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On Some Impulsive Fractional Integro-Differential Equation with Anti-Periodic Conditions

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Abstract: We investigate a class of boundary value problems (BVPs) involving an impulsive fractional integro-differential equation (IF-IDE) with the Caputo–Hadamard fractional derivative (C-HFD). We employ some fixed-point theorems (FPTs) to study the existence of this fractional BVP and its unique solution. The boundary conditions (BCs) established in this study are of a more general type and can be reduced to numerous specific examples by defining the parameters involved in the conditions. In this way, we extend some recent nice results. At the end, we use an example to verify our results.

Keywords: impulsive; integro-differential equations; Caputo–Hadamard fractional derivative; boundary value problems

MSC: 26A33; 34A09; 34A12; 47H10

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1. Introduction

Fractional derivatives (FDs) have several definitions, including those given by Hadamard (1891 [1,2]), Riemann (1849), Caputo (1997), Grunwald–Letnikov (1867), Riemann–Liouville (1832), and many more. The Hadamard fractional derivative (HFD) can be generated into the C-HFD by combining the differential and integral parts. The primary distinction between an HFD and a C-HFD, notwithstanding any differences in the function's standards, is that a constant C-HFD is zero [3].

The HFD, in contrast to the Caputo and Riemann–Liouville derivatives, has an arbitrary-order logarithmic function ($\log \alpha - \log \beta$) rather than ($\alpha - \beta$). The expression of the HFD can be understood as a generalization operator (refer, for example, to [4–11]). This is only one more crucial aspect of the HFD. The characteristics and uses of the HFD are covered in a number of articles (see, e.g., [12]). For example, Kilbas talked about fractional differential equations (FDEs) of the Hadamard type on the finite interval [a, b] in various spaces. See also the properties of Hadamard calculus, and the modification of the HFD with the Caputo one, known as the C–HFD [12]. It is obtained from the HFD by changing the order of its differentiation and integration.

Recently, the theory of FDEs has gained significant attention due to its numerous applications in fields such as physics, chemistry, aerodynamics, complex-medium electrodynamics, polymer rheology, and others. Another useful application of FDEs is the explanation of the inherited properties of different materials. Because of this, FDEs are becoming a very important and well-known topic. For details, see, e.g., [3,13–19].

The study of impulsive BVPs has developed during the past few decades. In the applied sciences and engineering, it has been shown to be extremely useful in the development of numerous applicable mathematical models for real processes. Some existence results of impulsive BVPs involving FDs of Caputo's type were studied by Tian and Bai [20]. It has been discovered recently that a large portion of the research on the FDEs of the

Caputo and Caputo–Hadamard types with various circumstances, including impulses, time delays, and BVCs, are in [3,13,21–26].

Several researchers have focused on some fascinating outcomes of solutions to FDEs with beginning and boundary conditions by applying various FPTs (see [27–29]). Many studies on FPTs, topology, and nonlinear analysis have contributed to the current rapid growth of FDEs with HFD (see [30–32]).

In 2008, Benchohra et al. [33] discussed the following solutions of the nonlinear FDE:

$$\begin{aligned}&\mathcal{D}^{\zeta}\vartheta(\tau)=\hbar(\tau,\vartheta(\tau)),\quad \tau\in\mathcal{J},\\&\mathfrak{a}\vartheta(0)+\mathfrak{b}\vartheta(\mathcal{T})=\mathfrak{c}.\end{aligned}$$

The following BVP has been studied in [21]:

$$\begin{aligned}&\mathcal{D}^{\zeta}\vartheta(\tau)=\hbar(\tau,\vartheta(\tau)),\\&\vartheta(0)=\omega\mathcal{I}^{\epsilon}\vartheta(\varsigma),0<\varsigma<1.\end{aligned}$$

In [34], Irguedi et al. studied the following Caputo–Hadamard fractional derivative equation (C-HFDE) and initial boundary condition (IBC) with fractional initial condition (FIC):

$$\begin{aligned}&{}^{\mathcal{CH}}\mathcal{D}^{\kappa}\vartheta(\tau)=\hbar(\tau,\vartheta(\tau)),\quad \tau\in\mathcal{J}=[\mathfrak{a},\mathcal{T}],\\&\Delta\vartheta|_{\tau=\tau_{\kappa}}=\mathcal{I}_{\kappa}(\vartheta(\tau_{\kappa}^{-})),\quad \tau=\tau_{\kappa},\\&\Delta\vartheta'|_{\tau=\tau_{\kappa}}=\overline{\mathcal{I}}_{\kappa}(\vartheta(\tau_{\kappa}^{-})),\\&\vartheta(\tau)=\mu(\tau),\\&\vartheta'(\mathcal{T})=\int_{\mathfrak{a}}^{\mathcal{T}}\ell(\nu,\vartheta(\nu)).\end{aligned}$$

In 2018, Benhamida et al. [17,35] studied the following nonlinear FDE:

$$\begin{aligned}&\mathcal{D}^{\varpi}\vartheta(\tau)=\hbar(\tau,\vartheta(\tau)),\\&\mathcal{A}\vartheta(1)+\mathcal{B}\vartheta(T)=\mathcal{C}.\end{aligned}$$

where $\mathcal{J}:=[1,T]$, and $\mathcal{D}^{\varpi}$ is the order of the C-HFD of ϖ.

In [36], Reunsumrit et al. studied the BVP for the following C-FIDE:

$$\begin{aligned}&{}^{\mathcal{ABC}}\mathcal{D}_{\tau}^{\zeta}[\omega(t)-\mathcal{U}(\tau,\omega(t))]=\ell(\tau,\omega(\tau),\mathcal{G}\omega(\tau)),\\&\Delta(\omega)|_{\tau=\tau_{\kappa}}=I_{\kappa}(\omega(\tau_{\kappa}^{-})),\\&\omega(0)=\int_0^{\iota}\frac{(\iota-p)^{\epsilon-1}}{\Gamma(\iota)}\wp(\rho,\omega(\rho)).\end{aligned}$$

The following FDE has been studied in [37]:

$$\begin{aligned}&\mathcal{D}^{\epsilon}\vartheta(\tau)=\ell(\tau,\vartheta(\tau)),\\&\vartheta(0)+\vartheta(T)=\mathcal{B}\int_0^T\vartheta(\nu)\nu,\end{aligned}$$

order $\epsilon\in[0,1)$.

