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New Uses of q-Generalized Janowski Function in q-Bounded Turning Functions

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Abstract: In this paper, we discussed a new subclass $\mathcal{JQ}^{B}_{\eth,A}(q)$ of bi-univalent functions in the unit disk $\mathbb{U}$ using q-generalized Janowski function and q-derivative. Additionally, certain properties were examined and effectively demonstrated, such as the second Hankel determinant, Fekete–Szegö estimates, and Coefficients Bounds. Each of these bounds were precise and were confirmed by finding the extremal function for the new class. Furthermore, there are in-depth conversations available regarding certain intriguing specific cases of the outcomes achieved.

Keywords: Hankel determinant; Fekete–Szegö estimates; bounded turning function; q-generalized Janowski function; generalized derivatives

MSC: 30C80; 30C45; 30C50; 47B38; 11B65

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1. Introduction

This paper refers to a mathematical branch known as quantum calculus or q-calculus, which extends traditional calculus to include quantum mechanics principles by introducing a new parameter, q. This field, which is known for its broad use in various mathematical areas, notably in geometric function theory, incorporates q to generalize traditional calculus concepts. The inclusion of the parameter q is a key component of derivatives, integrals, generalized derivatives, and functions in the realm of q-calculus. The q-derivative is an operator that employs q-analogs of traditional derivatives in a difference quotient.

The q-integral can be thought of as the q-version of the Riemann integral. q-calculus involves a range of q-special functions with important applications in mathematics and physics, including q-binomial coefficients and q-factorials. Overall, q-calculus is a useful tool for analyzing and solving problems related to discrete and quantum systems.

Utilizing fractional calculus operators is a common practice in solving problems in applied sciences and Geometric Function, as mentioned in reference [1]. Fractional q-calculus, an extension of traditional fractional calculus, is utilized in diverse areas such as ordinary fractional calculus, q-integral equations, optimal control problems, and q-difference. To delve deeper, it is advisable to refer to a published work [2] and recent literature, which could include references like [3,4].

This article provides a summary of q-calculus, first introduced by Jackson and further studied by various mathematicians [5–10]. It aims to present essential concepts and definitions in q-calculus and emphasizes the importance of the q-difference operator in fields like geometric function theory. Assuming q is between 0 and 1, the study heavily utilizes fundamental definitions and properties of q-calculus, as detailed in Gasper and Rahman's work [11].

Many scholars have shown interest in investigating q-calculus (or quantum calculus) due to its applications in various quantitative disciplines, which has sparked their curiosity and drive. The q-derivative research has motivated scholars to apply it in geometric function theory, as well as in other areas of mathematics and mathematical sciences. Jackson [12,13] played a significant role in pioneering and advancing the theory of q-calculus. The use of q-calculus has extended to various mathematical disciplines, with its formulas commonly employed to study the presence of various function theory structures. Ismail and colleagues [14] were the pioneers in linking the geometric properties of analytic functions with the q-derivative operator. Kanas and Raducanu [15] describe the initial features of the q-difference operator. The q-difference operator was utilized, the concept of convolution was employed, and the q-version of the Ruscheweyh differential operator was established, while [16] introduced the group of q-starlike functions linked to the q-version of the Ruscheweyh differential operator. Zang and colleagues [17] employed q-calculus symbols and subordination methods to establish a broad conic region, which was then utilized to analyze the category of q-starlike functions. Several authors have recently conducted studies on the categories of q-starlike functions, as referenced in articles [18–23]. In order to research different categories of analytic and bi-univalent functions, the initial step is to establish the definition of the q-difference operator.

2. Preliminaries

In this research, we use the symbol $\mathcal{A}$ to represent analytic functions within the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. The normalized type analytic function expansion is given by:

$$f(z) = z + \sum_{b=2}^{\infty} l_b z^b \tag{1}$$

and we define $\mathcal{A}$ as the subclass of $\mathcal{S}$ that contains functions which are univalent in $\mathbb{U}$.

The function $f(z)$ is bi-univalent in $\mathbb{U}$ if both it and its inverse are univalent in $\mathbb{U}$ and $\mathbb{U}_{e_0}$, indicated by Λ.

The equation

$$p(z) = 1 + \sum_{b=1}^{\infty} p_b z^b, \quad (z \in \mathbb{U}) \tag{2}$$

satisfying $\Re(p(z)) > 0$ is denoted as $\mathcal{P}$ in $\mathbb{U}$, where $|p_b| < 2$. For more details, see [24].

The implementation of differential subordination in analytic functions can provide substantial benefits to geometric function theory. Miller and Mocanu [25] introduced the original differential subordination problem, which was further explored in subsequent studies [26]. Their book [27] provides a detailed summary of advancements in the field, including publication dates. Classes $\mathcal{P}[A,B]$, $\mathcal{S}^{\star}[A,B]$, and $\mathcal{C}[A,B]$ are defined for real numbers A and B with $-1 \leq B < A \leq 1$.

$$\mathcal{P}[A,B] = \left(p \in \mathcal{P} : \text{ iff } p(z) \prec \frac{1+Az}{1+Bz}, \quad (z \in \mathbb{U}) \right),$$

$$\mathcal{S}^{\star}[A,B] = \left(f(z) \in \mathcal{A} : \text{ iff } \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, \quad (z \in \mathbb{U}) \right),$$

$$\mathcal{C}[A,B] = \left(f(z) \in \mathcal{A} : \text{ iff } 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz}, \quad (z \in \mathbb{U}) \right).$$

Janowski [28] examined and analyzed the subset $\mathcal{P}[A,B]$, which is part of $\mathcal{P}[A,B]$, with other classes like $\mathcal{S}^{\star}[A,B]$ and $\mathcal{C}[A,B]$ also investigated in prior research [29–32], among others.

In 2006, Polatoglu [33] and colleagues enhanced the class $\mathcal{P}[A,B]$ to create a new class:

$$\mathcal{P}[\eth; A,B] = \left(p(z) \in \mathcal{P}: \ p(z) \prec (1-\eth)\frac{1+Az}{1+Bz} + \eth\right),$$

where $\eth \in [0,1)$, $-1 \leq B < A \leq 1$, $z \in \mathbb{U}$.

The function $f^{-1}(w)$, known as the inverse of $f(z) \in \mathcal{S}$ given in (1), can be written as:

$$f^{-1}(f(z)) = z \,, z \in \mathbb{U}, f^{-1}(f(w)) = w, \ w \in \mathbb{U}_{e_0} = \{w \in \mathbb{U} : |w| < e_0(f)\}, \ e_0(f) \geq 1/4.$$

Also

$$f^{-1}(w) = w + \Delta_2 w^2 + \Delta_3 w^3 + \Delta_4 w^4 + \cdots, \quad w \in \mathbb{U}_{e_0}, \tag{3}$$

where

$$\Delta_2 = -l_2, \ \Delta_3 = 2l_2^2 - l_3, \ \Delta_4 = -5l_2^3 + 5l_2 l_3 - l_4.$$

