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Linear Generalized n-Derivations on C^*-Algebras

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Abstract: Let $n \geq 2$ be a fixed integer and $\mathcal{A}$ be a C^*-algebra. A permuting n-linear map $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ is known to be symmetric generalized n-derivation if there exists a symmetric n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ such that $\mathcal{G}(\varsigma_1, \varsigma_2, \ldots, \varsigma_i\varsigma_i', \ldots, \varsigma_n) = \mathcal{G}(\varsigma_1, \varsigma_2, \ldots, \varsigma_i, \ldots, \varsigma_n)\varsigma_i' + \varsigma_i\mathfrak{D}(\varsigma_1, \varsigma_2, \ldots, \varsigma_i', \ldots, \varsigma_n)$ holds $\forall\, \varsigma_i, \varsigma_i' \in \mathcal{A}$. In this paper, we investigate the structure of C^*-algebras involving generalized linear n-derivations. Moreover, we describe the forms of traces of linear n-derivations satisfying certain functional identity.

Keywords: linear derivation; generalized n-derivation; Lie ideal; Banach algebra; C^*-algebra

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1. Introduction

A Banach algebra is a linear associate algebra which, as a vector space, is a Banach space with norm $||.||$ satisfying the multiplicative inequality; $||\varsigma\varepsilon|| \leq ||\varsigma||||\varepsilon|| \;\forall\, \varsigma$ and ε in $\mathcal{A}$. "An involution on an algebra $\mathcal{A}$ is a linear map $\varsigma \mapsto \varsigma^*$ of $\mathcal{A}$ into itself such that the following conditions hold: (i) $(\varsigma\varepsilon)^* = \varepsilon^*\varsigma^*$, (ii) $(\varsigma^*)^* = \varsigma$, and (iii) $(\varsigma + \lambda\varepsilon)^* = \varsigma^* + \bar{\lambda}\varepsilon^* \;\forall\, \varsigma, \varepsilon \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, the field of complex numbers, where $\bar{\lambda}$ is the conjugate of λ. An algebra equipped with an involution $*$ is called a $*$-algebra or algebra with involution. A Banach $*$-algebra is a Banach algebra $\mathcal{A}$ together with an isometric involution $||\varsigma^*|| = ||\varsigma|| \;\forall\, \varsigma \in \mathcal{A}$. A C^*-algebra $\mathcal{A}$ is a Banach $*$-algebra with the additional norm condition $||\varsigma^*\varsigma|| = ||\varsigma||^2 \;\forall\, \varsigma \in \mathcal{A}$.

Throughout this discussion, unless otherwise mentioned, $\mathcal{A}$ will denote C^*-algebra with $\mathcal{Z}(\mathcal{A})$ as its center. However, $\mathcal{A}$ may or may not have unity. The symbols $[\varsigma, \varepsilon]$ and $\varsigma \circ \varepsilon$ denote the commutator $\varsigma\varepsilon - \varepsilon\varsigma$ and the anti-commutator $\varsigma\varepsilon + \varepsilon\varsigma$, respectively, for any $\varsigma, \varepsilon \in \mathcal{A}$. An algebra $\mathcal{A}$ is said to be prime if $\varsigma\mathcal{A}\varepsilon = \{0\}$ implies that either $\varsigma = 0$ or $\varepsilon = 0$, and semiprime if $\varsigma\mathcal{A}\varsigma = \{0\}$ implies that $\varsigma = 0$, where $\varsigma, \varepsilon \in \mathcal{A}$. An additive subgroup U of $\mathcal{A}$ is said to be a Lie ideal of $\mathcal{A}$ if $[u, r] \in U, \forall\, u \in U, r \in \mathcal{A}$. U is called a square-closed Lie ideal of $\mathcal{A}$ if U is a Lie ideal and $u^2 \in U \;\forall\, u \in U$. A linear operator $\mathcal{D}$ on a C^*-algebra $\mathcal{A}$ is called a derivation if $\mathcal{D}(\varsigma\varepsilon) = \mathcal{D}(\varsigma)\varepsilon + \varsigma\mathcal{D}(\varepsilon)$ holds $\forall\, \varsigma, \varepsilon \in \mathcal{A}$. Consider the inner derivation δ_a implemented by an element a in $\mathcal{A}$, which is defined as $\delta_a(\varsigma) = \varsigma a - a\varsigma$ for every ς in $\mathcal{A}$, as a typical example of a nonzero derivation in a noncommutative algebra.

In order to broaden the scope of derivation, Maksa [1] introduced the concept of symmetric bi-derivations. A bi-linear map $\mathfrak{D} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is said to be a bi-derivation if

$$\mathfrak{D}(\varsigma\varsigma', \varepsilon) = \mathfrak{D}(\varsigma, \varepsilon)\varsigma' + \varsigma\mathfrak{D}(\varsigma', \varepsilon)$$

$$\mathfrak{D}(\varsigma, \varepsilon\varepsilon') = \mathfrak{D}(\varsigma, \varepsilon)\varepsilon' + \varepsilon\mathfrak{D}(\varsigma, \varepsilon')$$

holds for any $\varsigma, \varsigma', \varepsilon, \varepsilon' \in \mathcal{A}$. The foregoing conditions are identical if $\mathfrak{D}$ is also a symmetric map, whereby if $\mathfrak{D}(\varsigma, \varepsilon) = \mathfrak{D}(\varepsilon, \varsigma)$ for every $\varsigma, \varepsilon \in \mathcal{A}$. In this case, $\mathfrak{D}$ is referred to as a symmetric bi-derivation of $\mathcal{A}$. Vukman [2] investigated symmetric bi-derivations in prime

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and semiprime rings. Argao and Yenigül ([3], Chapter 3) and Muthana [4] obtained the similar type of results on Lie ideals of ring R.