Abdo et al. in [13] discussed the positive solutions of the following FDE with IBC:

$$\begin{aligned}&\mathcal{D}^{\epsilon}\vartheta(\tau)=\ell(\tau,\vartheta(\tau)),\\&\vartheta(0)=\mathcal{B}\int_0^1\vartheta(\nu),\mathfrak{d}\nu.\end{aligned}$$

In 2019, Ardjouni et al. [23] discussed the following FDE with IBC:

$$\mathcal{D}_1^{\mathfrak{e}}\vartheta(\tau) = \wp(\tau, \vartheta(\tau)),$$
$$\vartheta(1) = \mathcal{B}\int_1^{\mathfrak{e}} \vartheta(\nu)\partial\nu.$$

Motivated, among other papers, by the aforementioned ones, we concentrate on the following IF-IDE with BVP:

$$^{\mathcal{C}}_{\mathcal{H}}\mathcal{D}^{\iota}_{1+}\aleph(\tau) = \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)), \tau \in \Im := [1, \mathcal{T}], 0 < \iota \leq 1, \tag{1}$$

$$\aleph(\tau_\kappa^+) = \aleph(\tau_\kappa^-) + \vartheta_\kappa, \quad \vartheta_\kappa \in \Re \quad \kappa = 1, \ldots, \mathcal{M}, \tag{2}$$

$$\nu\aleph(1) + \pi\aleph(\mathcal{T}) = \delta\mathcal{I}^{\varsigma}\aleph(\xi) + \varpi, \varsigma \in (0, 1], \tag{3}$$

where ${}^{\mathcal{CH}}\mathcal{D}^{\iota}$ is the C-HFD of order ι, $\ell : [1, \mathcal{T}] \times \Re \times \Re \to \Re$ is a continuous function, ν, π, δ, ϖ are real constants, $\xi \in (1, \mathcal{T})$, $\mathcal{G}\aleph(\tau) = \int_0^\tau \kappa(\tau, \upsilon, \aleph(\upsilon))\partial\upsilon$, and $\kappa : \Delta \times [1, \mathcal{T}] \to \Re$, $\Delta = \{(\tau, \upsilon) : 1 \leq \upsilon \leq \tau \leq \mathcal{T}\}$. Furthermore, $1 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_{\mathcal{M}} = 1, \Delta\aleph|_{\tau=\tau_\kappa} = \aleph(\tau_\kappa^+) - \aleph(\tau_\kappa^-)$ and $\aleph(\tau_\kappa^+) = lim_{\hbar\to 0^+}\aleph(\tau_\kappa + \hbar)$ and $\aleph(\tau_\kappa^-) = lim_{\hbar\to 0^-}\aleph(\tau_\kappa + \hbar)$ represent the right- and left-hand limits of $\aleph(\tau)$ at $\tau = \tau_\kappa$.

Motivations:

1. The main motivation for this work is to use the C-HFD to present a new class of IF-IDE with anti-periodic BC;
2. We investigate the existence and uniqueness of the solutions of (1)–(3) using Schauder's FPT, Krasnoselkii's FPT, and the Banach Contraction Principle;
3. We extend the results studied in [38] by including the C-HFD, impulsive conditions, and nonlinear integrals.

The remaining part of the paper is organized as follows. Section 2 discusses the integral operator associated with the problem that is presented and many more. The existence results that follow are based on the FPT of Krasnoselkii, Scheafer, and Schauder's nonlinear alternative. Furthermore, we obtain uniqueness results in Section 3 by applying BFPT. An example of the outcomes is provided in Section 4.

2. Preliminaries

Let the space $\mathcal{PC}_{\varpi}^{\mathcal{N}}([\mathcal{A}, \mathcal{B}], \Re) = \{\hbar : [\mathcal{A}, \mathcal{B}] \to \Re : \varpi^{\mathcal{N}-1}\hbar(\aleph) \in \mathcal{PC}([\mathcal{A}, \mathcal{B}], \Re)\}$ and $\mathcal{E} = \mathcal{PC}([1, \mathcal{T}], \Re)$, endowed with the norm $\| \wp \| = \max_{\tau\in[1,\mathcal{T}]} |\wp(\tau)|$, be the Banach space of all continuous functions from $[1, \mathcal{T}]$ into $\Re$. We recall from [39] both the Hadamard fractional integral and HFD concepts, respectively, as follows:

The fractional integral (Hadamard) of order $\eta > 0$ for function $\Psi : [1, \infty) \to \mathbb{R}$ is (with Γ the Gamma function)

$$I^{\eta}_{a^+}\Psi(t_1) = \frac{1}{\Gamma(\eta)}\int_a^{t_1}\left(\log\frac{t_1}{s}\right)^{\eta-1}\Psi(s)\frac{ds}{s}.$$

For function Ψ on $[1, +\infty)$, and $n - 1 < \eta < n$, the HFD of order η is

$$\begin{aligned} D^{\eta}_{a^+}\Psi(t) &= \frac{1}{\Gamma(n-\eta)}\left(t\frac{d}{dt}\right)^n \int_a^t \left(\log\frac{t}{s}\right)^{n-\eta-1}\Psi(s)\frac{ds}{s} \\ &= \delta^n I^{n-\eta}_{a^+}\Psi(t). \end{aligned}$$

where $[\eta]$ denotes the integer part of η, $n = [\eta] + 1$, and $\delta = t\frac{d}{dt}$, provided the convergence of the integral.

Jarad et al. in [12] generalized the HFDs and presented the properties of such derivatives. These generalizations are now known as the C-HFDs (see also the versions used in [40,41]) and are given by the following definition (C-HFD [12]): Let $\nu = 0$ and $\mathcal{N} = [\nu] + 1$ if $\hbar(\aleph) \in \mathcal{PC}_{\varpi}^{\mathcal{N}}[\mathcal{A}, \mathcal{B}]$, where $0 < \mathcal{A} < \mathcal{B} < \infty$ and

$$\mathcal{PC}_{\varpi}^{\mathcal{N}}[\mathcal{A}, \mathcal{B}] = \{\hbar : [\mathcal{A}, \mathcal{B}] \to \mathcal{C} : \varpi^{\mathcal{N}-1}\hbar(\aleph) \in \mathcal{PC}[\mathcal{A}, \mathcal{B}]\}.$$

The Caputo-type update of HFDs of order ν is given by

$$ {}^{\mathcal{C}}_{\mathcal{H}}\mathcal{D}^{\nu}_{\mathcal{A}^+}\hbar(\tau) = \mathcal{D}^{\nu}_{\mathcal{A}^+}\left(\hbar(\tau) - \sum_{\kappa=0}^{\mathcal{N}-1}\frac{\varpi^{\kappa}\hbar(\mathcal{A})}{\kappa!}\left(\log\frac{\tau}{v}\right)^{\kappa}\right) $$