The ϱth Hankel determinant of $f(z)$, a concept presented by [34] can be defined when $b \geq 1$ and $\varrho \geq 1$:

$$\mathcal{H}_\varrho(b) = \begin{vmatrix} l_b & l_{b+1} & l_{b+2} & \cdots & \cdots & l_{b+\varrho-1} \\ l_{b+1} & l_{b+2} & l_{b+3} & \cdots & \cdots & l_{b+l} \\ l_{b+2} & l_{b+3} & l_{b+4} & \cdots & \cdots & l_{b+\varrho+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{b+\varrho-1} & l_{b+\varrho} & l_{b+\varrho+1} & \cdots & \cdots & l_{b+2(\varrho-1)} \end{vmatrix}. \tag{4}$$

When $\varrho = 2$ and $b = 1$, we can see that $\mathcal{H}_2(1)$ equals $l_3 - l_2^2$. The second Hankel determinant $\mathcal{H}_2(2)$, which is equal to $|l_2 l_3 - l_3^2|$, is commonly used to represent a certain class in this area of research [5,35–37]. Srivastava et al. [5] recently characterized a fascinating class of bi-univalent functions incorporating Euler polynomials and determined the second Hankel determinant for this specific class. Fekete–Szegö [38] analyzed the Hankel determinant of the function $f(z)$ to have

$$|\mathcal{H}_2(1)| = \begin{vmatrix} l_1 & l_2 \\ l_2 & l_3 \end{vmatrix} = l_1 l_3 - l_2^2. \tag{5}$$

They built upon a previous research on estimates of $|l_3 - I_2^2|$, with the condition $l_1 = 1$ and $I \in \mathbb{R}$.

Definition 1 ([12,13])**.** *A function $f(z)$ can be defined with the q-derivative in the following manner:*

$$\eth_q f(z) \leq \begin{cases} \frac{f(z)-f(qz)}{z-qz} & \text{if } q \in (0,1), z \neq 0 \\ f'(0) & \text{if } z = 0, \ \ q \longrightarrow 1^- \\ f'(z) & \text{if } z \neq 0, \ \ q \longrightarrow 1^-. \end{cases} \tag{6}$$

Definition 2 ([14])**.** *Selecting values for A and B where $-1 \leq B < A \leq 1$. If the condition of surbordination is satisfied, then let $f \in \mathfrak{S}^\star[A,B;q]$ to have*

$$\frac{z\eth_q f(z)}{f(z)} \prec \frac{1+A_q z}{1+B_q z}, \quad (q \in (0,1), z \in \mathbb{U}),$$

where

$$A_q = \frac{(A+1)+q(A-1)}{2} \ \textit{and} \ B_q = \frac{(B+1)+q(B-1)}{2}.$$

In our recent research, we were inspired by Srivastava et al. [5] and Polatoglu [33] in 2006 and discovered a novel set of bi-univalent functions by utilizing q-generalized Janowski functions and q-derivative, leading to precise limits for the second Hankel determinant, Fekete–Szegö estimates, and Coefficients Bounds. Additionally, we identified the extremal function for this new class to validate our findings.

3. The New Defined Class and Lemmas

We now define a new subclass of bi-bounded turning function with q-derivative associated with q-generalized Janowski functions.

Definition 3. *Imposing the set conditions* $0 \leq \eth < 1$, $-1 \leq A < B \leq 1$, $A_q = \frac{(1+A)+q(A-1)}{2}$, $B_q = \frac{(1+B)+q(B-1)}{2}$ *and* $q \in (0,1)$, *then* $f(z) \in \mathcal{J}\mathcal{Q}^{B}_{\eth,A}(q)$ *if the expressions below*

$$\eth_q f(z) \prec (1-\eth)\frac{1+A_q z}{1+B_q z} + \eth \tag{7}$$

and

$$\eth_q f^{-1}(w) \prec (1-\eth)\frac{1+A_q w}{1+B_q w} + \eth, \tag{8}$$

are satisfied.

Remark 1. *This remark presents the bi-q-bounded turning function class alongside subordination when* $\eth = 0$, *which is denoted by* $\mathcal{J}\mathcal{Q}^{B}_{A}(q)$ *and satisfies the conditions below.*

$$\eth_q f(z) \prec \frac{1+A_q z}{1+B_q z}$$

and

$$\eth_q f^{-1}(w) \prec \frac{1+A_q w}{1+B_q w},$$

where $-1 \leq A < B \leq 1$, $A_q = \frac{(1+A)+q(A-1)}{2}$, $B_q = \frac{(1+B)+q(B-1)}{2}$ *and* $q \in (0,1)$.

Remark 2. *This remark provides the bi-bounded turning function class alongside subordination when* $\eth = 0$ *and* $q \uparrow 1$, *which is denoted by* $\mathcal{J}\mathcal{Q}^{B}_{A}$ *and satisfies the conditions below.*

$$f'(z) \prec \frac{1+Az}{1+Bz}$$

and

$$(f^{-1}(w))' \prec \frac{1+Aw}{1+Bw},$$

where $-1 \leq A < B \leq 1$.

This paper will explore the second Hankel determinant, Fekete–Szegö problem, and coefficient bound estimates in Geometric Function Theory within the new subclass $\mathcal{J}\mathcal{Q}^{B}_{\eth,A}(q)$.

The lemmas provided below are utilized to effectively demonstrate the theorems in the primary results.

Lemma 1 ([39])**.** *Let $p \in \mathcal{P}$ be of the form*

$$p(z) = 1 + \sum_{n=1}^{+\infty} p_n z^n \quad (z \in \mathbb{U}). \tag{9}$$

If $\Re(p(z)) > 0$, $z \in \mathbb{U}$, then

$$|p_n| \leq 2, \quad n \in \mathbb{N}.$$

Lemma 2 ([39])**.** *Let $p \in \mathcal{P}$ be of the form* (9)*. If $\Re(p(z)) > 0$, $z \in \mathbb{U}$, then*

$$p_2 = \frac{[p_1^2 + (4 - p_1^2)\varepsilon]}{2},$$
$$p_3 = \frac{p_1^3 + 2(4 - p_1^2)p_1\varepsilon - (4 - p_1^2)p_1\varepsilon^2 + 2(4 - p_1^2)(1 - |\varepsilon|^2)z}{4}.$$

4. Estimates for Bounds of Coefficients

In this section, the upcoming theorem concentrates on precise upper-limit estimations for functions belonging to the novel class $\mathcal{JQ}^{B}_{\eth,A}(q)$.