In this paper, we briefly discuss the various extensions of the notion of derivations on C^*-algebras. The most general and important one among them is the notion of symmetric linear generalized n-derivations on C^*-algebras. Suppose n is a fixed positive integer and $\mathcal{A}^n = \mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}$. A map $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ is said to be symmetric (permuting) if the relation $\mathfrak{D}(\varsigma_1, \varsigma_2, \ldots, \varsigma_n) = \mathfrak{D}(\varsigma_{\pi(1)}, \varsigma_{\pi(2)}, \ldots, \varsigma_{\pi(n)})$ holds $\forall\ \varsigma_i \in \mathcal{A}$, $1 \leq i \leq n$ and for every permutation $\{\pi(1), \pi(2), \ldots, \pi(n)\}$. The concept of derivation and symmetric bi-derivation was generalized by Park [5] as follows: a n-linear map $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ is said to be a symmetric (permuting) linear n-derivation if $\mathfrak{D}$ is permuting and $\mathfrak{D}(\varsigma_1, \varsigma_2, \ldots, \varsigma_i\varsigma_i', \ldots, \varsigma_n) = \mathfrak{D}(\varsigma_1, \varsigma_2, \ldots, \varsigma_n)\varsigma_i' + \varsigma_i\mathfrak{D}(\varsigma_1', \varsigma_2, \ldots, \varsigma_n)$ hold $\forall$ $\varsigma_i, \varsigma_i' \in \mathcal{A}$, $i = 1, 2, \ldots, n$. A map $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ defined by $\mathscr{d}(\varsigma) = \mathcal{D}(\varsigma, \varsigma, \ldots, \varsigma)$ is called the trace of $\mathfrak{D}$. If $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ is permuting and n-linear, then the trace $\mathscr{d}$ of $\mathfrak{D}$ satisfies the relation

$$\mathscr{d}(\varsigma + \varepsilon) = \mathscr{d}(\varsigma) + \mathscr{d}(\varepsilon) + \sum_{l=1}^{n-1} {}^nC_l\ h_l(\varsigma; \varepsilon)$$

$\forall\ \varsigma, \varepsilon \in \mathcal{A}$, where ${}^nC_l = \binom{n}{l}$ and

$$h_l(\varsigma; \varepsilon) = \mathfrak{D}(\underbrace{\varsigma, \ldots, \varsigma}_{(n-l)\text{-times}}, \underbrace{\varepsilon, \ldots, \varepsilon}_{l\text{-times}}).$$

Ashraf et al. [6] introduced the notion of symmetric generalized n-derivations in a ring, building upon the concept of generalized derivation. Let $n \geq 1$ be a fixed positive integer. A symmetric n-linear map $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ is known to be symmetric linear generalized n-derivation if there exists a symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ such that $\mathcal{G}(\varsigma_1, \varsigma_2, \ldots, \varsigma_i\varsigma_i', \ldots, \varsigma_n) = \mathcal{G}(\varsigma_1, \varsigma_2, \ldots, \varsigma_i, \ldots, \varsigma_n)\varsigma_i' + \varsigma_i\mathfrak{D}(\varsigma_1, \varsigma_2, \ldots, \varsigma_i', \ldots, \varsigma_n)$ holds $\forall$ $\varsigma_i, \varsigma_i' \in \mathcal{A}''$.

Example 1. *Let*

$$\mathcal{A} = \left\{ \begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix} \mid a \in \mathbb{C} \right\},$$

where $\mathbb{C}$ is a complex field. Next, define an involution $*$ to be the identity map. It is clear that $\mathcal{A}$ is a C^*-algebra under norm defined by $||A|| = |a|$ for all $A \in \mathcal{A}$. Denote $A_i = \begin{bmatrix} a_i & a_i \\ 0 & 0 \end{bmatrix} \in \mathcal{A}$, $a_i \in \mathbb{C}$, $1 \leq i \leq n$, and let us define $\mathcal{G} = \mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ by $\mathfrak{D}(A_1, A_2, \ldots, A_n) = \begin{bmatrix} 0 & a_1a_2\cdots a_n \\ 0 & 0 \end{bmatrix}$ with trace $g = \mathscr{d} : \mathcal{A} \to \mathcal{A}$ define by $g\left(\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & a^n \\ 0 & 0 \end{bmatrix}$. Then it is easy to see that $\mathcal{G}$ is a symmetric linear generalized n-derivation on $\mathcal{A}$.

There has been notable scholarly focus on the structure of linear derivations and linear bi-derivations within the context of C^*-algebras. Various authors have provided diverse expositions of derivations on C^*-algebras, showcasing a spectrum of perspectives and methodologies. For instance, Kadison's work in 1966 [7] demonstrated that every linear derivation acting on a C^*-algebra annihilates its center. In 1989, Mathieu [8] built upon Posner's first theorem [9] regarding C^*-algebras, extending its implications. Basically, he proved that "if the product of two linear derivations d and d' on a C^*-algebra is a linear derivation then $dd' = 0$". Very recently, Ekrami and Mirzavaziri [10] showed that "if $\mathcal{A}$ is a C^*-algebra admitting two linear derivations d and d' on $\mathcal{A}$, then there exists a linear derivation D on $\mathcal{A}$ such that $dd' + d'd = D^2$ if and only if d and d' are linearly dependent".

