and $\tau > 0$. Then, $\mathcal{A}, S, B$ and T have a UCFP in Ξ if we put $Q_0 = L$, $Q_1 = R$, $\zeta_2\zeta_4 \ldots \zeta_{2n} = ST$ and $\zeta_1\zeta_3 \ldots \zeta_{2n-1} = \mathcal{A}B$ into Theorem 4.

Corollary 2. *Let $\mathcal{A}, B, S, T, L$, and R be self-mappings on a complete NMS $(\Xi, M, N, O, *, \Diamond)$, with $\tau * \tau \geq \tau$ and $(1-\tau) \Diamond (1-\tau) \leq 1-\tau$ for all $\tau \in [0,1]$, satisfying the following conditions:*

(1) $L(\Xi) \subseteq \mathcal{A}B(\Xi), R(\Xi) \subseteq ST(\Xi)$;
(2) $\mathcal{A}B = B\mathcal{A}, ST = TS, LT = TL, RB = BR$;
(3) *Either L or ST is continuous;*
(4) *(L, ST) is compatible and $(R, \mathcal{A}B)$ is weakly compatible;*
(5) *There exists $\lambda \in (0,1)$ such that*

$$\begin{aligned}
M(L\varpi, R\omega, \lambda\tau) \geq\; & M(ST\varpi, L\varpi, \tau) * M(\mathcal{A}B\omega, R\omega, \tau) * M(\mathcal{A}B\omega, L\varpi, \beta\tau)\\
& * M(ST\varpi, R\omega, (2-\beta)\tau) * M(ST\varpi, \mathcal{A}B\omega, \tau), N(L\varpi, R\omega, \lambda\tau);\\
& \leq N(ST\varpi, L\varpi, \tau) \Diamond N(\mathcal{A}B\omega, R\omega, \tau) \Diamond N(\mathcal{A}B\omega, L\varpi, \beta\tau)\\
& \Diamond N(ST\varpi, R\omega, (2-\beta)\tau) \Diamond N(ST\varpi, \mathcal{A}B\omega, \tau), O(L\varpi, R\omega, \lambda\tau);\\
& \leq O(ST\varpi, L\varpi, \tau) \Diamond O(\mathcal{A}B\omega, R\omega, \tau) \Diamond O(\mathcal{A}B\omega, L\varpi, \beta\tau)\\
& \Diamond O(ST\varpi, R\omega, (2-\beta)\tau) \Diamond O(ST\varpi, \mathcal{A}B\omega, \tau).
\end{aligned}$$

for all $\varpi, \omega \in \Xi$, $\beta \in (0,2)$ and $\tau > 0$. Then, $\mathcal{A}, B, S, T, L$ and R have a UCFP in Ξ.

Definition 10. *Let A and S be self-maps on a NMS $(\Xi, M, N, *, \Diamond)$. If S is continuous, then (A, S) is semi-compatible if (A, S) is compatible.*

Theorem 6. *Let $(\Xi, M, N, O, *, \Diamond)$ be a complete NMS and A, B, S, T, P, and Q be mappings from Ξ into itself such that the following conditions are satisfied:*

1. $P(\Xi) \subset ST(\Xi)$, $Q(\Xi) \subset AB(\Xi)$;
2. $AB = BA, ST = TS, PB = BP, QT = TQ$;
3. *Either P or AB is continuous;*
4. *(P, AB) is compatible of type (β) and (Q, ST) is semi-compatible;*
5. *There exists $k \in (0,1)$ such that for every $\varpi, \omega \in \Xi$, $\alpha \in (0, 2)$ and $\tau > 0$*

$$M(P\varpi, Q\varpi, k\tau) \geq \min\left\{\begin{array}{c} M(AB\varpi, Q\omega, (2-\alpha)\tau), M(AB\varpi, ST\omega, \tau), \\ M(AB\varpi P\varpi, \tau), M(ST\omega, Q\omega, \tau) \end{array}\right\};$$
$$N(P\varpi, Q\varpi, k\tau) \leq \max\left\{\begin{array}{c} N(AB\varpi, Q\omega, (2-\alpha)\tau), N(AB\varpi, ST\omega, \tau), \\ N(AB\varpi P\varpi, \tau), N(ST\omega, Q\omega, \tau) \end{array}\right\};$$
$$O(P\varpi, Q\varpi, k\tau) \leq \max\left\{\begin{array}{c} O(AB\varpi, Q\omega, (2-\alpha)\tau), O(AB\varpi, ST\omega, \tau) \\ , O(AB\varpi P\varpi, \tau), O(ST\omega, Q\omega, \tau) \end{array}\right\}.$$

Then, the mappings AB, ST, P, and Q have unique common fixed points in Ξ, and A, B, P, Q, S, and T have a unique common fixed point in Ξ.

Proof. Let ϖ_0 *be* $\in \Xi$; then, in (1), there is $\varpi_1, \varpi_2 \in \Xi$ such that $P\varpi_0 = ST\varpi_1 = \omega_0$ and $Q\varpi_1 = AB\varpi_2 = \omega_1$. Inductively, we can construct sequences $\{\varpi_n\}$ and $\{\omega_n\}$ in Ξ such that

$$\omega_{2n} = P\varpi_{2n} = ST\varpi_{2n+1} \text{ and } \omega_{2n+1} = Q\varpi_{2n+1} = AB\varpi_{2n+2}$$

with $n = 0, 1, 2 \ldots$

If $\varpi = \varpi_{2n+2}$ and $\omega = \varpi_{2n+1}$ for all $\tau > 0$ and $\alpha = 1 - q$ are with $q \in (0,1)$ in (5), we have

$$\begin{aligned}
M(\omega_{2n+1},\ \omega_{2n+2}, k\tau) &= M(P\varpi_{2n+2}, Q\varpi_{2n+1}, k\tau) \\
&\geq \min\left\{\begin{array}{c} M(AB\varpi_{2n+2}, Q\varpi_{2n+1}, (2-(1-\alpha))\tau), \\ M(AB\varpi_{2n+2}, P\varpi_{2n+2}, \tau), M(ST\varpi_{2n+1}, Q\varpi_{2n+1}, \tau) \end{array}\right\} \\
&\geq \min\left\{\begin{array}{c} M(\omega_{2n+1}, \omega_{2n+1}, (1+\alpha))\tau, M(\omega_{2n+1}, \omega_{2n}, \tau), \\ M(\omega_{2n+1}, \omega_{2n+2}, \tau), M(\omega_{2n}, \omega_{2n+1}, \tau) \end{array}\right\} \\
&= \min\{1, M(\omega_{2n}, \omega_{2n+1}, \tau), M(\omega_{2n+1}, \omega_{2n+2}, \tau)\} \\
N(\omega_{2n+1},\ \omega_{2n+2}, k\tau) &= N(P\varpi_{2n+2}, Q\varpi_{2n+1}, k\tau) \\
&\leq \max\left\{\begin{array}{c} N(AB\varpi_{2n+2}, Q\varpi_{2n+1}, (2-(1-\alpha))\tau), \\ N(AB\varpi_{2n+2}, P\varpi_{2n+2}, \tau), N(ST\varpi_{2n+1}, Q\varpi_{2n+1}, \tau) \end{array}\right\} \\
&\leq \max\left\{\begin{array}{c} N(\omega_{2n+1}, \omega_{2n+1}, (1+\alpha))\tau, N(\omega_{2n+1}, \omega_{2n}, \tau), \\ N(\omega_{2n+1}, \omega_{2n+2}, \tau), N(\omega_{2n}, \omega_{2n+1}, \tau) \end{array}\right\} \\
&= \max\{0, N(\omega_{2n}, \omega_{2n+1}, \tau), N(\omega_{2n+1}, \omega_{2n+2}, \tau)\},
\end{aligned}$$