Theorem 1. *Suppose $f(z) \in \mathcal{JQ}^{B}_{\eth,A}(q)$. Then,*

$$|l_2| \leq \frac{(1-\eth)(A_q - B_q)}{[2]_q},$$

$$|l_3| \leq \frac{(1-\eth)(A_q - B_q)}{[3]_q},$$

$$|l_4| \leq \frac{(1-\eth)(A_q - B_q)}{[4]_q}.$$

This specific strict upper limits above the given function confirmed the sharpness of the equation:

$$f_1(z) = z + \frac{(1-\eth)(A_q - B_q)}{[2]_q} z^2 + \frac{(1-\eth)(A_q - B_q)}{[3]_q} z^3 + \cdots$$
$$f_2(z) = z + \frac{(1-\eth)(A_q - B_q)}{[3]_q} z^3 + \frac{(1-\eth)(A_q - B_q)}{[4]_q} z^4 + \cdots$$
$$f_3(z) = z + \frac{(1-\eth)(A_q - B_q)}{[4]_q} z^4 + \frac{(1-\eth)(A_q - B_q)}{[5]_q} z^5 + \cdots.$$

Proof. Suppose $f(z) \in \mathcal{JQ}^{B}_{\eth,A}(q)$, $0 \leq \eth < 1$, $-1 \leq A < B \leq 1$, $A_q = \frac{(1+A)+q(A-1)}{2}$, $B_q = \frac{(1+B)+q(B-1)}{2}$ and $q \in (0,1)$. Next, the specified conditions

$$\eth_q f(z) = (1-\eth)\frac{1 + A_q \xi_1(z)}{1 + B_q \xi_1(z)} + \eth \tag{10}$$

and

$$\eth_q f^{-1}(w) = (1-\eth)\frac{1 + A_q \xi_2(w)}{1 + B_q \xi_2(w)} + \eth, \tag{11}$$

are met by the analytic functions $\xi_1 : \mathbb{U} \longrightarrow \mathbb{U}$ and $\xi_2 : \mathbb{U}_{e_0} \longrightarrow \mathbb{U}_{e_0}$ with initial values of $\xi_1(0) = 0 = \xi_2(0)$, $|\xi_1(z)| < 1$, and $|\xi_2(w)| < 1$.

Utilizing Equations (10) and (11) and performing basic calculations results in

$$\eth_q f(z) = 1 - \left[\frac{(\Game - 1)(A_q - B_q)}{2}\right] p_1 z - \left[\frac{(\Game - 1)(A_q - B_q)}{2}\right]\left[p_2 - (1 + B_q)\frac{p_1^2}{2}\right] z^2 \\ - \left[\frac{(\Game - 1)(A_q - B_q)}{2}\right]\left[p_3 - (1 - B_q)p_1 p_2 + (1 + 2B_q + (B_q)^2)\frac{p_1^3}{4}\right] z^3 + \cdots \tag{12}$$

and

$$\eth_q f^{-1}(w) = 1 - \left[\frac{(\Game - 1)(A_q - B_q)}{2}\right] \psi_1 w - \left[\frac{(\Game - 1)(A_q - B_q)}{2}\right]\left[\psi_2 - (1 + B_q)\frac{\psi_1^2}{2}\right] w^2 \\ - \left[\frac{(\Game - 1)(A_q - B_q)}{2}\right]\left[\psi_3 - (1 - B_q)\psi_1 \psi_2 + (1 + 2B_q + (B_q)^2)\frac{\psi_1^3}{4}\right] w^3 + \cdots \tag{13}$$

where p, ψ are functions in class $\mathcal{P}$.

To find the following expression, the constants of variables with equal powers in (12) and (13) must be compared and equated, resulting in the determination of l_2, l_3, and l_4.

$$l_2[2]_q = \left[\frac{(1 - \Game)(A_q - B_q)}{2}\right] p_1, \tag{14}$$

$$l_3[3]_q = \left[\frac{(1 - \Game)(A_q - B_q)}{2}\right]\left[p_2 - (1 + B_q)\frac{p_1^2}{2}\right], \tag{15}$$

$$l_4[4]_q = \left[\frac{(1 - \Game)(A_q - B_q)}{2}\right]\left[p_3 - (1 - B_q)p_1 p_2 + (1 + 2B_q + (B_q)^2)\frac{p_1^3}{4}\right] \tag{16}$$

and

$$-[2]_q l_2 = \left[\frac{(1 - \Game)(A_q - B_q)}{2}\right] \psi_1, \tag{17}$$

$$-[3]_q l_3 + 2[3]_q l_2^2 = \left[\frac{(1 - \Game)(A_q - B_q)}{2}\right]\left[\psi_2 - (1 + B_q)\frac{\psi_1^2}{2}\right], \tag{18}$$

$$-5[4]_q l_2^3 + 5[4]_q l_2 l_3 - [4]_q l_4 = \left[\frac{(1 - \Game)(A_q - B_q)}{2}\right]\left[\psi_3 - (1 - B_q)\psi_1 \psi_2 \right. \\ \left. + (1 + 2B_q + (B_q)^2)\frac{\psi_1^3}{4}\right]. \tag{19}$$

The subsequent equation is derived using Equations (14) and (17) as follows

$$\left[\frac{(1 - \Game)(A_q - B_q)}{2[2]_q}\right] p_1 = -\left[\frac{(1 - \Game)(A_q - B_q)}{2[2]_q}\right] \psi_1 \Longrightarrow p_1 = -\psi_1 \Longrightarrow p_1^2 = \psi_1^2. \tag{20}$$

Utilizing Lemma 1 and basic calculations on the final equation makes the result of the theorem evident: it is

$$|l_2| \leq \frac{(1 - \Game)(A_q - B_q)}{[2]_q}. \tag{21}$$

Using Equations (15) and (18) allows us to derive the limit for l_3, while setting $p_1 = -\psi_1$ gives

$$l_3 = l_2^2 + \frac{(1 - \Game)(A_q - B_q)(p_2 - \psi_2)}{4[3]_q}$$

that is:
$$l_3 = \frac{(1-\eth)^2(A_q-B_q)^2}{4[2]_q^2}p_1^2 + \frac{(1-\eth)(A_q-B_q)(p_2-\psi_2)}{4[3]_q}. \tag{22}$$

Similarly, using (16) and (19) while referencing (20) and (22) to calculate l_4 results in the following expression:

$$\begin{aligned} l_4 =& \frac{(1-\eth)(A_q-B_q)(1+2B_2+(B_q)^2)}{8[4]_q}p_1^3 + \frac{5(1-\eth)^2(A_q-B_q)^2 p_1(p_2-p_2)}{16[2]_q[3]_q} \\ &+ \frac{(1-\eth)(A_q-B_q)(p_3-\psi_3)}{4[4]_q} - \frac{(1-\eth)(A_q-B_q)(1+B_q)p_1(p_2+\psi_2)}{4[4]_q}. \end{aligned} \tag{23}$$

It is evident from (20) and utilizing Lemma 2 where $|\varepsilon| \leq 1$, $|\varphi| \leq 1$, $|z| \leq 1$ and $|w| \leq 1$ yields

$$p_2 - \psi_2 = \frac{4-p_1^2}{2}(\varepsilon - \varphi), \quad p_2 + \psi_2 = p_1^2 + \frac{4-p_1^2}{2}(\varepsilon + \varphi) \tag{24}$$

and

$$p_3 - \psi_3 = \frac{p_1^3}{2} + \frac{(4-p_1^2)p_1}{2}(\varepsilon+\varphi) - \frac{(4-p_1^2)p_1}{2}(\varepsilon^2+\varphi^2) + \frac{4-p_1^2}{2}\left([1-|\varepsilon|^2]z - [1-|\varphi|^2]w\right). \tag{25}$$

Employing (22) and (24) results in

$$l_3 = \frac{(1-\eth)^2(A_q-B_q)^2}{4[2]_q^2}p_1^2 + \frac{(1-\eth)(A_q-B_q)(4-p_1^2)(\varepsilon-\varphi)}{8[3]_q}.$$

It is evident that $|p_1| = \epsilon$ is a positive result because of Lemma 1, leading to the expression $4 - p_1^2 = 4 - \epsilon^2$.