In [11], Ali and Khan proved that if $\mathcal{A}$ is a C^*-algebra admitting a symmetric bilinear generalized $*$-biderivation $\mathcal{H} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ with an associated symmetric bilinear $*$-biderivation $\mathcal{B} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, then $\mathcal{H}$ maps $\mathcal{A} \times \mathcal{A}$ into $Z(\mathcal{A})$. In [12], Rehman and Ansari provided a characterization of the trace of symmetric bi-derivations, and they proved more general results by examining different conditions on a subset of the ring R, specifically the Lie ideal of R. Basically, they proved that "let R be a prime ring with $charR \neq 2$ and U be a square closed Lie ideal of R. Suppose that $B : R \times R \to R$ is a symmetric bi-derivation and f, the trace of B. If $[f(x), x] = 0 \,\forall\, x \in U$, then either $U \subseteq Z(R)$ or $f = 0$" (see also [13–19] for recent results).

The motivation behind this research stems from the seminal works of Ali and Khan [11], as well as Rehman and Ansari [12], who explored the intricate connections between bilinear biderivations and algebraic structures within C^*-algebras and prime rings, respectively. In this study, we extend the above mentioned inquiry to the realm of linear generalized n-derivations in C^*-algebras. Focusing specifically on Lie ideals within these algebras, we aim to uncover broader outcomes and novel insights into the intricate relationships between linear generalized n-derivations and algebraic structure of C^*-algebras. By scrutinizing the behavior of linear generalized n-derivations within Lie ideals, our research seeks to elucidate their role in the algebraic landscape, contributing to a deeper understanding of the underlying principles governing linear generalized n-derivations in C^*-algebras. Precisely, we prove that if $\mathcal{A}$ is a C^*-algebra, U is a square closed Lie ideal of $\mathcal{A}$ admitting a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathcal{d} : \mathcal{A} \to \mathcal{A}$ satisfying the condition $(g(\varsigma\varepsilon) - g(\varepsilon\varsigma)) \pm [\varsigma, \varepsilon] \in Z(\mathcal{A}) \,\forall\, \varsigma, \varepsilon \in U$, then $U \subseteq Z(\mathcal{A})$.

2. The Results

To initiate the substantiation of our primary theorems, we first articulate a result that we frequently invoke in the demonstration of our principal outcomes.

Lemma 1 ([20], Corollary 2.1)**.** *"Let R be a 2-torsion free semiprime ring, U a Lie ideal of R such that $U \nsubseteq Z(R)$ and $a, b \in U$.*

1. *If $aUa = \{0\}$, then $a = 0$.*
2. *If $aU = \{0\}$ ($Ua = \{0\}$), then $a = 0$.*
3. *If U is a square closed Lie ideal and $aUb = \{0\}$, then $ab = 0$ and $ba = 0$.*

Lemma 2 ([21], Lemma 1)**.** *Let R be a semiprime, 2 torsion-free ring and let U be a Lie ideal of R. Suppose that $[U, U] \subseteq Z(R)$, then $U \subseteq Z(R)$.*

Lemma 3 ([22])**.** *Let n be a fixed positive integer and R a $n!$-torsion free ring. Suppose that $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in R$ satisfy $\lambda\varepsilon_1 + \lambda^2\varepsilon_2 + \cdots + \lambda^n\varepsilon_n = 0$ for $\lambda = 1, 2, \ldots, n$. Then $\varepsilon_i = 0$ for $i = 1, 2, \ldots, n$".*

Daif and Bell [23] proved that if a semiprime ring admits a derivation d such that either $\varsigma\varepsilon - d(\varsigma\varepsilon) = \varepsilon\varsigma - d(\varepsilon\varsigma)$ or $\varsigma\varepsilon + d(\varsigma\varepsilon) = \varepsilon\varsigma + d(\varepsilon\varsigma)$ holds $\forall\, \varsigma, \varepsilon \in R$, then R is commutative. In this section, apart from proving other results, we expand the previous result by demonstrating the following theorem for the traces of generalized linear n-derivation on well behaved subsets of $\mathcal{A}$.

Theorem 1. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra, U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathcal{d} : \mathcal{A} \to \mathcal{A}$ satisfying the condition of $(g(\varsigma\varepsilon) - g(\varepsilon\varsigma)) \pm [\varsigma, \varepsilon] \in Z(\mathcal{A}) \,\forall\, \varsigma, \varepsilon \in U$, then $U \subseteq Z(\mathcal{A})$.*

Proof. It is given that

$$(g(\varsigma\varepsilon) - g(\varepsilon\varsigma)) \pm [\varsigma, \varepsilon] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U.$$

Replacing ε by $\varsigma + m\varepsilon$, where $1 \leq m \leq n-1$ in the given condition, we obtain

$$g(\varsigma(\varsigma + m\varepsilon)) - g((\varsigma + m\varepsilon)\varsigma) \pm [\varsigma, \varsigma + m\varepsilon] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U$$

which on solving, we have

$$g(\varsigma m\varepsilon) - g(m\varepsilon\varsigma) + \sum_{l=1}^{n-1} {}^{n}C_{l}\mathcal{G}(\underbrace{\varsigma^2, \ldots, \varsigma^2}_{(n-l)-\text{times}}, \underbrace{\varsigma m\varepsilon, \ldots, \varsigma m\varepsilon}_{l-\text{times}}) - \sum_{l=1}^{n-1} {}^{n}C_{l}\mathcal{G}(\underbrace{\varsigma^2, \ldots, \varsigma^2}_{(n-l)-\text{times}}, \underbrace{m\varepsilon\varsigma, \ldots, m\varepsilon\varsigma}_{l-\text{times}})$$
$$\pm [\varsigma, m\varepsilon] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U. \quad (1)$$