Lemma 1. *Let* $\aleph$ *be an impulsive solution of the IBC with* $g : \mathcal{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *given by*

$$ {}^{\mathcal{C}}_{\mathcal{H}}\mathcal{D}^{\iota}_{1+}\aleph(\tau) = \mathrm{g}(\tau), \tau \in \Im := [1,\mathcal{T}], 0<\iota\le 1, \tag{4} $$

$$ \aleph(\tau_{\kappa}^{+}) = \aleph(\tau_{\kappa}^{-}) + \vartheta_{\kappa}, \quad \vartheta_{\kappa}\in\mathbb{R} \quad \kappa = 1,\dots,\mathcal{M}, \tag{5} $$

$$ \nu\aleph(1) + \pi\aleph(\mathcal{T}) = \delta\mathcal{I}^{\varsigma}\aleph(\xi) + \varpi, \varsigma\in(0,1] \tag{6} $$

if and only if

$$ x(t) = \begin{cases} \frac{1}{\Gamma(\iota)}\int_1^{\tau}\left(\log\frac{\tau}{v}\right)^{\iota-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v} \\ +\frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)}\int_1^{\xi}\left(\log\frac{\xi}{v}\right)^{\iota+\varsigma-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v} \\ +\frac{|\pi|}{|\Lambda|\Gamma(\iota)}\int_1^{\mathcal{T}}\left(\log\frac{\mathcal{T}}{v}\right)^{\iota-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v}, \quad \texttt{for} \quad \tau\in[1,\tau_1) \\ \vartheta_1 + \frac{1}{\Gamma(\iota)}\int_1^{\tau}\left(\log\frac{\tau}{v}\right)^{\iota-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v} \\ +\frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)}\int_1^{\xi}\left(\log\frac{\xi}{v}\right)^{\iota+\varsigma-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v} \\ +\frac{|\pi|}{|\Lambda|\Gamma(\iota)}\int_1^{\mathcal{T}}\left(\log\frac{\mathcal{T}}{v}\right)^{\iota-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v}, \quad \texttt{for} \quad \tau\in(\tau_1,\tau_2) \\ \vartheta_1+\vartheta_2 + \frac{1}{\Gamma(\iota)}\int_1^{\tau}\left(\log\frac{\tau}{v}\right)^{\iota-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v} \\ +\frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)}\int_1^{\xi}\left(\log\frac{\xi}{v}\right)^{\iota+\varsigma-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v} \\ +\frac{|\pi|}{|\Lambda|\Gamma(\iota)}\int_1^{\mathcal{T}}\left(\log\frac{\mathcal{T}}{v}\right)^{\iota-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v}, \quad \texttt{for} \quad \tau\in(\tau_2,\tau_3) \\ . \\ . \\ . \\ \sum_{\kappa=1}^{\mathtt{m}}\vartheta_{\mathtt{i}} + \frac{1}{\Gamma(\iota)}\int_1^{\tau}\left(\log\frac{\tau}{v}\right)^{\iota-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v} \\ +\frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)}\int_1^{\xi}\left(\log\frac{\xi}{v}\right)^{\iota+\varsigma-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v} \\ +\frac{|\pi|}{|\Lambda|\Gamma(\iota)}\int_1^{\mathcal{T}}\left(\log\frac{\mathcal{T}}{v}\right)^{\iota-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v}, \quad \texttt{for} \quad \tau\in(\tau_{\kappa},\tau_{\kappa+1}). \end{cases} \tag{7} $$

Proof. Assume that $\aleph$ satisfies (4) and (6). If $\tau\in[1,\tau_1)$, then

$$ \begin{aligned} &{}^{\mathcal{C}}_{\mathcal{H}}\mathcal{D}^{\iota}_{1+}\aleph(\tau) = \}(\tau), \tau\in\mathcal{J} := [1,\mathcal{T}], 0<\iota\le 1, \\ &\nu\aleph(1) + \pi\aleph(\mathcal{T}) = \delta\mathcal{I}^{\varsigma}\aleph(\xi) + \varpi, \varsigma\in(0,1]. \end{aligned} $$

Easily,

$$ \begin{aligned} \aleph(\tau) =& \frac{1}{\Gamma(\iota)}\int_1^{\tau}\left(\log\frac{\tau}{v}\right)^{\iota-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v} \\ &+\frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)}\int_1^{\xi}\left(\log\frac{\xi}{v}\right)^{\iota+\varsigma-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v} \\ &+\frac{|\pi|}{|\Lambda|\Gamma(\iota)}\int_1^{\mathcal{T}}\left(\log\frac{\mathcal{T}}{v}\right)^{\iota-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v}. \end{aligned} $$

If $\tau \in (\tau_1, \tau_2)$, then

$$\begin{aligned}
\aleph(\tau) =& \vartheta(\tau_1^+) - \frac{1}{\Gamma(\iota)} \int_1^{\tau_1} \left(\log \frac{\tau}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&+ \frac{1}{\Gamma(\iota)} \int_1^{\tau} \left(\log \frac{\tau}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&+ \frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)} \int_1^{\xi} \left(\log \frac{\xi}{v}\right)^{\iota+\varsigma-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&+ \frac{|\pi|}{|\Lambda|\Gamma(\iota)} \int_1^{\mathcal{T}} \left(\log \frac{\mathcal{T}}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&= \vartheta(\tau_1^+) + \vartheta_1 - \frac{1}{\Gamma(\iota)} \int_1^{\tau} \left(\log \frac{\tau}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&+ \frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)} \int_1^{\xi} \left(\log \frac{\xi}{v}\right)^{\iota+\varsigma-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&+ \frac{|\pi|}{|\Lambda|\Gamma(\iota)} \int_1^{\mathcal{T}} \left(\log \frac{\mathcal{T}}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&= \vartheta_1 + \frac{1}{\Gamma(\iota)} \int_1^{\tau} \left(\log \frac{\tau}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&+ \frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)} \int_1^{\xi} \left(\log \frac{\xi}{v}\right)^{\iota+\varsigma-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&+ \frac{|\pi|}{|\Lambda|\Gamma(\iota)} \int_1^{\mathcal{T}} \left(\log \frac{\mathcal{T}}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v}.
\end{aligned}$$