Selecting the range of ϵ from 0 to 2 and using the triangular inequality results in $|\varepsilon| = \sigma_1$, and $|\varphi| = \sigma_2$, gives

$$|l_3| \leq \frac{(1-\eth)^2(A_q-B_q)^2}{4[2]_q^2}\epsilon^2 + \frac{(1-\eth)(A_q-B_q)(4-\epsilon^2)(\sigma_1+\sigma_2)}{8[3]_q}, \quad (\sigma_1,\sigma_2) \in [0,1]^2.$$

Now, the function $K : \mathbb{R} \longrightarrow \mathbb{R}$ needs to be established, and the function should be tested to determine the maximum value within the closed square $\varpi = \{(\sigma_1,\sigma_2) : (\sigma_1,\sigma_2) \in [0,1]^2\}$ as shown:

$$\begin{aligned} K(\sigma_1,\sigma_2) =& \frac{(1-\eth)^2(A_q-B_q)^2}{4[2]_q^2}\epsilon^2 + \frac{(1-\eth)(A_q-B_q)(4-\epsilon^2)(\sigma_1+\sigma_2)}{8[3]_q}, \\ & (\sigma_1,\sigma_2) \in [0,1]^2. \end{aligned}$$

It can be inferred that the maximum parameter of $K(\sigma_1,\sigma_2)$ is located at the edge of ϖ, and when differentiating $K(\sigma_1,\sigma_2)$ with respect to σ_1, it can be further determined that

$$K_{\sigma_1}(\sigma_1,\sigma_2) = \frac{(1-\eth)(A_q-B_q)(4-\epsilon^2)}{8[3]_q}.$$

If $K_{\sigma_1}(\sigma_1,\sigma_2) \geq 0$, where $\sigma_2 \in [0,1]$ and $\epsilon \in [0,2]$. It can be deduced that the function $K(\sigma_1,\sigma_2)$ increases with σ_1 and reaches its peak at $\sigma_1 = 1$, implying that:

$$\max[K(\sigma_1,\sigma_2):\sigma_1\in[0,1]] = K(1,\sigma_2) = \frac{(1-\text{Э})^2(A_q-B_q)^2}{4[2]_q^2}\epsilon^2 + \frac{(1-\text{Э})(A_q-B_q)(4-\epsilon^2)}{8[3]_q}(1+\sigma_2).$$

Additional differentiation of $K(1,\sigma_2)$ results in

$$K'(1,\sigma_2) = \frac{(1-\text{Э})(A_q-B_q)(4-\epsilon^2)}{8[3]_q}.$$

If $K'(1,\sigma_2)\geq 0$, where $\epsilon\in[0,2]$. It can be deduced that the function $K(1,\sigma_2)$ increases and reaches its peak at $\sigma_2=1$, implying that

$$\max[K(\sigma_1,\sigma_2):\sigma_1\in[0,1]] = K(1,1) = \frac{(1-\text{Э})^2(A_q-B_q)^2}{4[2]_q^2}\epsilon^2 + \frac{(1-\text{Э})(A_q-B_q)(4-\epsilon^2)}{4[3]_q}.$$

Hence,

$$K(\sigma_1,\sigma_2)\leq \max[K(\sigma_1,\sigma_2):(\sigma_1,\sigma_2)\in\varpi] = K(1,1) = \frac{(1-\text{Э})^2(A_q-B_q)^2}{4[2]_q^2}\epsilon^2 + \frac{(1-\text{Э})(A_q-B_q)(4-\epsilon^2)}{4[3]_q}.$$

If $|l_3|\leq K(\sigma_1,\sigma_2)$, it is easy to see that

$$|l_3|\leq h(\text{Э},q)\epsilon^2 + \frac{(1-\text{Э})(A_q-B_q)}{[3]_q},\ \ \epsilon\in[0,2],$$

where

$$h(\text{Э},q) = \frac{1}{4}\left\{\frac{(1-\text{Э})^2(A_q-B_q)^2}{[2]_q^2} - \frac{(1-\text{Э})(A_q-B_q)}{[3]_q}\right\}.$$

We introduce function $K_1:\mathbb{R}\longrightarrow\mathbb{R}$ to determine its maximum value, defined as:

$$K_1(\epsilon) = h(\text{Э},q)\epsilon^2 + \frac{(1-\text{Э})(A_q-B_q)}{[3]_q},\ \ \epsilon\in[0,2].$$

When the derivative of $K_1(\epsilon)$ is calculated, the equation $K_1'(\epsilon)=2h(\text{Э},q)\epsilon,\ \epsilon\in[0,2]$ is true. If $h(\text{Э},q)\leq 0$, then $K_1'(\epsilon)\leq 0$, resulting in $K_1(\epsilon)$ being a decreasing function with a maximum at $\epsilon=0$. So

$$\max[K_1(\epsilon):\epsilon\in[0,2]] = K_1(0) = \frac{(1-\text{Э})(A_q-B_q)}{[3]_q}$$

and $h(\text{Э},q)\geq 0$, then $K_1'(\epsilon)\geq 0$, resulting in $K_1(\epsilon)$ being a increasing function with a maximum at $\epsilon=2$. So

$$\max[K_1(\epsilon):\epsilon\in[0,2]] = K_1(2) = \frac{(1-\text{Э})^2(A_q-B_q)^2}{[2]_q^2}.$$

Hence, a successful determination of the sharp upper bound for $|l_3|$ is provided as follows:

$$|l_3|\leq \frac{(1-\text{Э})(A_q-B_q)}{[3]_q}. \tag{26}$$

Utilizing Equations (23)–(25) along with the well-known triangular inequality, the inequality for the magnitude of l_4 can be expressed in the following manner:

$$|l_4| \leq o_1(\epsilon) + o_2(\epsilon)(\sigma_1 + \sigma_2) + o_3(\epsilon)(\sigma_1^2 + \sigma_2^2) = K_2(\sigma_1, \sigma_2),$$

where

$$\begin{aligned} o_1(\epsilon) &= \frac{(1-\eth)(A_q - B_q)(B_q)^2}{8[4]_q}\epsilon^3 + \frac{(1-\eth)(A_q - B_q)(4-\epsilon^2)}{4[4]_q}, \\ o_2(\epsilon) &= \frac{5(1-\eth)^2(A_q - B_q)^2(4-\epsilon^2)}{32[2]_q[3]_q}\epsilon + \frac{(1-\eth)(A_q - B_q)B_q(4-\epsilon^2)}{8[4]_q}, \\ o_3(\epsilon) &= \frac{(1-\eth)(A_q - B_q)(4-\epsilon^2)(\epsilon - 2)}{16[4]_q}. \end{aligned}$$

The coefficients $o_1(\epsilon)$, $o_2(\epsilon)$, and $o_3(\epsilon)$ of $K_2(\sigma_1, \sigma_2)$ vary based on the parameter ϵ, requiring maximization of $K_2(\sigma_1, \sigma_2)$ on ϖ for each $\epsilon \in [0, 2]$. Subsequently, it is crucial to determine the maximum value of $K_2(\sigma_1, \sigma_2)$ for various ϵ values.