By using hypothesis, we obtain

$$\sum_{l=1}^{n-1} {}^{n}C_{l}\mathcal{G}(\underbrace{\varsigma^2, \ldots, \varsigma^2}_{(n-l)-\text{times}}, \underbrace{\varsigma m\varepsilon, \ldots, \varsigma m\varepsilon}_{l-\text{times}}) - \sum_{l=1}^{n-1} {}^{n}C_{l}\mathcal{G}(\underbrace{\varsigma^2, \ldots, \varsigma^2}_{(n-l)-\text{times}}, \underbrace{m\varepsilon\varsigma, \ldots, m\varepsilon\varsigma}_{l-\text{times}}) \pm [\varsigma, m\varepsilon] \in Z(\mathcal{A})$$

$\forall \ \varsigma, \varepsilon \in U$. Making use of Lemma 3, we see that

$$\mathcal{G}(\varsigma^2, \ldots, \varsigma^2, \varsigma\varepsilon) - \mathcal{G}(\varsigma^2, \ldots, \varsigma^2, \varepsilon\varsigma) \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U. \quad (2)$$

For $1 \leq m \leq n$, (1) can also be written as

$$m^n g(\varsigma\varepsilon) - m^n g(\varepsilon\varsigma) + \sum_{l=1}^{n-1} {}^{n}C_{l}\mathcal{G}(\underbrace{\varsigma^2, \ldots, \varsigma^2}_{(n-l)-\text{times}}, \underbrace{\varsigma m\varepsilon, \ldots, \varsigma m\varepsilon}_{l-\text{times}}) - \sum_{l=1}^{n-1} {}^{n}C_{l}\mathcal{G}(\underbrace{\varsigma^2, \ldots, \varsigma^2}_{(n-l)-\text{times}}, \underbrace{m\varepsilon\varsigma, \ldots, m\varepsilon\varsigma}_{l-\text{times}})$$
$$\pm [\varsigma, m\varepsilon] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U.$$

Again making use of Lemma 3, we have

$$n\{\mathcal{G}(\varsigma^2, \ldots, \varsigma^2, \varsigma\varepsilon) - \mathcal{G}(\varsigma^2, \ldots, \varsigma^2, \varepsilon\varsigma)\} \pm [\varsigma, \varepsilon] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U. \quad (3)$$

From (2) and (3), we obtain $[\varsigma, \varepsilon] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U$. As every C^*-algebra is a semiprime ring, using Lemma 2, we obtain $U \subseteq Z(\mathcal{A})$. □

Theorem 2. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra and U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying the condition of $(g(\varsigma) \pm g(\varepsilon)) \pm \varsigma \circ \varepsilon \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U$, then $U \subseteq Z(\mathcal{A})$.*

Proof. Suppose on the contrary that $U \not\subseteq Z(\mathcal{A})$. We have given that

$$(g(\varsigma) - g(\varepsilon)) \pm \varsigma \circ \varepsilon \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U.$$

Replacing ε by $\varsigma + m\varepsilon$, where $z \in U$ and $1 \leq m \leq n-1$ in the given condition, we obtain

$$g(\varsigma) \pm g(\varsigma + m\varepsilon) \pm (\varsigma \circ \varsigma + m\varepsilon) \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon, z \in U$$

which on solving, we have

$$g(\varsigma) \pm g(\varsigma)g(m\varepsilon) \pm \sum_{l=1}^{n-1} {}^{n}C_{l}\mathcal{G}(\underbrace{\varsigma, \ldots, \varsigma}_{(n-l)-\text{times}}, \underbrace{m\varepsilon, \ldots, m\varepsilon}_{l-\text{times}}) \pm \varsigma \circ \varsigma \pm \varsigma \circ m\varepsilon \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon, z \in U. \quad (4)$$

Using the given condition, we obtain

$$g(\varsigma) \pm \varsigma^2 \pm \sum_{l=1}^{n-1} {}^nC_l \mathcal{G}(\underbrace{\varsigma, \ldots, \varsigma}_{(n-l)-\text{times}}, \underbrace{m\varepsilon, \ldots, m\varepsilon}_{l-\text{times}}) \in Z(\mathcal{A}) \;\forall\; \varsigma, \varepsilon, z \in U.$$

Multiply the above equation by m which implies that

$$mA_1(\varsigma, \varepsilon) + m^2 A_2(\varsigma, \varepsilon) + \cdots + m^{n-1} A_{n-1}(\varsigma, \varepsilon) \in Z(\mathcal{A})$$

$\forall\, \varsigma, \varepsilon, z \in U$ where $A_l(\varsigma, \varepsilon)$ represents the term in which z appears l-times.

Making use of Lemma 3, we see that

$$\mathcal{G}(\varsigma, \ldots, \varsigma, \varepsilon) \in Z(\mathcal{A}) \;\forall\; \varsigma, \varepsilon \in U.$$

Replace ε by ς, we obtain

$$g(\varsigma) \in Z(\mathcal{A}) \;\forall\; \varsigma, \varepsilon \in U.$$

From hypothesis, we have $\varsigma \circ \varepsilon \in Z(\mathcal{A}) \;\forall\; \varsigma, \varepsilon \in U$. Again replace ς by $\varepsilon\varsigma$, we have $\varepsilon(\varsigma \circ \varepsilon) \in Z(\mathcal{A})$ which imply $[\varepsilon(\varsigma \circ \varepsilon), z] \in Z(\mathcal{A})$. On solving, we obtain $[\varepsilon, z](\varsigma \circ \varepsilon) = 0$ $\forall\, \varsigma, \varepsilon, z \in U$. Again replace ς by ςz, we have $[\varepsilon, z]\varsigma[z, \varepsilon] = 0 \;\forall\; \varsigma, \varepsilon, z \in U$. By Lemma 1, we have $[z, \varepsilon] = 0 \;\forall\; \varepsilon, z \in U$. Again using Lemma 2, we obtain $U \subseteq Z(\mathcal{A})$, which is a contradiction. □