If $\tau \in (\tau_2, \tau_3)$, then

$$\begin{aligned}
\aleph(\tau) =& \vartheta(\tau_2^+) - \frac{1}{\Gamma(\iota)} \int_1^{\tau_2} \left(\log \frac{\tau}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&+ \frac{1}{\Gamma(\iota)} \int_1^{\tau} \left(\log \frac{\tau}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&+ \frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)} \int_1^{\xi} \left(\log \frac{\xi}{v}\right)^{\iota+\varsigma-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&+ \frac{|\pi|}{|\Lambda|\Gamma(\iota)} \int_1^{\mathcal{T}} \left(\log \frac{\mathcal{T}}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&= \vartheta(\tau_2^+) + \vartheta_2 - \frac{1}{\Gamma(\iota)} \int_1^{\tau} \left(\log \frac{\tau}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&+ \frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)} \int_1^{\xi} \left(\log \frac{\xi}{v}\right)^{\iota+\varsigma-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&+ \frac{|\pi|}{|\Lambda|\Gamma(\iota)} \int_1^{\mathcal{T}} \left(\log \frac{\mathcal{T}}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&= \vartheta_1 + \vartheta_2 + \frac{1}{\Gamma(\iota)} \int_1^{\tau} \left(\log \frac{\tau}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&+ \frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)} \int_1^{\xi} \left(\log \frac{\xi}{v}\right)^{\iota+\varsigma-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\
&+ \frac{|\pi|}{|\Lambda|\Gamma(\iota)} \int_1^{\mathcal{T}} \left(\log \frac{\mathcal{T}}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v}.
\end{aligned}$$

If $\tau \in (\tau_{\mathtt{m}}, \mathcal{T})$, then

$$\begin{aligned} \aleph(\tau) =& \sum_{\kappa=1}^{\mathtt{m}} \vartheta_{\mathtt{i}} + \frac{1}{\Gamma(\iota)} \int_1^{\tau} \left(\log \frac{\tau}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\ &+ \frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)} \int_1^{\xi} \left(\log \frac{\xi}{v}\right)^{\iota+\varsigma-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\ &+ \frac{|\pi|}{|\Lambda|\Gamma(\iota)} \int_1^{\mathcal{T}} \left(\log \frac{\mathcal{T}}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v}. \end{aligned} \tag{8}$$

Let us say that $\aleph$ satisfies the impulsive FBC of (8). □

3. Main Results

This section shows the main findings of this article.

Theorem 2. *If*

(A_1)*: $\exists$ constants $\mathcal{W}_1, \mathcal{W}_2 > 0$,*

$$\left|\ell(\tau, \wp, \mathcal{V}) - \ell(\tau, \overline{\wp}, \overline{\mathcal{V}})\right| \leq \mathcal{W}_1|\wp - \overline{\wp}| + \mathcal{W}_2\left|\mathcal{V} - \overline{\mathcal{V}}\right|$$

for any $\wp, \mathcal{V}, \overline{\wp}, \overline{\mathcal{V}} \in \mathbb{R}$ and $\tau \in [1,2]$.

(A_2)*: A constant $\mathcal{Y} > 0$ exists:*

$$|\kappa(\tau, \wp, v) - \kappa(\tau, v, \mathcal{V})| \leq \mathcal{Y}|\wp - v|.$$

If

$$(\mathcal{W}_1 + \mathcal{W}_2\mathcal{Y})\omega < 1,$$

with

$$\omega := \left\{ \frac{(\log \mathcal{T})^{\iota}}{\Gamma(\iota+1)} + \frac{|\delta|(\log \xi)^{\iota+\varsigma}}{|\Lambda|\Gamma(\iota+\varsigma+1)} + \frac{|\pi|(\log \mathcal{T})^{\iota}}{|\Lambda|\Gamma(\iota+1)} \right\},$$

then the problem (1)–(3) has a unique solution on $\mathcal{J}$.

Proof. Assign the operator $\mathfrak{R}$ described by (1)–(3) and turn them into an FP problem as follows:

$$\begin{aligned} \mathfrak{R}\aleph(\tau) =& \frac{1}{\Gamma(\iota)} \int_1^{\tau} \left(\log \frac{\tau}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\ &+ \frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)} \int_1^{\xi} \left(\log \frac{\xi}{v}\right)^{\iota+\varsigma-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} \\ &+ \frac{|\pi|}{|\Lambda|\Gamma(\iota)} \int_1^{\mathcal{T}} \left(\log \frac{\mathcal{T}}{v}\right)^{\iota-1} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) \frac{dv}{v} + \sum_{\kappa=1}^{\mathtt{m}} \vartheta_{\mathtt{i}}. \end{aligned} \tag{9}$$

Utilizing the Banach FPT to show that $\mathfrak{R}$ is a contraction, let $\aleph, \vartheta \in \mathcal{PC}(\mathcal{J}, \mathbb{R})$, and we have

$$
\begin{aligned}
|(\mathfrak{R}\aleph)(\tau)-(\mathfrak{R}\vartheta)(\tau)|\leq{}& \frac{1}{\Gamma(\iota)}\int_1^{\tau}\left(\log\frac{\tau}{v}\right)^{\iota-1}|\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))-\ell(\tau,\vartheta(\tau),\mathcal{G}\vartheta(\tau))|\frac{dv}{v}\\
&+\frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)}\int_1^{\xi}\left(\log\frac{\xi}{v}\right)^{\iota+\varsigma-1}|\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))-\ell(\tau,\vartheta(\tau),\mathcal{G}\vartheta(\tau))|\frac{dv}{v}\\
&+\frac{|\pi|}{|\Lambda|\Gamma(\iota)}\int_1^{\mathcal{T}}\left(\log\frac{\mathcal{T}}{v}\right)^{\iota-1}|\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))-\ell(\tau,\vartheta(\tau),\mathcal{G}\vartheta(\tau))|\frac{dv}{v}\\
&\leq\frac{1}{\Gamma(\iota)}\int_1^{\tau}\left(\log\frac{\tau}{v}\right)^{\iota-1}(\mathcal{W}_1+\mathcal{W}_2\mathcal{Y})|\aleph(\tau)-\vartheta(\tau)|\frac{dv}{v}\\
&+\frac{|\varpi|}{|\Lambda|\Gamma(\iota+\varsigma)}\int_1^{\xi}\left(\log\frac{\xi}{v}\right)^{\iota+\varsigma-1}(\mathcal{W}_1+\mathcal{W}_2\mathcal{Y})|\aleph(\tau)-\vartheta(\tau)|\frac{dv}{v}\\
&+\frac{|\pi|}{|\Lambda|\Gamma(\iota)}\int_1^{\mathcal{T}}\left(\log\frac{\mathcal{T}}{v}\right)^{\iota-1}(\mathcal{W}_1+\mathcal{W}_2\mathcal{Y})|\aleph(\tau)-\vartheta(\tau)|\frac{dv}{v}\\
&\leq(\mathcal{W}_1+\mathcal{W}_2\mathcal{Y})\left\{\frac{(\log\mathcal{T})^{\iota}}{\Gamma(\iota+1)}+\frac{|\delta|(\log\xi)^{\iota+\varsigma}}{|\Lambda|\Gamma(\iota+\varsigma+1)}+\frac{|\pi|(\log\mathcal{T})^{\iota}}{|\Lambda|\Gamma(\iota+1)}\right\}.
\end{aligned}
$$