Since $\epsilon = 0$, because $o_2(0) = 0$,

$$\begin{aligned} o_1(0) &= \frac{(1-\eth)(A_q - B_q)}{[4]_q} \quad \text{and} \\ o_2(0) &= -\frac{(1-\eth)(A_q - B_q)}{2[4]_q}, \end{aligned}$$

which yields

$$K_2(\sigma_1, \sigma_2) = \frac{(1-\eth)(A_q - B_q)}{[4]_q} - \frac{(1-\eth)(A_q - B_q)}{2[4]_q}(\sigma_1^2 + \sigma_2^2), \quad (\sigma_1, \sigma_2) \in [0, 1]^2.$$

As a result of this, we have

$$K_2(\sigma_1, \sigma_2) \leq \max[K(\sigma_1, \sigma_2) : (\sigma_1, \sigma_2) \in \varpi] = K(0, 0) = \frac{(1-\eth)(A_q - B_q)}{[4]_q}.$$

Assume $\epsilon = 2$. Given that $o_2(2) = o_3(2)$, then

$$o_1(2) = \frac{(1-\eth)(A_q - B_q)(B_q)^2}{[4]_q}.$$

We therefore have a fixed value for the function K_2 as described:

$$K_2(\sigma_1, \sigma_2) = o_1(2) = \frac{(1-\eth)(A_q - B_q)(B_q)^2}{[4]_q}.$$

Observing that $K_2(\sigma_1, \sigma_2)$ cannot reach a maximum value on ϖ, when ϵ lies within the interval $[0, 2]$, we conclude that

$$|l_4| \leq \frac{(1-\eth)(A_q - B_q)}{[4]_q}. \tag{27}$$

Effectively, we can confirm that the results obtained in (21), (26) and (27) apply to the functions listed below:

$$f_1(z) = z + \frac{(1-\eth)(A_q - B_q)}{[2]_q} z^2 + \frac{(1-\eth)(A_q - B_q)}{[3]_q} z^3 + \cdots$$
$$f_2(z) = z + \frac{(1-\eth)(A_q - B_q)}{[3]_q} z^3 + \frac{(1-\eth)(A_q - B_q)}{[4]_q} z^4 + \cdots$$
$$f_3(z) = z + \frac{(1-\eth)(A_q - B_q)}{[4]_q} z^4 + \frac{(1-\eth)(A_q - B_q)}{[5]_q} z^5 + \cdots .$$

□

We have the following Corollary when $\eth = 0$ in Theorem 1.

Corollary 1. *Let $f(z) \in \mathcal{JQ}_A^B(q)$. Then*

$$|l_2| \leq \frac{(A_q - B_q)}{[2]_q},\ |l_3| \leq \frac{(A_q - B_q)}{[3]_q},\ |l_4| \leq \frac{(A_q - B_q)}{[4]_q}.$$

The accuracy of the results is confirmed using the functions listed below:

$$f_1(z) = z + \frac{(A_q - B_q)}{[2]_q} z^2 + \frac{(A_q - B_q)}{[3]_q} z^3 + \cdots$$
$$f_2(z) = z + \frac{(A_q - B_q)}{[3]_q} z^3 + \frac{(A_q - B_q)}{[4]_q} z^4 + \cdots$$
$$f_3(z) = z + \frac{(A_q - B_q)}{[4]_q} z^4 + \frac{(A_q - B_q)}{[5]_q} z^5 + \cdots .$$

We have the following Corollary when $q \uparrow 1$ and $\eth = 0$ in Theorem 1.

Corollary 2. *Let $f(z) \in \mathcal{JQ}_A^B$. Then*

$$|l_2| \leq \frac{(A-B)}{2},\ |l_3| \leq \frac{(A-B)}{3},\ |l_4| \leq \frac{(A-B)}{4}.$$

The accuracy of the results is confirmed using the functions listed below:

$$f_1(z) = z + \frac{A-B}{2} z^2 + \frac{A-B}{3} z^3 + \cdots$$
$$f_2(z) = z + \frac{A-B}{3} z^3 + \frac{A-B}{4} z^4 + \cdots$$
$$f_3(z) = z + \frac{A-B}{4} z^4 + \frac{A-B}{5} z^5 + \cdots .$$

We have the following Corollary when $\eth = 0$, $A = 1$, $B = -1$ and $q \uparrow 1$ in Theorem 1.

Corollary 3. *Let $f(z) \in \mathcal{JQ}$. Then,*

$$|l_2| \leq 1, \quad |l_3| \leq \frac{2}{3}, \quad |l_4| \leq \frac{1}{2}.$$

The accuracy of the results is confirmed using the functions listed below:

$$f_1(z) = z + z^2 + \frac{2}{3} z^3 + \cdots$$
$$f_2(z) = z + \frac{2}{3} z^3 + \frac{1}{2} z^4 + \cdots$$
$$f_3(z) = z + \frac{1}{2} z^4 + \frac{2}{5} z^5 + \cdots .$$

5. Estimates for Second Hankel Determinant

This section focuses on accurate calculations for the second Hankel determinant of functions in the new class $\mathcal{J}\mathcal{Q}^{B}_{\eth,A}(q)$.

Theorem 2. *Suppose $f(z) \in \mathcal{J}\mathcal{Q}^{B}_{\eth,A}(q)$. Then,*

$$|l_2 l_4 - l_3^2| \leq \left[\frac{(1-\eth)(A_q - B_q)}{[3]_q}\right]^2,$$

where

$$f_2(z) = z + \frac{(1-\eth)(A_q - B_q)}{[3]_q} z^3 + \frac{(1-\eth)(A_q - B_q)}{[4]_q} z^4 + \cdots.$$

Proof. Suppose $f(z) \in \mathcal{J}\mathcal{Q}^{B}_{\eth,A}(q)$, $0 \leq \eth < 1$, $-1 \leq A < B \leq 1$, $A_q = \frac{(1+A)+q(A-1)}{2}$, $B_q = \frac{(1+B)+q(B-1)}{2}$ and $q \in (0,1)$. Utilizing Equations (20), (22) and (23) results in the equivalence of $l_2 l_4 - l_3^2$:

$$\begin{aligned} l_2 l_4 - l_3^2 &= \frac{(1-\eth)^2 (A_q - B_q)^2 (1 + 2B_q + (B_q)^2)}{16[4]_q [2]_q} p_1^4 - \frac{(1-\eth)^4 (A_q - B_q)^4}{16[2]_q^4} p_1^4 \\ &+ \frac{(1-\eth)^3 (A_q - B_q)^3 p_1^2 (p_2 - \psi_2)}{32[3]_q [2]_q^2} - \frac{(1-\eth)^2 (A_q - B_q)^2 (1 + B_q) p_1^2 (p_2 + \psi_2)}{8[4]_q [2]_q} \\ &+ \frac{(1-\eth)^2 (A_q - B_q)^2 p_1 (p_2 - \psi_2)}{8[2]_q [4]_q} - \frac{(1-\eth)^2 (A_q - B_q)^2 (p_2 - \psi_2)^2}{16[3]_q^2}. \end{aligned}$$