and

$$\begin{aligned}
O(\omega_{2n+1},\ \omega_{2n+2}, k\tau) &= O(P\varpi_{2n+2}, Q\varpi_{2n+1}, k\tau) \\
&\leq \max\left\{\begin{array}{c} O(AB\varpi_{2n+2}, Q\varpi_{2n+1}, (2-(1-\alpha))\tau), \\ O(AB\varpi_{2n+2}, P\varpi_{2n+2}, \tau), O(ST\varpi_{2n+1}, Q\varpi_{2n+1}, \tau) \end{array}\right\} \\
&\leq \max\left\{\begin{array}{c} O(\omega_{2n+1}, \omega_{2n+1}, (1+\alpha))\tau, O(\omega_{2n+1}, \omega_{2n}, \tau), \\ O(\omega_{2n+1}, \omega_{2n+2}, \tau), O(\omega_{2n}, \omega_{2n+1}, \tau) \end{array}\right\} \\
&= \max\{0, O(\omega_{2n}, \omega_{2n+1}, \tau), O(\omega_{2n+1}, \omega_{2n+2}, \tau)\}.
\end{aligned}$$

Therefore, based on Lemmas 1 and 3, we find that

$$\begin{aligned}
M(\omega_{2n+1}, \omega_{2n+2}, k\tau) &\geq M(\omega_{2n+1}, \omega_{2n}, \tau), \\
N(\omega_{2n+1}, \omega_{2n+2}, k\tau) &\leq N(\omega_{2n+1}, \omega_{2n}, \tau),
\end{aligned}$$

and

$$O(\omega_{2n+1}, \omega_{2n+2}, k\tau) \leq O(\omega_{2n+1}, \omega_{2n}, \tau).$$

Similarity, we have

$$\begin{aligned}
M(\omega_{2n+2}, \omega_{2n+3}, k\tau) &\geq M(\omega_{2n+1}, \omega_{2n+2}, \tau), \\
N(\omega_{2n+2}, \omega_{2n+3}, k\tau) &\leq N(\omega_{2n+1}, \omega_{2n+2}, \tau), \\
O(\omega_{2n+2}, \omega_{2n+3}, k\tau) &\leq O(\omega_{2n+1}, \omega_{2n+2}, \tau).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
M(\omega_{n+1}, \omega_{n+2}, k\tau) &\geq M(\omega_n, \omega_{n+1}, \tau), \\
N(\omega_{n+1}, \omega_{n+2}, k\tau) &\leq N(\omega_n, \omega_{n+1}, \tau), \\
O(\omega_{n+1}, \omega_{n+2}, k\tau) &\leq O(\omega_n, \omega_{n+1}, \tau).
\end{aligned}$$

for $n = 1, 2 \ldots$, etc., and so

$$\begin{aligned}
M(\omega_n, \omega_{n+1}, \tau) \geq M\left(\omega_n, \omega_{n-1}, \tfrac{\tau}{q}\right) \geq M\left(\omega_{n-2}, \omega_{n-1}, \tfrac{\tau}{q^2}\right) \ldots \geq M\left(\omega_1, \omega_2, \tfrac{\tau}{q^n}\right) \to 1, \text{ as } n \to \infty \\
N(\omega_n, \omega_{n+1}, \tau) \leq N\left(\omega_n, \omega_{n-1}, \tfrac{\tau}{q}\right) \leq N\left(\omega_{n-2}, \omega_{n-1}, \tfrac{\tau}{q^2}\right) \ldots \leq N\left(\omega_1, \omega_2, \tfrac{\tau}{q^n}\right) \to 0, \text{ as } n \to \infty
\end{aligned}$$

and

$$O(\omega_n, \omega_{n+1}, \tau) \leq O\left(\omega_n, \omega_{n-1}, \frac{\tau}{q}\right) \leq O\left(\omega_{n-2}, \omega_{n-1}, \frac{\tau}{q^2}\right) \ldots \leq O\left(\omega_1, \omega_2, \frac{\tau}{q^n}\right) \to 0, \text{ as } n \to \infty$$

Hence, $M(\omega_n, \omega_{n+1}, \tau) \to 1, N(\omega_n, \omega_{n+1}, \tau) \to 0$ and $N(\omega_n, \omega_{n+1}, \tau) \to 0$, as $n \to \infty$, for any $\tau > 0$.

For each $\varepsilon > 0$ and each $\tau > 0$, we can choose $n_0 \in N$, such that

$$\begin{aligned}
M(\omega_{2n}, \omega_{2n+1}, \tau) &> 1 - \varepsilon; \\
N(\mathrm{y}_{2n}, \mathrm{y}_{2n+1}, \mathrm{t}) &< \varepsilon; \\
O(\mathrm{y}_{2n}, \mathrm{y}_{2n+1}, \mathrm{t}) &< \varepsilon.
\end{aligned}$$

for all $n > n_0$. For $m, n \in \mathbb{N}$, we suppose that $m \geq n$; then, we find that

$$\begin{aligned}
M(\omega_n, \omega_m, \tau) &\geq M\left(\omega_n, \omega_{n+1}, \tfrac{\tau}{m-n}\right) * M\left(\omega_{n+1}, \omega_{n+2}, \tfrac{\tau}{m-n}\right) * \ldots * M\left(\omega_{m-1}, \omega_m \tfrac{\tau}{m-n}\right) \\
&\geq \{(1-\varepsilon) * (1-\varepsilon) * \ldots (1-\varepsilon)(m-n \text{ times })\} \\
&\geq (1-\varepsilon), \\
N(\omega_n, \omega_m, \tau) &\leq N\left(\omega_n, \omega_{n+1}, \tfrac{\tau}{m-n}\right) \lozenge N\left(\omega_{n+1}, \omega_{n+2}, \tfrac{\tau}{m-n}\right) \lozenge \ldots \lozenge N\left(\omega_{m-1}, \omega_m \tfrac{\tau}{m-n}\right) \\
&\leq \{(1-\varepsilon) \lozenge (1-\varepsilon) \lozenge \ldots (1-\varepsilon)\}(m-n \text{ times }) < \varepsilon, \\
O(\omega_n, \omega_m, \tau) &\leq O\left(\omega_n, \omega_{n+1}, \tfrac{\tau}{m-n}\right) \lozenge O\left(\omega_{n+1}, \omega_{n+2}, \tfrac{\tau}{m-n}\right) \lozenge \ldots \lozenge O\left(\omega_{m-1}, \omega_m \tfrac{\tau}{m-n}\right) \\
&\leq \{(1-\varepsilon) \lozenge (1-\varepsilon) \lozenge \ldots (1-\varepsilon)\}(m-n \text{ times }) < \varepsilon,
\end{aligned}$$

hence, $\{\omega_n\}$ is a Cauchy sequence in Ξ. Since $(\Xi, M, N, O, *, \lozenge)$ is complete, $\{\omega_n\}$ converges to some point with $z \in \Xi$ and has subsequences convergences at the same point, i.e., $z \in \Xi$

$$\begin{gathered}
\{Q\varpi_{2n+1}\} \to z \text{ and } \{ST\varpi_{2n+1}\} \to z \\
\{P\varpi_{2n}\} \to z \text{ and } \{AB\varpi_{2n}\} \to z
\end{gathered}$$