Theorem 3. *Let $\mathcal{A}$ be a C^*-algebra and U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying $g(\varsigma^2) \pm \varsigma^2 = 0 \;\forall$ $\varsigma \in U$, then $U \subseteq Z(\mathcal{A})$.*

Proof. Suppose on the contrary that $U \nsubseteq Z(\mathcal{A})$. We have given that $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ be symmetric linear generalized n-derivations associated with $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ of a C^*-algebra $\mathcal{A}$ such that $g(\varsigma^2) \pm \varsigma^2 = 0 \;\forall\; \varsigma \in U$. Therefore, $\mathcal{A}$ is semiprime as $\mathcal{A}$ is a C^*-algebra. Now replacing ς by $\varsigma + m\varepsilon$, $\varepsilon \in U$ for $1 \leq m \leq n-1$ in the given condition, we obtain

$$g(\varsigma + m\varepsilon)^2 \pm (\varsigma + m\varepsilon)^2 = 0 \;\forall\; \varsigma, \varepsilon \in U.$$

Further solving, we have

$$\begin{aligned} g(\varsigma^2) + g(m(\varsigma\varepsilon + \varepsilon\varsigma)) + \sum_{l=1}^{n-1} {}^nC_l \mathcal{G}(\underbrace{\varsigma^2, \ldots, \varsigma^2}_{(n-l)-\text{times}}, \underbrace{m(\varsigma\varepsilon + \varepsilon\varsigma), \ldots, m(\varsigma\varepsilon + \varepsilon\varsigma)}_{l-\text{times}}) + \\ g((m\varepsilon)^2) + \sum_{l=1}^{n-1} {}^nC_l \mathcal{G}(\underbrace{\varsigma^2 + m(\varsigma\varepsilon + \varepsilon\varsigma), \ldots, \varsigma^2 + m(\varsigma\varepsilon + \varepsilon\varsigma)}_{(n-l)-\text{times}}, \underbrace{(m\varepsilon)^2, \ldots, (m\varepsilon)^2}_{l-\text{times}}) \\ \pm \varsigma^2 \pm (m\varepsilon)^2 \pm m(\varsigma\varepsilon + \varepsilon\varsigma) = 0 \;\forall\; \varsigma, \varepsilon \in U. \end{aligned}$$

In accordance of the given condition and Lemma 3, we obtain

$$n\mathcal{G}(\varsigma^2, \ldots, \varsigma^2, \varsigma\varepsilon + \varepsilon\varsigma) \pm (\varsigma\varepsilon + \varepsilon\varsigma) = 0 \;\forall\; \varsigma, \varepsilon \in U.$$

Replacing ε by ς, we find that

$$2ng(\varsigma^2) \pm 2\varsigma^2 = 0,$$

or

$$\varsigma^2 = 0.$$

This implies that $\varsigma\varepsilon + \varepsilon\varsigma = 0 \ \forall \ \varsigma, \varepsilon \in U$. Replacing ε by εz, where $z \in U$, we obtain $[\varsigma, \varepsilon]z = 0$. Again replacing z by $z[\varsigma, \varepsilon]$, we obtain $[\varsigma, \varepsilon]z[\varsigma, \varepsilon] = 0 \ \forall \ \varsigma, \varepsilon, z \in U$. Using the Lemma 1, we obtain $[\varsigma, \varepsilon] = 0 \ \forall \ \varsigma, \varepsilon \in U$. By Lemma 2, we obtain $U \subseteq Z(\mathcal{A})$, a contradiction. □

Corollary 1. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra and U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $d : \mathcal{A} \to \mathcal{A}$ satisfying $g(\varsigma \circ \varepsilon) \pm \varsigma \circ \varepsilon = 0 \ \forall \ \varsigma, \varepsilon \in U$, then $U \subseteq Z(\mathcal{A})$.*

Theorem 4. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra and U be a square closed Lie ideal of $\mathcal{A}$. Let $\mathcal{A}$ admit a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $d : \mathcal{A} \to \mathcal{A}$ satisfying one of the following conditions:*

(i) $[g(\varsigma), g(\varepsilon)] - [\varsigma, \varepsilon] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U$

(ii) $[g(\varsigma), g(\varepsilon)] - [\varepsilon, \varsigma] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U$.

Then, $U \subseteq Z(\mathcal{A})$.

Proof. (i) Given that

$$[g(\varsigma), g(\varepsilon)] - [\varsigma, \varepsilon] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U. \tag{5}$$

Consider a positive integer m; $1 \leq m \leq n-1$. Replacing ε by $\varepsilon + mz$, where $z \in U$ in (5), we obtain

$$[g(\varsigma), g(\varepsilon + mz)] - [\varsigma, \varepsilon + mz] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon, z \in U.$$

On further solving, we obtain

$$[g(\varsigma), g(\varepsilon)] + [g(\varsigma), g(mz)] + [g(\varsigma), \sum_{l=1}^{n-1} {}^nC_l \mathcal{G}(\underbrace{\varepsilon, \ldots, \varepsilon}_{(n-l)-\text{times}}, \underbrace{mz, \ldots, mz}_{l-\text{times}})] - [\varsigma, \varepsilon] - [\varsigma, mz] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon, z \in U.$$