Thus,

$$\|\,(\mathfrak{R}\aleph)(\tau)-(\mathfrak{R}\vartheta)(\tau)\,\|_{\infty}\leq(\mathcal{W}_1+\mathcal{W}_2\mathcal{Y})\tau\|\,\aleph-\vartheta\,\|_{\infty}$$

Consequently, (9) is a contraction. Hence, using the Banach FPT, the problem (1)–(3) has a unique solution. □

Theorem 3. *If:*

(A_3): $\exists$ a constant $\mathcal{W}_{\mathtt{g}}>0$, $\forall\,|\ell(\tau,\wp,\mathcal{V})|\leq\mathcal{W}_{\mathtt{g}}$ for $\tau\in\mathcal{J}$ and each $\wp,\mathcal{V}\in\mathbb{R}$.
(A_4): $\exists$ a constant $\mathcal{K}^*>0\;\forall\sum_{\mathtt{i}=1}^{\mathtt{m}}|\vartheta_{\mathtt{i}}|\leq\mathcal{K}^*$,
then the problem (1)–(2) has at least one solution on $\mathbb{J}$.

Proof. To demonstrate that $\mathfrak{R}$, as described by (9), has an FP, carry out the following steps:

Step 1: $\mathfrak{R}$ is continuous. Let $\aleph_{\mathtt{n}}$ be a sequence $\forall\,\aleph_{\mathtt{n}}\to\aleph$ in $\mathcal{PC}(\mathcal{J},\mathbb{R})$. Then, for each $\tau\in\mathcal{J}$,

$$
\begin{aligned}
\|\,(\mathfrak{R}\aleph_{\mathtt{n}})(\tau)-(\mathfrak{R}\aleph)(\tau)\,\|\leq{}&\frac{1}{\Gamma(\iota)}\int_1^{\tau}\left(\log\frac{\tau}{v}\right)^{\iota-1}\|\,\ell(v,\aleph_{\mathtt{n}}(v),\mathcal{G}\aleph_{\mathtt{n}}(v))-\ell(v,\aleph(v),\mathcal{G}\aleph(v))\,\|\frac{dv}{v}\\
&+\frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)}\int_1^{\xi}\left(\log\frac{\xi}{v}\right)^{\iota+\varsigma-1}\|\,\ell(v,\aleph_{\mathtt{n}}(v),\mathcal{G}\aleph_{\mathtt{n}}(v))-\ell(v,\aleph(v),\mathcal{G}\aleph(v))\,\|\frac{dv}{v}\\
&+\frac{|\pi|}{|\Lambda|\Gamma(\iota)}\int_1^{\mathcal{T}}\left(\log\frac{\mathcal{T}}{v}\right)^{\iota-1}\|\,\ell(v,\aleph_{\mathtt{n}}(v),\mathcal{G}\aleph_{\mathtt{n}}(v))-\ell(v,\aleph(v),\mathcal{G}\aleph(v))\,\|\frac{dv}{v}\\
&+\sum_{\mathtt{i}=1}^{\mathtt{m}}\vartheta_{\mathtt{i}}\\
&\leq\left\{\frac{(\log\mathcal{T})^{\iota}}{\Gamma(\iota+1)}+\frac{|\delta|(\log\xi)^{\iota+\varsigma}}{|\Lambda|\Gamma(\iota+\varsigma+1)}+\frac{|\pi|(\log\mathcal{T})^{\iota}}{|\Lambda|\Gamma(\iota+1)}+\sum_{\mathtt{i}=1}^{\mathtt{m}}\vartheta_{\mathtt{i}}\right\}\times\\
&\|\,\ell(v,\aleph_{\mathtt{n}}(v),\mathcal{G}\aleph_{\mathtt{n}}(v))-\ell(v,\aleph(v),\mathcal{G}\aleph(v))\,\|.
\end{aligned}
$$

Therefore, $\mathfrak{R}$ is continuous, and $\|\,(\mathfrak{R}\aleph_{\mathtt{n}})(\tau)-(\mathfrak{R}\aleph)(\tau)\,\|_{\infty}\to 0$ as $\mathtt{n}\to\infty$.

Step 2: $\mathfrak{R}$ maps bounded sets into bounded sets in $\mathcal{PC}(\mathcal{J},\mathbb{R})$, and it is enough to show that $\iota>0$.

$$\wp\in\mathcal{B}_{\iota}=\{\aleph\in\mathcal{PC}(\mathcal{J},\mathbb{R}),\|\,\aleph\,\|_{\infty}\leq\iota\}$$

For $\aleph \in \mathcal{B}_\iota$ and for each $\tau \in [1, \mathcal{T}]$, we have

$$\begin{aligned}|(\Re\aleph)(\tau)| \leq & \frac{1}{\Gamma(\iota)} \int_1^\tau \left(\log \frac{\tau}{v}\right)^{\iota-1} |\ell(v, \aleph(v), \mathcal{G}\aleph(v))| \frac{dv}{v} \\ & + \frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)} \int_1^\xi \left(\log \frac{\xi}{v}\right)^{\iota+\varsigma-1} |\ell(v, \aleph(v), \mathcal{G}\aleph(v))| \frac{dv}{v} \\ & + \frac{|\pi|}{|\Lambda|\Gamma(\iota)} \int_1^{\mathcal{T}} \left(\log \frac{\mathcal{T}}{v}\right)^{\iota-1} |\ell(v, \aleph(v), \mathcal{G}\aleph(v))| \frac{dv}{v} + \sum_{\mathtt{i}=1}^{\mathtt{m}} \vartheta_{\mathtt{i}},\end{aligned}$$

$$\begin{aligned}&\leq \frac{\mathcal{W}_{\mathtt{g}}}{\Gamma(\iota)} \int_1^\tau \left(\log \frac{\tau}{v}\right)^{\iota-1} \frac{dv}{v} + \frac{|\delta|(\mathcal{W}_{\mathtt{g}})}{|\Lambda|\Gamma(\iota+\varsigma)} \int_1^\xi \left(\log \frac{\xi}{v}\right)^{\iota+\varsigma-1} \frac{dv}{v} \\ &+ \frac{|\pi|(\mathcal{W}_{\mathtt{g}})}{|\Lambda|\Gamma(\iota)} \int_1^{\mathcal{T}} \left(\log \frac{\mathcal{T}}{v}\right)^{\iota-1} \frac{dv}{v} + \mathcal{K}^*, \\ &\leq (\mathcal{W}_{\mathtt{g}}) \left\{ \frac{(\log \mathcal{T})^\iota}{\Gamma(\iota+1)} + \frac{|\delta|(\log \xi)^{\iota+\varsigma}}{|\Lambda|\Gamma(\iota+\varsigma+1)} + \frac{|\pi|(\log \mathcal{T})^\iota}{|\Lambda|\Gamma(\iota+1)} \right\} + \mathcal{K}^*, \\ &\leq (\mathcal{W}_{\mathtt{g}})\omega + \mathcal{K}^*.\end{aligned}$$

Thus,

$$\| (\Re\aleph)(\tau) \| \leq (\mathcal{W}_{\mathtt{g}})\omega + \mathcal{K}^*.$$

Step 3: $\Re$ maps bounded sets into equicontinuous sets of $\mathcal{PC}(\mathcal{J}, \mathbb{R})$.