□

Case I. *We suppose that AB is continuous. Since AB is continuous, we have*

$$(AB)^2\varpi_{2n} \to ABz \text{ and } ABP\varpi_{2n} \to ABz.$$

as (P, AB) is compatible with type (β), we have

$$\lim_{n\to\infty} M(PP\varpi_{2n},\ AB(AB)\varpi_{2n}, \tau) = 1,$$

for all $\tau > 0$, This gives $M(Pz, ABz, \tau) = 1$ or $Pz = ABz$. Also, based on the semi-compatibility of (P, AB), we find that $ABP\varpi_{2n} \to ABz$. Now, we will show that $Pz = ABz = z$.

Step 1. *If $\varpi = AB\varpi_{2n}$ and $\omega = \varpi_{2n+1}$ with $\alpha = 1$ in (5), we find that*

$$M(PAB\varpi_{2\varpi}, Q\varpi_{2n+1}, k\tau) \geq \min\left\{\begin{matrix} M(AB(AB)\varpi_{2n}, Q\varpi_{2n+1}, \tau), M(AB(AB)\varpi_{2n}, ST\varpi_{2n+1}, \tau), \\ M(AB(AB)\varpi_{2n}, P(AB)\varpi_{2n}, \tau), M(ST\varpi_{2n+1}, Q\varpi_{2n+1}, \tau) \end{matrix}\right\}$$

letting $n \to \infty$

$$\begin{gathered}
M(Pz, z, k\tau) \geq \min\left\{\begin{matrix} M(ABz, z, \tau), M(ABz, z, \tau), \\ M(ABz, Pz, \tau), M(z, z, \tau) \end{matrix}\right\} M(ABz, z, k\tau) \\
M(ABz, z, k\tau) \geq \min\left\{\begin{matrix} M(ABz, z, \tau), M(ABz, z, \tau), \\ M(ABz, ABz, \tau), M(z, z, \tau) \end{matrix}\right\} M(ABz, z, k\tau) \geq M(ABz, z, \tau)
\end{gathered}$$

and similarly

$$N(PAB\varpi_{2\varpi}, Q\varpi_{2n+1}, k\tau) \leq \max\left\{\begin{matrix} N(AB(AB)\varpi_{2n}, Q\varpi_{2n+1}, \tau), N(AB(AB)\varpi_{2n}, ST\varpi_{2n+1}, \tau), \\ N(AB(AB)\varpi_{2n}, P(AB)\varpi_{2n}, \tau), N(ST\varpi_{2n+1}, Q\varpi_{2n+1}, \tau) \end{matrix}\right\}$$

letting $n \to \infty$

$$\begin{gathered}
N(Pz, z, k\tau) \leq \max\left\{\begin{matrix} N(ABz, z, \tau), N(ABz, z, \tau), \\ N(ABz, Pz, \tau), N(z, z, \tau) \end{matrix}\right\} N(ABz, z, k\tau) \\
N(ABz, z, k\tau) \leq \max\left\{\begin{matrix} N(ABz, z, \tau), N(ABz, z, \tau), \\ N(ABz, ABz, \tau), N(z, z, \tau) \end{matrix}\right\} N(ABz, z, k\tau) \leq N(ABz, z, \tau) \\
N(ABz, z, k\tau) \leq N(ABz, z, \tau)
\end{gathered}$$

and

$$O(PAB\varpi_{2\varpi}, Q\varpi_{2n+1}, k\tau) \leq \max\left\{\begin{matrix} O(AB(AB)\varpi_{2n}, Q\varpi_{2n+1}, \tau), O(AB(AB)\varpi_{2n}, ST\varpi_{2n+1}, \tau), \\ O(AB(AB)\varpi_{2n}, P(AB)\varpi_{2n}, \tau), O(ST\varpi_{2n+1}, Q\varpi_{2n+1}, \tau) \end{matrix}\right\}$$

letting $n \to \infty$

$$O(Pz,z,k\tau) \le \max\left\{\begin{array}{c} O(ABz,z,\tau), O(ABz,z,\tau), \\ O(ABz,Pz,\tau), O(z,z,\tau) \end{array}\right\} O(ABz,z,k\tau)$$
$$O(ABz,z,k\tau) \le \max\left\{\begin{array}{c} O(ABz,z,\tau), O(ABz,z,\tau), \\ O(ABz,ABz,\tau), O(z,z,\tau) \end{array}\right\} O(ABz,z,k\tau) \le O(ABz,z,\tau)$$
$$O(ABz,z,k\tau) \le O(ABz,z,\tau).$$

Therefore, using Lemma 3, we find that

$$Pz = ABz = z.$$

Step 2. *If* $\varpi = z$ *and* $\omega = \varpi_{2n+1}$ *with* $\alpha = 1$ *in (5), we have*

$$M(Pz, Q\varpi_{2n+1}, k\tau) \ge \min\left\{\begin{array}{c} M(ABz, Q\varpi_{2n+1}, \tau), M(ABz, ST\varpi_{2n+1}, \tau), \\ M(ABz, Pz, \tau), M(ST\varpi_{2n+1}, Q\varpi_{2n+1}, \tau) \end{array}\right\}$$

letting $n \to \infty$*, we have*

$$M(Pz,z,k\tau) \ge \min\{M(Pz,z,\tau), M(Pz,z,\tau), M(Pz,Pz,\tau), M(z,z,\tau)\}$$
$$M(Pz,z,k\tau) \ge M(Pz,z,\tau)$$
$$N(Pz, Q\varpi_{2n+1}, k\tau) \le \max\left\{\begin{array}{c} N(ABz, Q\varpi_{2n+1}, \tau), N(ABz, ST\varpi_{2n+1}, \tau), \\ N(ABz, Pz, \tau), N(ST\varpi_{2n+1}, Q\varpi_{2n+1}, \tau) \end{array}\right\}$$

letting $n \to \infty$*, we have*

$$N(Pz,z,k\tau) \le \max\{N(Pz,z,\tau), N(Pz,z,\tau), N(Pz,Pz,\tau), N(z,z,\tau)\}$$
$$N(Pz,z,k\tau) \le N(Pz,z,\tau)$$
$$O(Pz, Q\varpi_{2n+1}, k\tau) \le \max\left\{\begin{array}{c} O(ABz, Q\varpi_{2n+1}, \tau), O(ABz, ST\varpi_{2n+1}, \tau), \\ O(ABz, Pz, \tau), O(ST\varpi_{2n+1}, Q\varpi_{2n+1}, \tau) \end{array}\right\}$$

letting $n \to \infty$*, we have*

$$O(Pz,z,k\tau) \le \max\{O(Pz,z,\tau), O(Pz,z,\tau), O(Pz,Pz,\tau), O(z,z,\tau)\}$$
$$O(Pz,z,k\tau) \le O(Pz,z,\tau)$$

Therefore, based on Lemma 3, we find that

$$Pz = z$$

i.e., $z = Pz = ABz$.