On taking account of hypothesis, we see that

$$mA_1(\varsigma, \varepsilon, z) + m^2 A_2(\varsigma, \varepsilon, z) + \cdots + m^{n-1} A_{n-1}(\varsigma, \varepsilon, z) \in Z(\mathcal{A})$$

where $A_l(\varsigma, \varepsilon, z)$ represents the term in which z appears l-times.
Using Lemma 3, we have

$$[g(\varsigma), \mathcal{G}(\varepsilon, \ldots, \varepsilon, z)] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon, z \in U.$$

In particular, for $z = \varepsilon$, we obtain

$$[g(\varsigma), g(\varepsilon)] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U.$$

Now using the given condition, we find that

$$[\varsigma, \varepsilon] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U.$$

From Lemma 2, $U \subseteq Z(\mathcal{A})$.
(ii) Follows from the first implication with a slight modification. □

Corollary 2. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra and U be a square closed Lie ideal of $\mathcal{A}$. Let $\mathcal{A}$ admit a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $d : \mathcal{A} \to \mathcal{A}$ satisfying one of the following conditions:*

(i) $g(\varsigma)g(\varepsilon) \pm \varsigma\varepsilon \in Z(\mathcal{A})\ \forall\ \varsigma, \varepsilon \in U$
(ii) $g(\varsigma)g(\varepsilon) \pm \varepsilon\varsigma \in Z(\mathcal{A})\ \forall\ \varsigma, \varepsilon \in U$.
Then, $U \subseteq Z(\mathcal{A})$.

Corollary 3. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra and U be a square closed Lie ideal of $\mathcal{A}$. Let $\mathcal{A}$ admit a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying one of the following conditions:*
(i) $[g(\varsigma), g(\varepsilon)] = [\varsigma, \varepsilon]\ \forall\ \varsigma, \varepsilon \in U$
(ii) $[g(\varsigma), g(\varepsilon)] = [\varepsilon, \varsigma]\ \forall\ \varsigma, \varepsilon \in U$.
Then, $U \subseteq Z(\mathcal{A})$.

Theorem 5. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra and U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying the condition $g(\varsigma \circ \varepsilon) \pm [\varsigma, \varepsilon] \in Z(\mathcal{A})\ \forall\ \varsigma, \varepsilon \in U$, then $U \subseteq Z(\mathcal{A})$.*

Proof. Replacing ε by $\varepsilon + mz$ for $1 \leq m \leq n-1$, $z \in U$ in the given condition, we obtain

$$g(\varsigma \circ (\varepsilon + mz)) \pm [\varsigma, \varepsilon + mz] \in Z(\mathcal{A})\ \forall\ \varsigma, \varepsilon, z \in U.$$

On further solving and using the specified condition, we obtain

$$\sum_{l=1}^{n-1} {}^{n}C_l \mathcal{G}(\underbrace{\varsigma \circ \varepsilon, \ldots, \varsigma \circ \varepsilon}_{(n-l)-\text{times}}, \underbrace{\varsigma \circ mz, \ldots, \varsigma \circ mz}_{l-\text{times}}) \in Z(\mathcal{A})\ \forall\ \varsigma, \varepsilon, z \in U$$

which implies that

$$mA_1(\varsigma, \varepsilon, z) + m^2 A_2(\varsigma, \varepsilon, z) + \cdots + m^{n-1} A_{n-1}(\varsigma, \varepsilon, z) \in Z(\mathcal{A})$$

$\forall\ \varsigma, \varepsilon, z \in U$ where $A_l(\varsigma, \varepsilon, z)$ represents the term in which z appears l-times. Using Lemma 3, we obtain

$$\mathcal{G}(\varsigma \circ \varepsilon, \ldots, \varsigma \circ \varepsilon, \varsigma \circ z) \in Z(\mathcal{A})\ \forall\ \varsigma, \varepsilon, z \in U. \tag{6}$$

For $z = \varepsilon$, we obtain $g(\varsigma \circ \varepsilon) \in Z(\mathcal{A})$ then our hypothesis reduces to $[\varsigma, \varepsilon] \in Z(\mathcal{A})$. Using the Lemma 2, we obtain $U \subseteq Z(\mathcal{A})$. □

Corollary 4. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra and U be a square closed Lie ideal of $\mathcal{A}$. If $\mathcal{A}$ admits a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying the condition $\mathscr{d}(\varsigma \circ \varepsilon) \pm [\varsigma, \varepsilon] \in Z(\mathcal{A})\ \forall\ \varsigma, \varepsilon \in U$, then $U \subseteq Z(\mathcal{A})$.*

Theorem 6. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra and U be a square closed Lie ideal of $\mathcal{A}$. Let $\mathcal{A}$ admit a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\mathscr{d} : \mathcal{A} \to \mathcal{A}$ satisfying one of the following conditions:*
(i) $g([\varsigma, \varepsilon]) \pm g(\varsigma) \pm [\varsigma, \varepsilon] \in Z(\mathcal{A})\ \forall\ \varsigma, \varepsilon \in U$
(ii) $g([\varsigma, \varepsilon]) \pm g(\varepsilon) \pm [\varsigma, \varepsilon] \in Z(\mathcal{A})\ \forall\ \varsigma, \varepsilon \in U$.
Then, $U \subseteq Z(\mathcal{A})$.