Let $\tau_1, \tau_2 \in \mathcal{J}, \tau_1 < \tau_2, \mathcal{B}_\iota$ be a bounded set of $\mathcal{PC}(\mathcal{J}, \mathbb{R})$ as in Step 2, and let $\aleph \in \mathcal{B}_\aleph$. Then,

$$\begin{aligned}\| \mathcal{R}\aleph(\tau_2) - \mathcal{R}\aleph(\tau_1) \| \leq & \frac{1}{\Gamma(\iota)} \int_1^{\tau_1} \left[\left(\log \frac{\tau_2}{v}\right)^{\iota-1} - \left(\log \frac{\tau_1}{v}\right)^{\iota-1} \right] \| \ell(v, \aleph(v), \mathcal{G}\aleph(v)) \| \frac{dv}{v} \\ & + \frac{1}{\Gamma(\iota)} \int_{\tau_1}^{\tau_2} \left(\log \frac{\tau_2}{v}\right)^{\iota-1} \| \ell(v, \aleph(v), \mathcal{G}\aleph(v)) \| \frac{dv}{v} \\ & \leq \frac{\mathcal{W}_{\mathtt{g}}}{\Gamma(\iota)} \int_1^{\tau_1} \left[\left(\log \frac{\tau_2}{v}\right)^{\iota-1} - \left(\log \frac{\tau_1}{v}\right)^{\iota-1} \right] \frac{dv}{v} \\ & + \frac{\mathcal{W}_{\mathtt{g}}}{\Gamma(\iota)} \int_{\tau_1}^{\tau_2} \left(\log \frac{\tau_2}{v}\right)^{\iota-1} \frac{dv}{v} \\ & \leq \frac{\mathcal{W}_{\mathtt{g}}}{\Gamma(\iota+1)} \left[(\log \tau_2)^\iota - (\log \tau_1)^\iota \right],\end{aligned}$$

which implies $\| \Re\aleph(\tau_2) - \Re\aleph(\tau_1) \|_\infty \to 0$ as $\tau_1 \to \tau_2$. Therefore, with the consequence of Steps 1–3, we conclude that $\Re$ is continuous and completely continuous.

Step 4: A priori bounds. Show that the set

$$\Lambda = \{\aleph \in \mathcal{PC}(\mathcal{J}, \mathbb{R}) : \aleph = \rho\mathcal{R}(\aleph) \text{forsome} 0 < \rho < 1\}$$

is bounded. For $\aleph \in \Lambda$ and $\tau \in \mathcal{J}$, we have

$$\begin{aligned}\aleph(\tau) \leq \rho \{ & \frac{1}{\Gamma(\iota)} \int_1^\tau \left(\log \frac{\tau}{v}\right)^{\iota-1} \ell(v, \aleph(v), \mathcal{G}\aleph(v)) \frac{dv}{v} \\ & + \frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)} \int_1^\xi \left(\log \frac{\xi}{v}\right)^{\iota+\varsigma-1} \ell(v, \aleph(v), \mathcal{G}\aleph(v)) \frac{dv}{v} \\ & + \frac{|\pi|}{|\Lambda|\Gamma(\iota)} \int_1^{\mathcal{T}} \left(\log \frac{\mathcal{T}}{v}\right)^{\iota-1} \ell(v, \aleph(v), \mathcal{G}\aleph(v)) \frac{dv}{v} + \sum_{\mathtt{i}=1}^{\mathtt{m}} \vartheta_{\mathtt{i}} \}\end{aligned}$$

For $\rho \in [0,1]$,

$$\begin{aligned}
\| \Re\aleph(\tau) \| \leq & \frac{1}{\Gamma(\iota)} \int_1^\tau \left(\log \frac{\tau}{v}\right)^{\iota-1} |\ell(v,\aleph(v),\mathcal{G}\aleph(v))| \frac{dv}{v} \\
& + \frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)} \int_1^\xi \left(\log \frac{\xi}{v}\right)^{\iota+\varsigma-1} |\ell(v,\aleph(v),\mathcal{G}\aleph(v))| \frac{dv}{v} \\
& + \frac{|\pi|}{|\Lambda|\Gamma(\iota)} \int_1^{\mathcal{T}} \left(\log \frac{\mathcal{T}}{v}\right)^{\iota-1} |\ell(v,\aleph(v),\mathcal{G}\aleph(v))| \frac{dv}{v} + \sum_{\mathtt{i}=1}^{\mathtt{m}} \vartheta_{\mathtt{i}}, \\
& \leq (\mathcal{W}_{\mathtt{g}})\omega + \mathcal{K}^*.
\end{aligned}$$

Thus

$$\| \Re\aleph(\tau) \| \leq \infty$$

This implies that Λ is a bounded set. Now, utilizing the Ascoli–Arzela theorem, we conclude that $\mathcal{R}$ has an FP, which is a solution of the problem (1)–(3) on $\mathcal{J}$. □

Theorem 4. *Assume the following hypotheses:*
(A_5)*:* $\exists\, \phi \in \mathcal{L}^1(\mathcal{J},\mathbb{R}^+)$ *and* $\psi : [0,\infty) \to (0,\infty)$ *continuous and non-decreasing:*

$$|\ell(\tau,\aleph,\vartheta)| \leq \phi(\tau)\psi(\| \aleph \| + \| \vartheta \|),\ \textit{for}\ \tau \in \mathcal{J} \textit{andeach} \aleph, \vartheta \in \mathbb{R}.$$

Then, the BVP (1)–(2) has at least one solution on J.