Step 3. *If* $\varpi = Bz$ *and* $\omega = \varpi_{2n+1}$ *with* $\alpha = 1$*, we have*

$$M(PBz, Q\varpi_{2n+1}, k\tau) \ge \min\left\{\begin{array}{c} M(AB(Bz), Q\varpi_{2n+1}, \tau), M(AB(Bz), ST\varpi_{2n+1}, \tau), \\ M(AB(AB), P(Bz), \tau), M(ST\varpi_{2n+1}, Q\varpi_{2n+1}, \tau) \end{array}\right\}$$

Since $PB = B\ P$, $AB = BA$,

$$P(Bz) = B(Pz) = Bz$$

and $AB(Bz) = B(AB)z = Bz$.
If $n \to \infty$, *we have*

$$M(Bz,z,k\tau) \ge \min\{M(Bz,z,\tau), M(Bz,z,\tau), M(Bz,Bz,\tau), M(z,z,\tau)\}$$
$$M(Bz,z,k\tau) \ge M(Bz,z,\tau),$$
$$N(PBz, Q\varpi_{2n+1}, k\tau) \le \max\left\{\begin{array}{c} N(AB(Bz), Q\varpi_{2n+1}, \tau), N(AB(Bz), ST\varpi_{2n+1}, \tau), \\ N(AB(AB), P(Bz), \tau), N(ST\varpi_{2n+1}, Q\varpi_{2n+1}, \tau) \end{array}\right\}$$

Since $PB = BP$, $AB = BA$,
So $P(Bz) = B(Pz) = Bz$
and $AB(Bz) = B(AB)z = Bz$.
If $n \to \infty$, *we have*

$$N(Bz,z,k\tau) \le \max\{N(Bz,z,\tau), N(Bz,z,\tau), N(Bz,Bz,\tau), N(z,z,\tau)\}$$
$$N(Bz,z,k\tau) \le N(Bz,z,\tau),$$

and

$$O(PBz, Q\varpi_{2n+1}, k\tau) \leq \max\left\{\begin{array}{c} O(AB(Bz), Q\varpi_{2n+1}, \tau), O(AB(Bz), ST\varpi_{2n+1}, \tau), \\ O(AB(AB), P(Bz), \tau), O(ST\varpi_{2n+1}, Q\varpi_{2n+1}, \tau) \end{array}\right\}$$

Since $PB = BP, AB = BA$,

So $P(Bz) = B(Pz) = Bz$

and $AB(Bz) = B(AB)z = Bz$.

If $n \to \infty$, *we have*

$$O(Bz, z, k\tau) \leq \max\{O(Bz, z, \tau), O(Bz, z, \tau), O(Bz, Bz, \tau), O(z, z, \tau)\}$$
$$O(Bz, z, k\tau) \leq O(Bz, z, \tau).$$

Therefore, using Lemmas 1 and 3, we find that

$$Bz = z.$$

Thus, $ABz = Pz = z$ *and* $ABz = Az$

$$z = ABz \Rightarrow Az = z.$$

Therefore, $ABz = Az = Bz = Pz = z$.

Step 4. *As* $P(\Xi) \subset ST(\Xi)$, $u \in \Xi$ *is*

$$z = Pz = STu.$$

If we combine $\varpi = \varpi_{2n}$ *and* $\omega = u$ *with* $\alpha = 1$, *we find that*

$$M(P\varpi_{2n}, Qu, k\tau) \geq \min\left\{\begin{array}{c} M(AB\varpi_{2n}, Qu, \tau), M(AB\varpi_{2n}, STu, \tau), \\ M(AB\varpi_{2n}, P\varpi_{2n}, \tau), M(STu, Qu, \tau) \end{array}\right\}$$

Taking $n \to \infty$

$$M(z, Qu, k\tau) \geq \min\{M(z, Qu, \tau), M(z, z, \tau), M(z, z, \tau), M(z, Qu, \tau)\}$$
$$M(z, Qu, k\tau) \geq M(z, Qu, \tau),$$
$$N(P\varpi_{2n}, Qu, k\tau) \leq \max\left\{\begin{array}{c} N(AB\varpi_{2n}, Qu, \tau), N(AB\varpi_{2n}, STu, \tau), \\ N(AB\varpi_{2n}, P\varpi_{2n}, \tau), N(STu, Qu, \tau) \end{array}\right\}$$

Taking $n \to \infty$

$$N(z, Qu, k\tau) \leq \max\{N(z, Qu, \tau), N(z, z, \tau), N(z, z, \tau), N(z, Qu, \tau)\}$$
$$N(z, Qu, k\tau) \leq N(z, Qu, \tau),$$

and

$$O(P\varpi_{2n}, Qu, k\tau) \leq \max\left\{\begin{array}{c} O(AB\varpi_{2n}, Qu, \tau), O(AB\varpi_{2n}, STu, \tau), \\ O(AB\varpi_{2n}, P\varpi_{2n}, \tau), O(STu, Qu, \tau) \end{array}\right\}$$

Taking $n \to \infty$

$$O(z, Qu, k\tau) \leq \max\{O(z, Qu, \tau), O(z, z, \tau), O(z, z, \tau), O(z, Qu, \tau)\}$$
$$O(z, Qu, k\tau) \leq O(z, Qu, \tau).$$

Therefore, using Lemmas 1 and 3,

we find that $Qu = z$.

Hence, $STu = z = Qu$.

Since (Q, ST) *is semi-compatible,*

$$\lim_{n\to\infty} STQ\varpi_{2n} = Qz$$
$$\lim_{n\to\infty} ST\varpi_{2n} = \lim_{n\to\infty} Q\varpi_{2n} = z$$
$$\lim_{n\to\infty} STQ\varpi_{2n} = Qz$$

Thus, $Qz = STz$.

Step 5. *If we combine* $\varpi = \varpi_{2n}$ *and* $\omega = z$ *with* $\alpha = 1$, *we find that*

$$M(P\varpi_{2n}, Qz, k\tau) \geq \min\left\{\begin{array}{c} M(AB\varpi_{2n}, Qz, \tau), M(AB\varpi_{2n}, STz, \tau), \\ M(AB\varpi_{2n}, P\varpi_{2n}, \tau), M(STu, Qz, \tau) \end{array}\right\}$$

Taking $n \to \infty$

$$M(z, Qz, k\tau) \geq \min\{M(z, Qz, \tau), M(z, Qz, \tau), M(z, z, \tau), M(Qz, Qz, \tau)\}$$
$$M(z, Qz, k\tau) \geq M(z, Qz, \tau),$$
$$N(P\varpi_{2n}, Qz, k\tau) \leq \max\left\{\begin{array}{c} N(AB\varpi_{2n}, Qz, \tau), N(AB\varpi_{2n}, STz, \tau), \\ N(AB\varpi_{2n}, P\varpi_{2n}, \tau), N(STu, Qz, \tau) \end{array}\right\}$$