Applying (24) and (25) and selecting the range of ϵ from 0 to 2 and using the triangular inequality results in $|\varepsilon| = \sigma_1$, and $|\varphi| = \sigma_2$, gives

$$|l_2 l_4 - l_3^2| \leq \Xi_1(\epsilon) + \Xi_2(\epsilon)(\sigma_1 + \sigma_2) + \Xi_3(\epsilon)(\sigma_1^2 + \sigma_2^2) + \Xi_4(\epsilon)(\sigma_1 + \sigma_2)^2, \tag{28}$$

where

$$\begin{aligned} \Xi_1(\epsilon) &= \frac{(1-\eth)^2 (A_q - B_q)^2 (B_q)^2}{16[4]_q [2]_q} \epsilon^4 - \frac{(1-\eth)^4 (A_q - B_q)^4}{16[2]_q^4} \epsilon^4 \\ &\quad + \frac{(1-\eth)^2 (A_q - B_q)^2 (4 - \epsilon^2)}{8[2]_q [4]_q} \geq 0, \\ \Xi_2(\epsilon) &= \frac{(1-\eth)^3 (A_q - B_q)^3 (4 - \epsilon^2)}{16[3]_q [2]_q^3} \epsilon^2 + \frac{(1-\eth)^2 (A_q - B_q)^2 B_q (4 - \epsilon^2)}{16[4]_q [2]_q} \epsilon^2 \geq 0, \\ \Xi_3(\epsilon) &= \frac{(1-\eth)^2 (A_q - B_q)^2 (4 - \epsilon^2) \epsilon (\epsilon - 2)}{32[2]_q [4]_q} \leq 0, \\ \Xi_4(\epsilon) &= \frac{(1-\eth)^2 (A_q - B_q)^2 (4 - \epsilon^2)^2}{64[3]_q^2} \geq 0. \end{aligned}$$

We introduce function $K_3 : \mathbb{R} \longrightarrow \mathbb{R}$ to determine its maximum value for $\epsilon \in [0,2]$, which is defined as:

$$K_3(\sigma_1, \sigma_2) = \Xi_1(\epsilon) + \Xi_2(\epsilon)(\sigma_1 + \sigma_2) + \Xi_3(\epsilon)(\sigma_1^2 + \sigma_2^2) + \Xi_4(\epsilon)(\sigma_1 + \sigma_2)^2, \quad (\sigma_1, \sigma_2) \in [0,1]^2.$$

The coefficients $\Xi_1(\epsilon)$, $\Xi_2(\epsilon)$, $\Xi_3(\epsilon)$ and $\Xi_4(\epsilon)$ of $K_2(\sigma_1, \sigma_2)$ vary based on the parameter ϵ, requiring maximization of $K_3(\sigma_1, \sigma_2)$ on ϖ for each $\epsilon \in [0,2]$. Subsequently, it is essential to establish the highest possible value of $K_2(\sigma_1, \sigma_2)$ for various ϵ values.

(1) Since $\epsilon = 0$, because $\Xi_1(0) = \Xi_2(0) = \Xi_3(0) = 0$ and

$$\Xi_4(0) = \frac{(1-\eth)^2(A_q - B_q)^2}{4[3]_q^2}$$

which yields

$$K_3(\sigma_1, \sigma_2) = \frac{(1-\eth)^2(A_q - B_q)^2}{4[3]_q^2}(\sigma_1 + \sigma_2)^2, \;\; (\sigma_1, \sigma_2) \in \varpi.$$

It can be inferred that the maximum parameter of $K_3(\sigma_1, \sigma_2)$ is located at the edge of ϖ, and when differentiating $K_3(\sigma_1, \sigma_2)$ with respect to σ_1, it can be further determined that

$$(K_3)_{\sigma_1}(\sigma_1, \sigma_2) = \frac{(1-\eth)^2(A_q - B_q)^2}{2[3]_q^2}(\sigma_1 + \sigma_2), \;\; \sigma_2 \in [0,1].$$

If $(K_3)_{\sigma_1}(\sigma_1, \sigma_2) \geq 0$, where $\sigma_2 \in [0,1]$ and $\epsilon \in [0,2]$. It can be deduced that the function $K_3(\sigma_1, \sigma_2)$ increases with σ_1 and reaches its peak at $\sigma_1 = 1$, implying that

$$\max[K_3(\sigma_1, \sigma_2) : \sigma_1 \in [0,1]] = K_3(1, \sigma_2) = \frac{(1-\eth)^2(A_q - B_q)^2}{2[3]_q^2}(1 + \sigma_2).$$

Additional differentiation of $K_3(1, \sigma_2)$ results in

$$K_3'(1, \sigma_2) = \frac{(1-\eth)^2(A_q - B_q)^2}{2[3]_q^2}, \;\; \epsilon \in [0,2].$$

If $(K_3)_{\sigma_1}(1, \sigma_2) \geq 0$, where $\epsilon \in [0,2]$, it can be deduced that the function $K_3(1, \sigma_2)$ increases and reaches its peak at $\sigma_2 = 1$, implying that

$$\max[K_3(1, \sigma_2) : \sigma_2 \in [0,1]] = K(1,1) = \frac{(1-\eth)^2(A_q - B_q)^2}{[3]_q^2}.$$

Hence, for $\epsilon = 0$ yields:

$$K_3(\sigma_1, \sigma_2) \leq \max[K_3(\sigma_1, \sigma_2); (\sigma_1, \sigma_2) \in [0,1]^2] = K_3(1,1) = \frac{(1-\eth)^2(A_q - B_q)^2}{[3]_q^2}.$$

Because $|l_2 l_4 - l_3^2| \leq K_3(\sigma_1, \sigma_2)$, gives

$$|l_2 l_4 - l_3^2| \leq \frac{(1-\eth)^2(A_q - B_q)^2}{[3]_q^2}.$$

(2) Assume $\epsilon = 2$. Given that $\Xi_2(2) = \Xi_3(2) = \Xi_4(2) = 0$, then

$$\Xi_1(2) = \frac{(1-\eth)^2(A_q - B_q)^2(B_q)^2}{[4]_q[2]_q} - \frac{(1-\eth)^4(A_q - B_q)^4}{[2]_q^4}.$$

which provides the constant function shown below:

$$K_3(\sigma_1, \sigma_2) = \Xi_1(2) = \frac{(1-\eth)^2(A_q - B_q)^2(B_q)^2}{[4]_q[2]_q} - \frac{(1-\eth)^4(A_q - B_q)^4}{[2]_q^4}.$$

Hence, it gives

$$|l_2 l_4 - l_3^2| \leq \frac{(1-\eth)^2(A_q - B_q)^2(B_q)^2}{[4]_q[2]_q} - \frac{(1-\eth)^4(A_q - B_q)^4}{[2]_q^4}.$$

(3) In order to analyze the peak of $K_3(\sigma_1, \sigma_2)$ within the range $\epsilon \in (0, 2)$, the operation $\rho(K_3) = (K_3)_{\sigma_1\sigma_1}(\sigma_1, \sigma_2)(K_3)_{\sigma_2\sigma_2}(\sigma_1, \sigma_2) - ((K_3)_{\sigma_1\sigma_2}(\sigma_1, \sigma_2))^2$ will be utilized. Additionally, two scenarios will be examined to determine the desired outcome for the expression $\rho(K_3) = 4\Xi_3(\epsilon)\{\Xi_3(\epsilon) + 2\Xi_4(\epsilon)\}$ here.

   (a) If $\Xi_3(\epsilon) + 2\Xi_4(\epsilon) \leq 0$ for $\epsilon \in (0, 2)$, the function K_3 will not have a maximum on ϖ because $(K_3)_{\sigma_1,\sigma_2}(\sigma_1, \sigma_2) = (K_3)_{\sigma_2,\sigma_1}(\sigma_1, \sigma_2) = 2\Xi_4(\epsilon) \geq 0$, and $\rho(K_3) \geq 0$.