Proof. Let $\aleph$ be such that for each $\tau \in \mathcal{J}$, we take $\aleph = \delta\,\mathrm{Im}\,\aleph$ for $\delta \in (0,1)$, and let $\aleph$ be a solution. Then,

$$\begin{aligned}
|\aleph(\tau)| \leq & \frac{1}{\Gamma(\iota)} \int_1^\tau \left(\log \frac{\tau}{v}\right)^{\iota-1} \phi(\tau)\psi(\| \aleph \| + \| \vartheta \|) \frac{dv}{v} \\
& + \frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)} \int_1^\xi \left(\log \frac{\xi}{v}\right)^{\iota+\varsigma-1} \phi(\tau)\psi(\| \aleph \| + \| \vartheta \|) \frac{dv}{v} \\
& + \frac{|\pi|}{|\Lambda|\Gamma(\iota)} \int_1^{\mathcal{T}} \left(\log \frac{\mathcal{T}}{v}\right)^{\iota-1} \phi(\tau)\psi(\| \aleph \| + \| \vartheta \|) \frac{dv}{v} + \mathcal{K}^* \\
& \leq \| \phi \| \psi(\| \aleph \|)\mathcal{Y} + \mathcal{K}^*.
\end{aligned}$$

and, consequently,

$$\frac{\| \aleph \|_\infty}{\| \phi \| \psi(\| \aleph \| + \| \vartheta \|)\mathcal{Y} + \mathcal{K}^*} \leq 1$$

Then, by condition (A_5), $\exists \epsilon$: $\| \aleph \|_\infty \neq \varpi$, set

$$\gamma = \{\aleph \in \mathcal{PC}(\mathcal{J},\mathbb{R}) : \| \aleph \| < \varpi\}.$$

Obviously, the operator $\mathrm{Im} : \overline{\gamma} \to \mathcal{PC}(\mathcal{J},\mathbb{R})$ is completely continuous. From the choice of γ, there is no $\aleph \in \partial\gamma$: $\aleph = \delta\,\mathrm{Im}(\aleph)$ for some $\delta \in (0,1)$. As a result, by Leray–Schauder's nonlinear alternative theorem, $\mathcal{R}$ has an FP $\aleph \in \gamma$, which is a solution of (1)–(2). □

Theorem 5. *If Assumptions 3 and 4 hold, then the problem (1)–(3) has at least one solution for* $\mathcal{J}$*.*

Proof. Take $\mathcal{B}_\iota = \{\aleph \in \mathcal{PC}(\mathcal{J},\mathbb{R}) : |\aleph| \leq \iota\}$. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the two operators explained for $\mathcal{B}_\iota$ by

$$(\mathcal{T}_1\aleph)(\tau) = \tfrac{1}{\Gamma(\iota)} \int_1^\tau (\log \tfrac{\tau}{v})^{\iota-1} \ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau)) \tfrac{dv}{v} \tag{10}$$

and

$$
\begin{aligned}
(\mathcal{T}_2\aleph)(\tau) = & \frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)}\int_1^{\xi}\left(\log\frac{\xi}{v}\right)^{\iota+\varsigma-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{dv}{v} \\
& +\frac{|\pi|}{|\Lambda|\Gamma(\iota)}\int_1^{\mathcal{T}}\left(\log\frac{\mathcal{T}}{v}\right)^{\iota-1}\ell(\tau,\aleph(\tau),\mathcal{G}\aleph(\tau))\frac{\mathcal{G}v}{v}+\sum_{\mathtt{k}=1}^{\mathtt{m}}\vartheta_{\mathtt{i}},
\end{aligned}
\tag{11}
$$

respectively.

Note that $\aleph, \vartheta \in \mathcal{B}_\iota$, then $\mathcal{T}_1\aleph + \mathcal{T}_2\vartheta \in \mathcal{B}_\iota$. Check the inequality in the above equation as follows:

$$
\begin{aligned}
|\mathcal{T}_1\aleph+\mathcal{T}_2\vartheta| \leq & \frac{1}{\Gamma(\iota)}\int_1^{\tau}\left(\log\frac{\tau}{v}\right)^{\iota-1}\ell(\vartheta,\aleph(\vartheta),\mathcal{G}\aleph(v))\frac{dv}{v} \\
& +\frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)}\int_1^{\xi}\left(\log\frac{\xi}{v}\right)^{\iota+\varsigma-1}\ell(v,\aleph(v),\mathcal{G}\aleph(v))\frac{dv}{v} \\
& +\frac{|\pi|}{|\Lambda|\Gamma(\iota)}\int_1^{\mathcal{T}}\left(\log\frac{\mathcal{T}}{v}\right)^{\iota-1}\ell(v,\aleph(v),\mathcal{G}\aleph(v))\frac{\mathcal{D}v}{v}+\sum_{\mathtt{i}=1}^{\mathtt{m}}\vartheta_{\mathtt{i}}, \\
& \leq \frac{1}{\Gamma(\iota)}\int_1^{\tau}\left(\log\frac{\tau}{v}\right)^{\iota-1}|\ell(v,\aleph(v),\mathcal{G}\aleph(v))|\frac{dv}{v} \\
& +\frac{|\delta|}{|\Lambda|\Gamma(\iota+\varsigma)}\int_1^{\xi}\left(\log\frac{\xi}{v}\right)^{\iota+\varsigma-1}|\ell(v,\aleph(v),\mathcal{G}\aleph(v))|\frac{dv}{v} \\
& +\frac{|\pi|}{|\Lambda|\Gamma(\iota)}\int_1^{\mathcal{T}}\left(\log\frac{\mathcal{T}}{v}\right)^{\iota-1}|\ell(v,\aleph(v),\mathcal{G}\aleph(v))|\frac{dv}{v}+\left|\sum_{\mathtt{i}=1}^{\mathtt{m}}\vartheta_{\mathtt{i}}\right|, \\
& \leq \frac{\mathcal{W}_{\mathtt{g}}}{\Gamma(\iota)}\int_1^{\tau}\left(\log\frac{\tau}{v}\right)^{\iota-1}\frac{\mathcal{D}v}{v}+\frac{|\delta|(\mathcal{W}_{\mathtt{g}})}{|\Lambda|\Gamma(\iota+\varsigma)}\int_1^{\xi}\left(\log\frac{\xi}{v}\right)^{\iota+\varsigma-1}\frac{dv}{v} \\
& +\frac{|\pi|(\mathcal{W}_{\mathtt{g}})}{|\Lambda|\Gamma(\iota)}\int_1^{\mathcal{T}}\left(\log\frac{\mathcal{T}}{v}\right)^{\iota-1}\frac{dv}{v}+\mathcal{K}^*, \\
& \leq (\mathcal{W}_{\mathtt{g}})\left\{\frac{(\log\mathcal{T})^{\iota}}{\Gamma(\iota+1)}+\frac{|\delta|(\log\xi)^{\iota+\varsigma}}{|\Lambda|\Gamma(\iota+\varsigma+1)}+\frac{|\pi|(\log\mathcal{T})^{\iota}}{|\Lambda|\Gamma(\iota+1)}\right\}+\mathcal{K}^*, \\
& \leq (\mathcal{W}_{\mathtt{g}})\omega+\mathcal{K}^*, \\
& \leq \iota.
\end{aligned}
$$

Thus, $\mathcal{T}_1\aleph + \mathcal{T}_2\vartheta \in \mathcal{B}_\iota$ for all $\aleph, \vartheta \in \mathcal{B}_\iota$.