Taking $n \to \infty$

$$N(z, Qz, k\tau) \leq \max\{N(z, Qz, \tau), N(z, Qz, \tau), N(z, z, \tau), N(Qz, Qz, \tau)\}$$
$$N(z, Qz, k\tau) \leq N(z, Qz, \tau),$$

and

$$O(P\varpi_{2n}, Qz, k\tau) \leq \max\left\{\begin{array}{c} O(AB\varpi_{2n}, Qz, \tau), O(AB\varpi_{2n}, STz, \tau), \\ O(AB\varpi_{2n}, P\varpi_{2n}, \tau), O(STu, Qz, \tau) \end{array}\right\}$$

Taking $n \to \infty$

$$O(z, Qz, k\tau) \leq \max\{O(z, Qz, \tau), O(z, Qz, \tau), O(z, z, \tau), O(Qz, Qz, \tau)\}$$
$$O(z, Qz, k\tau) \leq O(z, Qz, \tau)$$

Therefore, using Lemma 3,
we find that $Qz = z$

Step 6. *If we combined* $\varpi = \varpi_{2n}$ *and* $\omega = Tz$ *with* $\alpha = 1$, *we find that*

$$M(P\varpi_{2n}, QTz, k\tau) \geq \min\left\{\begin{array}{c} M(AB\varpi_{2n}, STz, \tau), M(AB\varpi_{2n}, ST(Tz), \tau), \\ M(AB\varpi_{2n}, P\varpi_{2n}, \tau), M(ST(Tz), Q(Tz), \tau) \end{array}\right\}$$

as $QT = TQ$ *and* $ST = TS$,
we have $QTz = TQz = Tz$ *and* $ST(Tz) = T(STz) = Tz$.
Taking $n \to \infty$, *we find that*

$$M(z, Tz, k\tau) \geq \min\{N(z, Tz, \tau), M(z, Tz, \tau), M(z, z, \tau), M(Tz, Tz, \tau)\}$$
$$M(z, Tz, k\tau) \geq M(z, Tz, \tau)$$
$$N(P\varpi_{2n}, QTz, k\tau) \leq \max\left\{\begin{array}{c} N(AB\varpi_{2n}, P\varpi_{2n}, , \tau), N(AB\varpi_{2n}, ST(Tz), \tau), \\ N(AB\varpi_{2n}, P\varpi_{2n}, \tau), N(ST(Tz), Q(Tz), \tau) \end{array}\right\}$$

Taking $n \to \infty$, *we find that*

$$N(z, Tz, k\tau) \leq \max\{N(z, Tz, \tau), M(z, Tz, \tau), N(z, z, \tau), N(Tz, Tz, \tau)\}$$
$$N(z, Tz, k\tau) \leq N(z, Tz, \tau),$$

and

$$O(P\varpi_{2n}, QTz, k\tau) \leq \max\left\{\begin{array}{c} O(AB\varpi_{2n}, P\varpi_{2n}, , \tau), O(AB\varpi_{2n}, ST(Tz), \tau), \\ O(AB\varpi_{2n}, P\varpi_{2n}, \tau), O(ST(Tz), Q(Tz), \tau) \end{array}\right\}$$

Taking $n \to \infty$, *we find that*

$$O(z, Tz, k\tau) \leq \max\{O(z, Tz, \tau), M(z, Tz, \tau), O(z, z, \tau), N(Tz, Tz, \tau)\}$$
$$O(z, Tz, k\tau) \leq N(z, Tz, \tau)$$

Therefore, using Lemma 3, we have

$$Tz = z$$

Now $STz = Tz = z$, *this implies that* $Sz = z$.
Hence, $Sz = Tz = Qz = z = STz$.
Combining (A) and (B), we find that

$$ABz = STz = Az = Bz = Pz = Qz = Sz = Tz = z$$

Hence, z *is the common fixed point of* $A, B, S, T, P,$ *and* Q.

Case II. *We suppose* P *is continuous.*
Since P *is continuous, we have* $(P)^2\varpi_{2n} \to Pz$ *and* $P(AB)\varpi_{2n} \to Pz$.
As (P, AB) *is a compatible of type* (β), *we have*

$\lim_{n\to\infty} M(PP\varpi_{2n}(AB)(AB)\varpi_{2n},\tau) = 1$, *for all* $\tau > 0$
Or $M(Pz, ABz, \tau) = 1$.
i.e., $Pz = ABz$

Step 7. *If we combined* $\varpi = P\varpi_{2n}$ *and* $\omega = \varpi_{2n+1}$ *with* $\alpha = 1$, *we find that*

$$M(P(P\varpi_{2n}), Q\varpi_{2n+1}, k\tau) \geq \min\begin{Bmatrix} M(AB(P)\varpi_{2n}, Q\varpi_{2n+1}, \tau), M(AB(P)\varpi_{2n}, ST(\varpi_{2n+1}), \tau), \\ M(AB(P)\varpi_{2n}, P(P)\varpi_{2n}, \tau), M(ST\varpi_{2n+1}, Q\varpi_{2n+1}, \tau) \end{Bmatrix}$$

Taking $n \to \infty$, *we find that*

$$M(Pz, z, k\tau) \geq \min\{M(Pz, z, \tau), M(Pz, z, \tau), M(Pz, Pz, \tau), M(z, z, \tau)\}$$
$$M(Pz, z, k\tau) \geq M(Pz, z, \tau),$$
$$N(P(P\varpi_{2n}), Q\varpi_{2n+1}, k\tau) \leq \max\begin{Bmatrix} N(AB(P)\varpi_{2n}, Q\varpi_{2n+1}, \tau), N(AB(P)\varpi_{2n}, ST(\varpi_{2n+1}), \tau), \\ N(AB(P)\varpi_{2n}, P(P)\varpi_{2n}, \tau), N(ST\varpi_{2n+1}, Q\varpi_{2n+1}, \tau) \end{Bmatrix}$$

Taking $n \to \infty$, *we find that*

$$N(Pz, z, k\tau) \leq \max\{N(Pz, z, \tau), N(Pz, z, \tau), N(Pz, Pz, \tau), N(z, z, \tau)\}$$
$$N(Pz, z, k\tau) \leq N(Pz, z, \tau),$$

and

$$O(P(P\varpi_{2n}), Q\varpi_{2n+1}, k\tau) \leq \max\begin{Bmatrix} O(AB(P)\varpi_{2n}, Q\varpi_{2n+1}, \tau), O(AB(P)\varpi_{2n}, ST(\varpi_{2n+1}), \tau), \\ O(AB(P)\varpi_{2n}, P(P)\varpi_{2n}, \tau), O(ST\varpi_{2n+1}, Q\varpi_{2n+1}, \tau) \end{Bmatrix}$$

Taking $n \to \infty$, *we find that*

$$O(Pz, z, k\tau) \leq \max\{O(Pz, z, \tau), O(Pz, z, \tau), O(Pz, Pz, \tau), O(z, z, \tau)\}$$
$$O(Pz, z, k\tau) \leq O(Pz, z, \tau).$$

Therefore, using Lemma 3, we find that

$$Pz = z.$$

Hence, $ABz = Pz = z$.