   (b) For the maximum of function K_3 on ϖ to be true, the condition must also be met for $\rho(K_3) \leq 0$ if $\Xi_3(\epsilon) + 2\Xi_4(\epsilon) \geq 0$ for $\epsilon \in (0, 2)$.

Consequently, due to the results of the three instances, we formulate

$$|l_2 l_4 - l_3^2| \leq \left[\frac{(1-\eth)(A_q - B_q)}{[3]_q}\right]^2. \tag{29}$$

Effectively, we can confirm that the result obtained in (29) apply to the function listed below:

$$f_2(z) = z + \frac{(1-\eth)(A_q - B_q)}{[3]_q} z^3 + \frac{(1-\eth)(A_q - B_q)}{[4]_q} z^4 + \cdots.$$

□

The Corollary mentioned in Theorem 2 holds true when the condition $\eth = 0$ is satisfied.

Corollary 4. *Let $f(z) \in \mathcal{JQ}_A^B(q)$. Then,*

$$|l_2 l_4 - l_3^2| \leq \left[\frac{(A_q - B_q)}{[3]_q}\right]^2.$$

The accuracy of the results is confirmed using the functions listed below:

$$f_2(z) = z + \frac{(A_q - B_q)}{[3]_q} z^3 + \frac{(A_q - B_q)}{[4]_q} z^4 + \cdots.$$

The Corollary stated in Theorem 2 is valid if the requirement $\eth = 0$ and $q \uparrow 1$ is met.

Corollary 5. *Let $f(z) \in \mathcal{JQ}_A^B$. Then,*

$$|l_2 l_4 - l_3^2| \leq \left[\frac{(A - B)}{3}\right]^2.$$

The accuracy of the results is confirmed using the functions listed below:

$$f_2(z) = z + \frac{(A - B)}{3} z^3 + \frac{(A - B)}{4} z^4 + \cdots.$$

The Corollary referred to in Theorem 1 is valid if $\eth = 0$, $A = 1$, $B = -1$, and $q \uparrow 1$.

Corollary 6. *Let $f(z) \in \mathcal{JQ}$. Then,*

$$|l_2 l_4 - l_3^2| \leq \frac{4}{9}$$

The accuracy of the results is confirmed using the functions listed below:

$$f_2(z) = z + \frac{2}{3}z^3 + \frac{1}{2}z^4 + \cdots.$$

6. Estimates for Fekete–Szegö Inequality

This section focuses on accurate calculations for the Fekete–Szegö Inequality of functions in the new class $\mathcal{JQ}^B_{\eth,A}(q)$.

Theorem 3. *Suppose $f(z) \in \mathcal{JQ}^B_{\eth,A}(q)$, $I \in \mathbb{C}$. Then,*

$$\left|l_3 - Il_2^2\right| \leq \begin{cases} \frac{(1-\eth)^2(A_q-B_q)^2 R(\eth,q)}{[2]_q^2}, & |1-I| \leq R(\eth,q) \\ \frac{(1-\eth)^2(A_q-B_q)^2|1-I|}{[2]_q^2} & |1-I| \geq R(\eth,q). \end{cases}$$

where

$$R(\eth,q) = \frac{[2]_q^2}{(1-\eth)(A_q-B_q)[3]_q}. \tag{30}$$

This specific strict upper limits above the given function confirmed the sharpness of the equation:

$$f_2(z) = z + \frac{(1-\eth)(A_q-B_q)}{[3]_q}z^3 + \frac{(1-\eth)(A_q-B_q)}{[4]_q}z^4 + \cdots.$$

Proof. Suppose $f(z) \in \mathcal{JQ}^B_{\eth,A}(q)$, $0 \leq \eth < 1$, $-1 \leq A < B \leq 1$, $A_q = \frac{(1+A)+q(A-1)}{2}$, $B_q = \frac{(1+B)+q(B-1)}{2}$, $I \in \mathbb{C}$ and $q \in (0,1)$. Utilizing Equations (20), (22) and (24) results in the equivalence of $l_3 - Il_2^2$:

$$l_3 - Il_2^2 = (1-I)p_1^2\frac{(1-\eth)^2(A_q-B_q)^2}{4[2]_q^2} + \frac{(4-p_1^2)(1-\eth)(A_q-B_q)(\varepsilon-\varphi)}{8[3]_q}. \tag{31}$$

Applying (31) and selecting the range of ϵ from 0 to 2 and using the triangular inequality results in $|\varepsilon| = \sigma_1$, and $|\varphi| = \sigma_2$, gives

$$|l_3 - Il_2^2| \leq |1-I|\epsilon^2\frac{(1-\eth)^2(A_q-B_q)^2}{4[2]_q^2} + \frac{(4-\epsilon^2)(1-\eth)(A_q-B_q)(\sigma_1+\sigma_2)}{8[3]_q}. \tag{32}$$

We introduce function $K_4 : \mathbb{R} \longrightarrow \mathbb{R}$ to determine its maximum value for $\epsilon \in [0,2]$, defined as:

$$\begin{aligned} K_4(\sigma_1,\sigma_2) = {} & |1-I|\epsilon^2\frac{(1-\eth)^2(A_q-B_q)^2}{4[2]_q^2} \\ & + \frac{(4-\epsilon^2)(1-\eth)(A_q-B_q)(\sigma_1+\sigma_2)}{8[3]_q}, \quad (\sigma_1,\sigma_2) \in \varpi, \quad \epsilon \in [0,2]. \end{aligned}$$

It can be inferred that the maximum parameter of $K_4(\sigma_1,\sigma_2)$ is located at the edge of ϖ, and when differentiating $K_4(\sigma_1,\sigma_2)$ with respect to σ_1, it can be further determined that

$$(K_4)_{\sigma_1}(\sigma_1,\sigma_2) = \frac{(4-\epsilon^2)(1-\eth)(A_q-B_q)}{8[3]_q} \quad \epsilon \in [0,2].$$

If $(K_4)_{\sigma_1}(\sigma_1, \sigma_2) \geq 0$, where $\sigma_2 \in [0,1]$ and $\epsilon \in [0,2]$, it can be deduced that the function $K_4(\sigma_1, \sigma_2)$ increases with σ_1 and reaches its peak at $\sigma_1 = 1$, implying that

$$\max[K_4(\sigma_1, \sigma_2) : \sigma_1 \in [0,1]] = K_4(1, \sigma_2) = |1 - I| \frac{(1-\eth)^2 (A_q - B_q)^2}{4[2]_q^2} \epsilon^2 \\ + \frac{(4-\epsilon^2)(1-\eth)(A_q - B_q)(1+\sigma_2)}{8[3]_q}, \quad \sigma_2 \in [0,1], \quad \epsilon \in [0,2].$$

Additional differentiation of $K_4(1, \sigma_2)$ results in

$$K_4'(1, \sigma_2) = \frac{(4-\epsilon^2)(1-\eth)(A_q - B_q)(1+\sigma_2)}{8[3]_q} \quad \epsilon \in [0,2].$$