It is clear that $(\mathcal{T}_2)$ is a contraction map, and the continuity of $\aleph$ and the operator $(\mathcal{T}_1\aleph)(\tau)$ is continuous and observe that

$$
\begin{aligned}
|(\mathcal{T}_1\aleph)(\tau)| \leq & \frac{1}{\Gamma(\iota)}\int_1^{\tau}\left(\log\frac{\tau}{v}\right)^{\iota-1}\ell(v,\aleph(v),\mathcal{G}\aleph(v))\frac{dv}{v}, \\
\leq & \frac{(\log\mathcal{T})^{\iota}}{\Gamma(\iota+1)}(\mathcal{W}_{\mathtt{g}}).
\end{aligned}
$$

Hence, $(\mathcal{T}_1)$ is uniformly bounded on $\mathcal{B}_\iota$.

Now, we prove that $(\mathcal{T}_1\aleph)(\tau)$ is equicontinuous, and $\tau_1, \tau_2 \in \mathcal{J}, \tau_2 \leq \tau_1$ and $\aleph \in \mathcal{B}_\iota$. Since $\mathcal{T}_1$ is bounded on compact set

$$
\sup_{(\tau,\aleph,\vartheta)\in\mathcal{J}\times\mathcal{B}_\iota}|\ell(v,\aleph(v),\mathcal{G}\aleph(v))| := \mathcal{C}_0 < \infty,
$$

we will obtain

$$\begin{aligned}\| \mathcal{T}\aleph(\tau_2) - \mathcal{T}\aleph(\tau_1) \| \leq & \frac{1}{\Gamma(\iota)} \int_1^{\tau_1} \left[\left(\log \frac{\tau_2}{v} \right)^{\iota-1} - \left(\log \frac{\tau_1}{v} \right)^{\iota-1} \right] \| \ell(v, \aleph(v), \mathcal{G}\aleph(v)) \| \frac{dv}{v} \\ & + \frac{1}{\Gamma(\iota)} \int_{\tau_1}^{\tau_2} \left(\log \frac{\tau_2}{v} \right)^{\iota-1} \| \ell(v, \aleph(v), \mathcal{G}\aleph(v)) \| \frac{dv}{v} \\ & \leq \frac{\mathcal{C}_0}{\Gamma(\iota)} \int_1^{\tau_1} \left[\left(\log \frac{\tau_2}{v} \right)^{\iota-1} - \left(\log \frac{\tau_1}{v} \right)^{\iota-1} \right] \frac{dv}{v} \\ & + \frac{\mathcal{C}_0}{\Gamma(\iota)} \int_{\tau_1}^{\tau_2} \left(\log \frac{\tau_2}{v} \right)^{\iota-1} \frac{dv}{v} \\ & \leq \frac{\mathcal{C}_0}{\Gamma(\iota+1)} \left[(\log \tau_2)^{\iota} - (\log \tau_1)^{\iota} \right] \\ & \to 0 \quad \texttt{as} \quad \tau_2 \to \tau_1. \end{aligned}$$

Consequently, $\mathcal{T}_1(\mathcal{B}_\iota)$ is relatively compact, and, according to the Ascoli–Arzela theorem, $\mathcal{T}_1$ is compact. Then, the problem (1)–(3) has at least one FP on $\mathcal{J}$. □

The following example is used to verify our main results.

4. Example

Consider the problem for the C-HFDE:

$$ {}^{\mathcal{C}}_{\mathcal{H}}\mathcal{D}^{\frac{2}{3}} \aleph(\tau) = \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)), \quad (\tau, \aleph) \in \left([1, e], \mathbb{R}^+ \right), \tag{12}$$

$$\aleph(\tau_\kappa^+) = \aleph(\tau_\kappa^-) + \frac{1}{6}, \tag{13}$$

$$\aleph(1) + \aleph(e) = \frac{1}{2} \left(\mathcal{I}^{\frac{1}{2}} \aleph(2) \right) + \frac{3}{4}. \tag{14}$$

Here,

$$\begin{aligned} &\iota = \tfrac{2}{3}, \quad \varsigma = \tfrac{1}{2}, \quad \nu = 1, \quad \pi = 1, \\ &\varpi = \tfrac{3}{4}, \quad \delta = \tfrac{1}{2}, \quad \xi = 2, \quad \mathcal{T} = e. \end{aligned}$$

with

$$\begin{aligned} \ell(\tau, \aleph(\tau), \mathcal{G}\aleph(\tau)) &= \frac{1}{\tau^2 + 4} \cos \aleph + \mathcal{G}\aleph(\tau), \tau \in [1, e] \\ \mathcal{G}\aleph(\tau) &= \frac{1}{2} \int_0^\tau \mathfrak{e}^{-(v-\tau)} \aleph(v) dv. \end{aligned}$$

Hence, the hypothesis (A_1)–(A_2) is satisfied with $\mathcal{W}_1 = \mathcal{W}_2 = \frac{1}{4}, \quad \mathcal{Y} = \frac{1}{3}$.
Further,

$$\omega := \left\{ \frac{(\log \mathcal{T})^\iota}{\Gamma(\iota+1)} + \frac{|\delta|(\log \xi)^{\iota+\varsigma}}{|\Lambda|\Gamma(\iota+\varsigma+1)} + \frac{|\pi|(\log \mathcal{T})^\iota}{|\Lambda|\Gamma(\iota+1)} \right\} \simeq 2.0286,$$

and

$$(\mathcal{W}_1 + \mathcal{W}_2 \mathcal{Y})\omega \approx 0.6745 < 1.$$

Hence, (12)–(14) has a unique solution on $[1, e]$.

5. Conclusions

We obtained some existence results for nonlinear C-HFDEs with Hadamard IBCs by means of some standard FPTs and a nonlinear alternative of the Leray–Schauder type. The method was utilized to prove the problem's existence is a common one; however, it is presented in a novel way in the current framework. By providing some examples, the current work is also illustrated. Potential future directions could be to investigate a much more complicated class of BVPs. In addition, we intend to study the Ulam stability, the generalized Ulam stability of the problem.

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