Similarly, we can apply step 4 to find that $Bz = z$; *therefore,*

$$Az = Bz = Pz = z.$$

Furthermore, we get

$$Qz = STz = Sz = Tz = z.$$

z *is also the common fixed point of the six self-maps* $A, B, S, T, P,$ *and* Q *in this case.*

4. Fixed-Point Results for Four Self-Mappings

Definition 11 ([20]). *Two families of self-mappings* $\{\mathcal{A}_i\}_{i=1}^{m}$ *and* $\left\{B_j\right\}_{j=1}^{n}$ *are said to be pairwise commuting if*

(i) $\mathcal{A}_i\mathcal{A}_j = \mathcal{A}_j\mathcal{A}_i,\ i, j \in \{1, 2, \ldots, m\}$;
(ii) $B_iB_j = B_jB_i,\ i, j \in \{1, 2, \ldots, n\}$;
(iii) $\mathcal{A}_iB_j = B_j\mathcal{A}_i,\ i, j \in \{1, 2, \ldots, m\},\ j \in \{1, 2, 3 \ldots, n\}$.

We demonstrate a CFP theorem for four finite families of mappings using complete NMS as an application of Theorem 4. Definition 11, a logical extension of the commutatively requirement to fit two finite families, is used to demonstrate our conclusion.

Theorem 7. *Let* $\{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m\}$, $\{B_1, B_2, \ldots, B_n\}$, $\{S_1, S_2, \ldots, S_\zeta\}$ *and* $\{T_1, T_2, \ldots, T_q\}$ *be four finite families of self-mappings of a complete NMS* $(\Xi, M, N, O, *, \Diamond)$ *with* $\tau * \tau \geq \tau$ *and* $(1-\tau) \Diamond (1-\tau) \leq 1-\tau$ *for all* $\tau \in [0,1]$, *such that* $\mathcal{A} = \mathcal{A}_1\mathcal{A}_2 \ldots \mathcal{A}_m$, $B = B_1B_2 \ldots B_m$, $S = S_1, S_2 \ldots S_\zeta$, *and* $T = T_1T_2 \ldots T_q$ *satisfy the following conditions:*

(1) $\mathcal{A}(\Xi) \subseteq T(\Xi),\ B(\Xi) \subseteq S(\Xi)$;

(2) *Either $\mathcal{A}$ or S is continuous;*
(3) *The pairs of families $(\{\mathcal{A}_i\}, \{S_\lambda\})$ and $(\{B_r\}, \{T_\tau\})$ commute pairwise;*
(4) $\exists\, \lambda \in (0,1)$ *such that*

$$
\begin{gathered}
M(\mathcal{A}\varpi, B\omega, \lambda\tau) \geq M(S\varpi, \mathcal{A}\varpi, \tau) * M(T\omega, B\omega, \tau);\\
*M(T\omega, \mathcal{A}\varpi, \beta\tau) * M(S\varpi, B\omega, (2-\beta)\tau) * M(S\varpi, T\omega, \tau);\\
N(\mathcal{A}\varpi, B\omega, \lambda\tau) \leq N(S\varpi, \mathcal{A}\varpi, \tau) \diamondsuit N(T\omega, B\omega, \tau);\\
\diamondsuit\, N(T\omega, \mathcal{A}\varpi, \beta\tau) \diamondsuit N(S\varpi, B\omega, (2-\beta)\tau) \diamondsuit N(S\varpi, T\omega, \tau);\\
O(\mathcal{A}\varpi, B\omega, \lambda\tau) \leq O(S\varpi, \mathcal{A}\varpi, \tau) \diamondsuit O(T\omega, B\omega, \tau);\\
\diamondsuit\, O(T\omega, \mathcal{A}\varpi, \beta\tau) \diamondsuit O(S\varpi, B\omega, (2-\beta)\tau) \diamondsuit O(S\varpi, T\omega, \tau).
\end{gathered}
$$

$\forall\, \varpi, \omega \in \Xi$, $\beta \in (0,2)$ *and* $\tau > 0$ *. Then,* $\{\mathcal{A}_i\}_{i=1}^{m}$, $\{S_\lambda\}_{\lambda=1}^{p}$, $\{B_r\}_{r=1}^{n}$ *and* $\{T_\tau\}_{\tau=1}^{q}$ *have a UCFP in* Ξ.

Proof. Now, we can prove that $\mathcal{A}S = S\mathcal{A}$ as

$$
\begin{gathered}
\mathcal{A}S = (\mathcal{A}_1\mathcal{A}_2\ldots\mathcal{A}_m)(S_1S_2\ldots S_p) = (\mathcal{A}_1\mathcal{A}_2\ldots\mathcal{A}_{m-1})(\mathcal{A}_mS_1S_2\ldots S_p)\\
= (\mathcal{A}_1\mathcal{A}_2\ldots\mathcal{A}_{m-1})(S_1S_2\ldots S_p\mathcal{A}_m)\\
= (\mathcal{A}_1\mathcal{A}_2\ldots\mathcal{A}_{m-2})(\mathcal{A}_{m-1}S_1S_2\ldots S_p\mathcal{A}_m)\\
= (\mathcal{A}_1\mathcal{A}_2\ldots\mathcal{A}_{m-2})(S_1S_2\ldots S_p\mathcal{A}_{m-1}\mathcal{A}_m) = \ldots = \mathcal{A}_1(S_1S_2\ldots S_p\mathcal{A}_2\ldots\mathcal{A}_{m-1}\mathcal{A}_m)\\
= (S_1S_2\ldots S_p)(\mathcal{A}_1\mathcal{A}_2\ldots\mathcal{A}_m) = S\mathcal{A}.
\end{gathered}
$$

Likewise, we may demonstrate that $BT = TB$. Thus, it follows naturally that the pair $(\mathcal{A}, S)$ is compatible and the pair (B, T) is weakly compatible. We must now demonstrate that z continues to be the fixed point of all component mappings. For this, we think about

$$
\begin{gathered}
\mathcal{A}(\mathcal{A}_i\sigma) = ((\mathcal{A}_1\mathcal{A}_2\ldots.\mathcal{A}_m)\mathcal{A}_i)\sigma = (\mathcal{A}_1\mathcal{A}_2\ldots.\mathcal{A}_{m-1})(\mathcal{A}_m\mathcal{A}_i)\sigma\\
= (\mathcal{A}_1\mathcal{A}_2\ldots.\mathcal{A}_{m-1})(\mathcal{A}_i\mathcal{A}_m)\sigma = (\mathcal{A}_1\mathcal{A}_2\ldots.\mathcal{A}_{m-2})(\mathcal{A}_{m-1}\mathcal{A}_i\mathcal{A}_m)\sigma\\
= (\mathcal{A}_1\mathcal{A}_2\ldots.\mathcal{A}_{m-2})(\mathcal{A}_i\mathcal{A}_{m-1}\mathcal{A}_m)\sigma = \ldots. = \mathcal{A}_1(\mathcal{A}_i\mathcal{A}_2\ldots\ldots.\mathcal{A}_m)\sigma\\
= (\mathcal{A}_1\mathcal{A}_i)(\mathcal{A}_2\ldots.\mathcal{A}_m)\sigma\\
= (\mathcal{A}_i\mathcal{A}_1)(\mathcal{A}_2\ldots..\mathcal{A}_m)\sigma = \mathcal{A}_i(\mathcal{A}_1\mathcal{A}_2\ldots..\mathcal{A}_m)\sigma = \mathcal{A}_i\sigma.
\end{gathered}
$$