Proof. (i) Given that

$$g([\varsigma, \varepsilon]) \pm g(\varsigma) \pm [\varsigma, \varepsilon] \in Z(\mathcal{A})\ \forall\ \varsigma, \varepsilon \in U.$$

Replacing ς by $\varsigma + mz$, where $z \in U$ and $1 \le m \le n-1$ in the given condition, we obtain

$$g([\varsigma + mz, \varepsilon]) \pm g(\varsigma + mz) \pm [\varsigma + mz, \varepsilon] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U$$

which on solving and using hypothesis, we obtain

$$\sum_{l=1}^{n-1} {}^{n}C_{l}\mathcal{G}(\underbrace{[\varsigma,\varepsilon],\ldots,[\varsigma,\varepsilon]}_{(n-l)-\text{times}},\underbrace{[mz,\varepsilon],\ldots,[mz,\varepsilon]}_{l-\text{times}}) \pm \sum_{l=1}^{n-1} {}^{n}C_{l}\mathcal{G}(\underbrace{\varsigma,\ldots,\varsigma}_{(n-l)-\text{times}},\underbrace{mz,\ldots,mz}_{l-\text{times}}) \in Z(\mathcal{A}) \ \forall \ \varsigma,\varepsilon,z \in U$$

which implies that

$$mA_1(\varsigma,\varepsilon,z) + m^2 A_2(\varsigma,\varepsilon,z) + \cdots + m^{n-1}A_{n-1}(\varsigma,\varepsilon,z) \in Z(\mathcal{A})$$

$\forall \ \varsigma,\varepsilon,z \in U$ where $A_l(\varsigma,\varepsilon,z)$ represents the term in which z appears l-times.

Making use of Lemma 3 and torsion restriction, we see that

$$\mathcal{G}([\varsigma,\varepsilon],\ldots,[\varsigma,\varepsilon],[z,\varepsilon]) \pm \mathcal{G}(\varsigma,\ldots,\varsigma,z) \in Z(\mathcal{A}) \ \forall \ \varsigma,\varepsilon,z \in U.$$

Replace z by ς to obtain

$$g([\varsigma,\varepsilon]) \pm g(\varsigma) \in Z(\mathcal{A}) \ \forall \ \varsigma,\varepsilon,z \in U.$$

Hence, by using the given condition, we find that $[\varsigma,\varepsilon] \in Z(\mathcal{A})$. On taking account of Lemma 2, we obtain $U \subseteq Z(\mathcal{A})$.
(ii) Given that

$$g([\varsigma,\varepsilon]) \pm g(\varsigma) \pm [\varsigma,\varepsilon] \in Z(\mathcal{A}) \ \forall \ \varsigma,\varepsilon \in U.$$

Replacing ε by $\varepsilon + mz$, where $z \in U$ and $1 \le m \le n-1$ in the given condition, we obtain

$$g([\varsigma, \varepsilon + mz]) \pm g(\varepsilon + mz) \pm [\varsigma, \varepsilon + mz] \in Z(\mathcal{A}) \ \forall \ \varsigma, \varepsilon \in U$$

which on solving and using hypothesis, we obtain

$$\sum_{l=1}^{n-1} {}^{n}C_{l}\mathcal{G}(\underbrace{[\varsigma,\varepsilon],\ldots,[\varsigma,\varepsilon]}_{(n-l)-\text{times}},\underbrace{[\varsigma,mz],\ldots,[\varsigma,mz]}_{l-\text{times}}) \pm \sum_{l=1}^{n-1} {}^{n}C_{l}\mathcal{G}(\underbrace{\varepsilon,\ldots,\varepsilon}_{(n-l)-\text{times}},\underbrace{mz,\ldots,mz}_{l-\text{times}}) \in Z(\mathcal{A}) \ \forall \ \varsigma,\varepsilon,z \in U$$

which implies that

$$mA_1(\varsigma,\varepsilon,z) + m^2 A_2(\varsigma,\varepsilon,z) + \cdots + m^{n-1}A_{n-1}(\varsigma,\varepsilon,z) \in Z(\mathcal{A})$$

$\forall \ \varsigma,\varepsilon,z \in U$ where $A_l(\varsigma,\varepsilon,z)$ represents the term in which z appears l-times.

Making use of Lemma 3 and torsion restriction, we see that

$$\mathcal{G}([\varsigma,\varepsilon],\ldots,[\varsigma,\varepsilon],[\varsigma,z]) \pm \mathcal{G}(\varepsilon,\ldots,\varepsilon,z) \in Z(\mathcal{A}) \ \forall \ \varsigma,\varepsilon,z \in U.$$

Replace z by ε to obtain

$$g([\varsigma,\varepsilon]) \pm g(\varepsilon) \in Z(\mathcal{A}) \ \forall\ \varsigma,\varepsilon,z \in U.$$

Hence, by using the given condition, we find that $[\varsigma,\varepsilon] \in Z(\mathcal{A})$. On taking account of Lemma 2, we obtain $U \subseteq Z(\mathcal{A})$.
(iii) Follows from the first implication with a slight modification. □

Corollary 5. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra and U be a square closed Lie ideal of $\mathcal{A}$. Let $\mathcal{A}$ admit a nonzero symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\ell : \mathcal{A} \to \mathcal{A}$ satisfying one of the following conditions:*
(i) $\ell([\varsigma,\varepsilon]) \pm \ell(\varsigma) \pm [\varsigma,\varepsilon] \in Z(\mathcal{A}) \ \forall\ \varsigma,\varepsilon \in U$
(ii) $\ell([\varsigma,\varepsilon]) \pm \ell(\varepsilon) \pm [\varsigma,\varepsilon] \in Z(\mathcal{A}) \ \forall\ \varsigma,\varepsilon \in U$
Then, $U \subseteq Z(\mathcal{A})$.