Similar to that, we may demonstrate that

$$
\begin{gathered}
\mathcal{A}(S_\lambda\sigma) = S_\lambda(\mathcal{A}\sigma) = S_\lambda\sigma, S(S_\lambda\sigma) = S_\lambda(S\sigma) = S_\lambda\sigma, S(\mathcal{A}_i\sigma)\\
= \mathcal{A}_i(S\sigma) = \mathcal{A}_i\sigma, B(B_r\sigma) = B_r(B\sigma) = B_r\sigma,\\
B(T_\tau\sigma) = T_\tau(B\sigma) = T_\tau\sigma, \quad T(T_\tau\sigma) = T_\tau(T_\sigma) = T_\tau\sigma\\
T(B_r\sigma) = B_r(T\sigma) = B_r\sigma.
\end{gathered}
$$

Which show that for all $(i, r, \lambda$ and $\tau)$, $\mathcal{A}_i\sigma$ and $S_\lambda\sigma$ are the other fixed points of the pair $(\mathcal{A}, S)$ and $B_r\sigma$ and $T_\tau Z$ are other fixed points of the pair (B, T). Now, by appealing the UCFP of mappings $\mathcal{A}, S, B$ and T, we find that

$$
\begin{gathered}
\sigma = \mathcal{A}_i\sigma = S_\lambda\sigma = B_r\sigma = T_\tau\sigma, \text{ for all } i = 1, 2, \ldots., m,\\
\lambda = 1, 2, \ldots\ldots p, r = 1, 2, \ldots..n, \tau = 1, 2, , \ldots\ldots\ldots, q,
\end{gathered}
$$

Which shows that σ is a UCFP of $\{\mathcal{A}_i\}_{i=1}^{m}, \{S_\lambda\}_{\lambda=1}^{p}, \{B_r\}_{r=1}^{n}$, and $\{T_\tau\}_{\tau=1}^{q}$. □

Remark 2. *The commutativity criteria of Theorem 7 are a little more stringent than those in Theorem 4; hence, it is a tiny but partial generalization of Theorem 4. If we put*

$$
\mathcal{A}_1 = \mathcal{A}_2 = \ldots = \mathcal{A}_m = \mathcal{A}, \quad B_1 = B_2 = \ldots = B_n = B,\ S_1 = S_2 = \ldots = S_p = S
$$

and $T_1 = T_2 = \ldots = T_\tau = T$ *into Theorem 7, we have the following outcome:*

Corollary 3. *Let $\mathcal{A}, B, S$, and T be four self-mappings of a complete NMS $(\Xi, M, N, O, *, \diamondsuit)$ with $\tau * \tau \geq \tau$ and $(1-\tau) \diamondsuit (1-\tau) \leq 1-\tau$ for all $\tau \in [0,1]$, thus satisfying the following conditions:*

(1) $\mathcal{A}(\Xi) \subseteq T(\Xi)$, $B(\Xi) \subseteq S(\Xi)$;
(2) *either $\mathcal{A}^m$ or S^p is continuous, where $m, p \in N$;*
(3) $\mathcal{A}S = S\mathcal{A}, BT = TB$;
(4) $\exists\lambda \in (0,1)$ *s. t*

$$
\begin{gathered}
M(\mathcal{A}^m\varpi, B^n\omega, \lambda\tau) \geq M(S^p\varpi, \mathcal{A}^m\varpi, \tau) * M(T^q\omega, B^n\omega, \tau); \\
* M(T^q\omega, \mathcal{A}^m\varpi, \beta\tau) * M(S^p\varpi, B^n\omega, (2-\beta)\tau) * M(S^p\varpi, T^q\omega, \tau); \\
N(\mathcal{A}^m\varpi, B^n\omega, \lambda\tau) \leq N(S^p\varpi, \mathcal{A}^m\varpi, \tau) \lozenge N(T^q\omega, B^n\omega, \tau); \\
\lozenge N(T^q\omega, \mathcal{A}^m\varpi, \beta\tau) \lozenge N(S^p\varpi, B^n\omega, (2-\beta)\tau) \lozenge N(S^p\varpi, T^q\omega, \tau); \\
O(\mathcal{A}^m\varpi, B^n\omega, \lambda\tau) \leq O(S^p\varpi, \mathcal{A}^m\varpi, \tau) \lozenge O(T^q\omega, B^n\omega, \tau); \\
\lozenge O(T^q\omega, \mathcal{A}^m\varpi, \beta\tau) \lozenge O(S^p\varpi, B^n\omega, (2-\beta)\tau) \lozenge O(S^p\varpi, T^q\omega, \tau).
\end{gathered}
$$

$\forall\, \varpi, \omega \in \Xi$, $\beta \in (0,2)$ and $\tau > 0$ and $m, n, p, q \in \mathbb{N}$. Then, $\mathcal{A}, B, S$ and T have a UCFP on Ξ. If we put

$$
\mathcal{A}_1 = \mathcal{A}_2 = \ldots = \mathcal{A}_m = \mathcal{A}, \qquad B_1 = B_2 = \ldots = B_n = B,\ S_1 = S_2 = \ldots = S_p = SL
$$

and $T_1 = T_2 = \ldots = T_\tau = T$ into Theorem 7.

Example 5. *Let $\Xi = [0,1]$, with the metric $d(\varpi, \omega) = |\varpi - \omega|$ and defined as*

$$
\begin{gathered}
M(\varpi, \omega, \tau) = \tfrac{\tau}{\tau + |\varpi - \omega|}, \\
N(\varpi, \omega, \tau) = \tfrac{|\varpi - \omega|}{\tau + |\varpi - \omega|}, \\
O(\varpi, \omega, \tau) = \tfrac{|\varpi - \omega|}{\tau},
\end{gathered}
$$

Clearly, $(\Xi, M, N, O, *, \lozenge)$ is a NMS where $a * b = min\{a, b\}$ and $a \lozenge b = max\{a, b\}$. Assuming that A, B, S and T be $A(\varpi) = B(\varpi) = 0$ if $\varpi = 0$ or 1,

$$
A\varpi = B\varpi = \frac{1}{\varpi + 1}, \text{if } \frac{1}{n+1} \leq \varpi < \frac{1}{n}\ ,\ n \in \mathbb{N}
$$

and

$$
S\varpi = T\varpi = \varpi, \qquad \forall \varpi \in \Xi.
$$

Then, it is easy to see that all the conditions of Corollary 3 have been satisfied.