Theorem 7. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra and U be a square closed Lie ideal of $\mathcal{A}$. Let $\mathcal{A}$ admit a nonzero symmetric linear generalized n-derivation $\mathcal{G} : \mathcal{A}^n \to \mathcal{A}$ with trace $g : \mathcal{A} \to \mathcal{A}$ associated with symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $\ell : \mathcal{A} \to \mathcal{A}$ satisfying one of the following conditions:*
(i) $g(\varsigma) \circ g(\varepsilon) \pm \varsigma \circ \varepsilon \in Z(\mathcal{A}) \ \forall\ \varsigma,\varepsilon \in U$
(ii) $g(\varsigma) \circ g(\varepsilon) \pm [\varsigma,\varepsilon] \in Z(\mathcal{A}) \ \forall\ \varsigma,\varepsilon \in U$
Then, $U \subseteq Z(\mathcal{A})$.

Proof. (i) Suppose on the contrary that $U \nsubseteq Z(\mathcal{A})$. It is given that

$$g(\varsigma) \circ g(\varepsilon) \pm \varsigma \circ \varepsilon \in Z(\mathcal{A}) \ \forall\ \varsigma,\varepsilon \in U.$$

Replacing ε by $\varepsilon + mz$, where $z \in U$ and $1 \leq m \leq n-1$ in the given condition, we obtain

$$g(\varsigma) \circ g(\varepsilon + mz) \pm \varsigma \circ (\varepsilon + mz) \in Z(\mathcal{A}) \ \forall\ \varsigma,\varepsilon,z \in U$$

which on solving, we have

$$g(\varsigma) \circ g(\varepsilon) + g(\varsigma) \circ g(mz) + g(\varsigma) \circ \sum_{l=1}^{n-1} {}^{n}C_l \mathcal{G}(\underbrace{\varepsilon,\ldots,\varepsilon}_{(n-l)-\text{times}}, \underbrace{mz,\ldots,mz}_{l-\text{times}}) \pm \varsigma \circ \varepsilon \pm \varsigma \circ mz \in Z(\mathcal{A}) \ \forall\ \varsigma,\varepsilon,z \in U.$$

By using hypothesis, we obtain

$$g(\varsigma) \circ \sum_{l=1}^{n-1} {}^{n}C_l \mathfrak{D}(\underbrace{\varepsilon,\ldots,\varepsilon}_{(n-l)-\text{times}}, \underbrace{mz,\ldots,mz}_{l-\text{times}}) \in Z(\mathcal{A}) \ \forall\ \varsigma,\varepsilon,z \in U$$

which implies that

$$mA_1(\varsigma,\varepsilon,z) + m^2 A_2(\varsigma,\varepsilon,z) + \cdots + m^{n-1} A_{n-1}(\varsigma,\varepsilon,z) \in Z(\mathcal{A})$$

$\forall\ \varsigma,\varepsilon,z \in U$ where $A_l(\varsigma,\varepsilon,z)$ represents the term in which z appears l-times.

Making use of Lemma 3, we see that

$$g(\varsigma) \circ \mathcal{G}(\varepsilon,\ldots,\varepsilon,z) \in Z(\mathcal{A}) \ \forall\ \varsigma,\varepsilon,z \in U.$$

In particular, $z = \varepsilon$, we obtain

$$g(\varsigma) \circ g(\varepsilon) \in Z(\mathcal{A}) \ \forall\ \varsigma, \varepsilon \in U.$$

Hence, by using the given condition, we find that $\varsigma \circ \varepsilon \in Z(\mathcal{A}) \ \forall\ \varsigma, \varepsilon \in U$. Replacing ς by $\varepsilon\varsigma$, we obtain $\varepsilon(\varsigma \circ \varepsilon) \in Z(\mathcal{A}) \ \forall\ \varsigma, \varepsilon \in U$. We can also write it as

$$[\varepsilon(\varsigma \circ \varepsilon), z] \ \forall\ \varsigma, \varepsilon, z \in U$$

which on solving, we obtain $[\varepsilon, z]\varsigma \circ \varepsilon = 0 \ \forall\ \varsigma, \varepsilon, z \in U$. Again replace ς by ςz and using the same equation, we obtain $[\varepsilon, z]\varsigma[z, \varepsilon] = 0 \ \forall\ \varsigma, \varepsilon, z \in U$. Using Lemma 1, we have $[z, \varepsilon] = 0 \ \forall\ z, \varepsilon \in U$. By Lemma 2, we have $U \subseteq Z(\mathcal{A})$ which is a contradiction.
(ii) Proceeding in the same way as in (i), we conclude. □

Corollary 6. *For any fixed integer $n \geq 2$, let $\mathcal{A}$ be a C^*-algebra and U be a square closed Lie ideal of $\mathcal{A}$. Let $\mathcal{A}$ admit a nonzero symmetric linear n-derivation $\mathfrak{D} : \mathcal{A}^n \to \mathcal{A}$ with trace $d : \mathcal{A} \to \mathcal{A}$ satisfying one of the following conditions:*
(i) $d(\varsigma) \circ d(\varepsilon) \pm \varsigma \circ \varepsilon = 0 \ \forall\ \varsigma, \varepsilon \in U$
(ii) $d(\varsigma) \circ d(\varepsilon) \pm [\varsigma, \varepsilon] = 0 \ \forall\ \varsigma, \varepsilon \in U$.
Then, $U \subseteq Z(\mathcal{A})$.

3. Conclusions

In this study, we have explored the structural properties of C^*-algebras through the lens of generalized linear n-derivations. In fact, our investigation delves into the structure of C^*-algebras, focusing particularly on the intricate interplay between symmetric generalized n-derivations $\mathcal{A}$ and Lie ideals of $\mathcal{A}$. By elucidating the functional identity governing the behavior of linear generalized n-derivations, we provided insights into their forms of traces, thus shedding light on their intrinsic properties and behaviors.

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