Corollary 4. *Let $\mathcal{A}, B, S, T, L$ and R be six self-mappings of a complete NMS $(\Xi, M, N, O, *, \lozenge)$ with $\tau * \tau \geq \tau$ and $(1-\tau) \lozenge (1-\tau) \leq 1 - \tau$ for all $\tau \in [0,1]$, thus satisfying the following conditions:*

(1) $\mathcal{A}(\Xi) \subseteq TR(\Xi)$, $B(\Xi) \subseteq SL(\Xi)$;
(2) Either $\mathcal{A}$ or SL is continuous;
(3) $\mathcal{A}L = L\mathcal{A}, \mathcal{A}S = S\mathcal{A}, LS = SL, BR = RB, BT = TB,$ and $TR = RT$;
(4) $\exists\, \lambda \in (0,1)$ such that

$$
\begin{gathered}
M(\mathcal{A}\varpi, B\omega, \lambda\tau) \geq M(SL\varpi, \mathcal{A}\varpi, \tau) * M(TR\omega, B\omega, \tau) \\
* M(TR\omega, \mathcal{A}\varpi, \beta\tau) * M(SL\varpi, B\omega, (2-\beta)\tau) * M(SL\varpi, TR\omega, \tau), \\
N(\mathcal{A}\varpi, B\omega, \lambda\tau) \leq N(SL\varpi, \mathcal{A}\varpi, \tau) \lozenge N(TR\omega, B\omega, \tau) \\
\lozenge N(TR\omega, \mathcal{A}\varpi, \beta\tau) \lozenge N(SL\varpi, B\omega, (2-\beta)\tau) \lozenge N(SL\varpi, T\omega, \tau), \\
O(\mathcal{A}\varpi, B\omega, \lambda\tau) \leq O(SL\varpi, \mathcal{A}\varpi, \tau) \lozenge O(TR\omega, B\omega, \tau) \\
\lozenge O(TR\omega, \mathcal{A}\varpi, \beta\tau) \lozenge O(SL\varpi, B\omega, (2-\beta)\tau) \lozenge O(SL\varpi, T\omega, \tau).
\end{gathered}
$$

$\forall \varpi, \omega \in \Xi, \beta \in (0,2)$ and $\tau > 0$. Then, $\mathcal{A}$, B, S, T, L and R have a UCFP in Ξ. If we put

$$
\mathcal{A}_1 = \mathcal{A}_2 = \ldots = \mathcal{A}_m = \mathcal{A}, \qquad B_1 = B_2 = \ldots = B_n = B,\ S_1 = S_2 = \ldots = S_p = SL
$$

and $T_1 = T_2 = \ldots = T_\tau = T$ in Theorem 7. we have the following result:

Remark 3. *Corollaries 3 and 4 are generalizations of Corollaries 3.1 and 3.2, respectively, because they have slightly strict commutativity criteria.*

5. Conclusions

The article represented two common fixed-point theorems in which we utilized even number of mappings on a complete NMS with some contractive conditions. We satisfied a common fixed-point theorem for four finite families of mappings on a complete NMS. These results generalized the results provided in [12]. In [12] author used only membership function and proved common fixed-point results for four self-mappings. We used a generalized set namely a neutrosophic set in which we used membership, non-membership and neutral functions. Further, we proved common fixed-point results for any even number of mappings. This work can easily be extended to fit the context of neutrosophic b-metric space, neutrosophic partial metric space, neutrosophic cone metric space, and many other structures.

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References

1. Smarandache, F. *A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic*; American Research Press: Rehoboth, DE, USA, 1999.
2. Zadeh, L. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353. [CrossRef]
3. Turksen, I. Interval valued fuzzy sets based on normal forms. *Fuzzy Sets Syst.* **1986**, *20*, 191–210. [CrossRef]
4. Atanassov, K. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
5. Banach, S. Sur les op'erations dans les ensembles abstraits et leur application aux ´equations int´egrales. *Fundam. Math.* **1922**, *3*, 133–181. [CrossRef]
6. Caccioppoli, R. Un teorema generale sull' esistenza di elementi uniti in una trasformazione funzionale. *Rend. Lincei* **1930**, *11*, 794–799.
7. Arora, S.C.; Sharma, V. Fixed points for fuzzy mappings. *Fuzzy Sets Syst.* **2000**, *110*, 127–130. [CrossRef]
8. Park, J.H. Intuitionistic fuzzy metric spaces. *Chaos Solit. Fract.* **2004**, *22*, 1039–1046. [CrossRef]
9. Argyros, I.K. *The Theory and Applications of Iterative Methods*; CRC Press: Boca Raton, FL, USA, 2022.
10. Argyros, I.K.; Argyros, C.I.; Regmi, S.; George, S. *Contemporary Algorithms: Theory and Applications Volume III*; Nova Science Publishers, Inc.: Hauppauge, NY, USA, 2023.
11. Jungck, G. Compatible mappings and common fixed points. *Int. J. Math. Math. Sci.* **1986**, *9*, 771–779. [CrossRef]
12. Mishra, S.N.; Sharma, N.; Singh, S.L. Common fixed points of maps on fuzzy metric spaces. *Int. J. Math. Math. Sci.* **1994**, *17*, 253–258. [CrossRef]
13. Turkoglu, D.; Alaca, C.; Yildiz, C. Compatible maps and compatible maps of types (α) and (β) in intuitionistic fuzzy metric spaces. *Demonstr. Math.* **2006**, *39*, 671–684. [CrossRef]
14. Alaca, C.; Altun, I.; Turkoglu, D. On compatible mappings of type (I) and (II) in intuitionistic fuzzy metric spaces. *Commun. Korean Math. Soc.* **2008**, *23*, 427–446. [CrossRef]
15. Kirişci, M.; Şimşek, N. Neutrosophic metric spaces. *Math. Sci.* **2020**, *14*, 241–248. [CrossRef]
16. Ishtiaq, U.; Simsek, N.; Khalil Javed, K.A.; Uddin, F.; Kirişçi, M. Fixed point theorem for neutrosophic extended metric-like spaces and their application. *Sigma J. Eng. Nat. Sci.* **2023**, *41*, 750–757. [CrossRef]
17. Uddin, F.; Ishtiaq, U.; Hussain, A.; Javed, K.; Al Sulami, H.; Ahmed, K. Neutrosophic Double Controlled Metric Spaces and Related Results with Application. *Fractal Fract.* **2022**, *6*, 318. [CrossRef]
18. Schweizer, B.; Sklar, A. Statistical metric spaces. *Pac. J. Math.* **1960**, *10*, 313–334. [CrossRef]
19. Alaca, C.; Turkoglu, D.; Yildiz, C. Fixed points in intuitionistic fuzzy metric spaces. *Chaos Solitons Fractals* **2006**, *29*, 1073–1078. [CrossRef]
20. Riaz, M.; Ishtiaq, U.; Park, C.; Ahmad, K.; Uddin, F. Some fixed point results for xi-chainable neutrosophic and generalized neutrosophic cone metric spaces with application. *Aims Math.* **2022**, *7*, 14756–14784. [CrossRef]
21. Manro, S.; Kumar, S.; Bhatia, S.S.; Tas, K. Common Fixed Point Theorems in Modified Intuitionistic Fuzzy Metric Spaces. *J. Appl. Math.* **2013**, *2013*, 189321. [CrossRef]
22. Sengar, P.; Vikram University; Jain, S.; Paria, A.; Badshah, V. Compatibility of type (β) and Common fixed point theorem in intuitionistic fuzzy metric space. *Int. J. Math. Trends Technol.* **2016**, *38*, 58–69. [CrossRef]
23. George, A.; Veeramani, P. On some results in fuzzy metric spaces. *Fuzzy Set. Syst.* **1994**, *64*, 395–399. [CrossRef]