If $(K_4)_{\sigma_1}(1, \sigma_2) \geq 0$, where $\epsilon \in [0,2]$, it can be deduced that the function $K_4(1, \sigma_2)$ increases and reaches its peak at $\sigma_2 = 1$, implying that

$$\max[K_3(1, \sigma_2) : \sigma_2 \in [0,1]] = K(1,1) = |1 - I| \frac{(1-\eth)^2 (A_q - B_q)^2}{4[2]_q^2} \epsilon^2 \\ + \frac{(4-\epsilon^2)(1-\eth)(A_q - B_q)}{4[3]_q}, \quad \epsilon \in [0,2].$$

Thus, we obtain

$$K_4(\sigma_1, \sigma_2) \leq \max[(\sigma_1, \sigma_2) : (\sigma_1, \sigma_2) \in \varpi] = K_4(1,1) \\ = |1 - I| \frac{(1-\eth)^2 (A_q - B_q)^2}{4[2]_q^2} \epsilon^2 + \frac{(4-\epsilon^2)(1-\eth)(A_q - B_q)}{4[3]_q}.$$

Because $|l_3 - I l_2^2| \leq K_3(\sigma_1, \sigma_2)$ gives

$$|l_3 - I l_2^2| \leq \frac{(1-\eth)^2 (A_q - B_q)^2}{4} \left[\frac{|1 - I| - R(\eth, q)}{[2]_q^2} \right] \epsilon^2 + \frac{(1-\eth)^2 (A_q - B_q)^2 R(\eth, q)}{[2]_q^2}$$

where

$$R(\eth, q) = \frac{[2]_q^2}{(1-\eth)(A_q - B_q)[3]_q}. \tag{33}$$

We introduce function $K_5 : [0,2] \longrightarrow \mathbb{R}$ to determine its maximum value for $\epsilon \in [0,2]$, defined as:

$$K_5(\epsilon) = \frac{(1-\eth)^2 (A_q - B_q)^2}{4} \left[\frac{|1 - I| - R(\eth, q)}{[2]_q^2} \right] \epsilon^2 + \frac{(1-\eth)^2 (A_q - B_q)^2 R(\eth, q)}{[2]_q^2}.$$

Additional differentiation of $K_5(\epsilon)$ results in

$$K_5'(\epsilon) = \frac{(1-\eth)^2 (A_q - B_q)^2}{2} \left[\frac{|1 - I| - R(\eth, q)}{[2]_q^2} \right] \epsilon.$$

If $K_5'(\epsilon) \leq 0$, then $K_5(\epsilon)$ is a decreasing function. The function peaks at $\epsilon = 0$ when $|1 - I| \leq R(\eth, q)$. So

$$\max[K_5(\epsilon); \epsilon \in [0,2]] = K(0) = \frac{(1-\eth)^2 (A_q - B_q)^2 R(\eth, q)}{[2]_q^2}.$$

and $K_5'(\epsilon) \geq 0$, and consequently, $K_5(\epsilon)$ will increase. The function peaks at $\epsilon = 2$ when $|1 - I| \geq R(\eth, q)$. So

$$\max[K_5(\epsilon); \epsilon \in [0,2]] = K(0) = \frac{(1-\eth)^2(A_q - B_q)^2|1-I|}{[2]_q^2}.$$

Thus, we have

$$\left|l_3 - Il_2^2\right| \leq \begin{cases} \frac{(1-\eth)^2(A_q-B_q)^2 R(\eth,q)}{[2]_q^2}, & |1-I| \leq R(\eth,q) \\ \frac{(1-\eth)^2(A_q-B_q)^2|1-I|}{[2]_q^2} & |1-I| \geq R(\eth,q). \end{cases} \tag{34}$$

Effectively, we can confirm that the result obtained in (34) applies to the function listed below:

$$f_2(z) = z + \frac{(1-\eth)(A_q - B_q)}{[3]_q} z^3 + \frac{(1-\eth)(A_q - B_q)}{[4]_q} z^4 + \cdots.$$

□

The Corollary mentioned in Theorem 3 holds true when the condition $\eth = 0$ is satisfied.

Corollary 7. *Let $f(z) \in \mathcal{JQ}_A^B(q)$. Then,*

$$\left|l_3 - Il_2^2\right| \leq \begin{cases} \frac{(A_q-B_q)^2 R(q)}{[2]_q^2}, & |1-I| \leq R(q) \\ \frac{(A_q-B_q)^2|1-I|}{[2]_q^2} & |1-I| \geq R(q). \end{cases}$$

where

$$R(q) = \frac{[2]_q^2}{(A_q - B_q)[3]_q}.$$

The accuracy of the results is confirmed using the functions listed below:

$$f_2(z) = z + \frac{(A_q - B_q)}{[3]_q} z^3 + \frac{(A_q - B_q)}{[4]_q} z^4 + \cdots.$$

The Corollary stated in Theorem 3 is valid if the requirement $\eth = 0$ and $q \uparrow 1$ is met.

Corollary 8. *Let $f(z) \in \mathcal{JQ}_A^B$. Then,*

$$\left|l_3 - Il_2^2\right| \leq \begin{cases} \frac{A-B}{3}, & |1-I| \leq \frac{4}{3(A-B)} \\ \frac{(A-B)^2|1-I|}{4} & |1-I| \geq \frac{4}{3(A-B)}. \end{cases}$$

The accuracy of the results is confirmed using the functions listed below:

$$f_2(z) = z + \frac{(A - B)}{3} z^3 + \frac{(A - B)}{4} z^4 + \cdots.$$

Theorem 4. *Suppose $f(z) \in \mathcal{J}\mathcal{Q}^{B}_{\eth,A}(q)$. Then,*

$$
\left|l_3 - Il_2^2\right| \leq \begin{cases} \frac{(1-\eth)^2(A_q-B_q)^2(1-I)}{[2]_q^2} & \text{if } I \leq 1 - R(\eth,q) \\ \\ \frac{(1-\eth)^2(A_q-B_q)^2 R(\eth,q)}{[2]_q^2} & \text{if } 1 - R(\eth,q) \leq I \leq 1 + R(\eth,q) \\ \\ \frac{(1-\eth)^2(A_q-B_q)^2(I-1)}{[2]_q^2} & \text{if } 1 + R(\eth,q) \leq I. \end{cases} \tag{35}
$$

where

$$
R(\eth,q) = \frac{[2]_q^2}{(1-\eth)(A_q - B_q)[3]_q}.
$$

Proof. Suppose $f(z) \in \mathcal{J}\mathcal{Q}^{B}_{\eth,A}(q)$, $0 \leq \eth < 1$, $-1 \leq A < B \leq 1$, $A_q = \frac{(1+A)+q(A-1)}{2}$, $B_q = \frac{(1+B)+q(B-1)}{2}$ and $q \in (0,1)$. We have $|1-I| \geq R(\eth,q)$ and $|1-I| \leq R(\eth,q)$ when $I \in \mathbb{R}$. This implies

$$
I \leq 1 - R(\eth,q) \text{ or } I \geq 1 + R(\eth,q)
$$

and

$$
1 - R(\eth,q) \leq I \leq 1 + R(\eth,q).
$$

□

The Corollary mentioned in Theorem 4 holds true when the condition $\eth = 0$ is satisfied.

Corollary 9. *Let $f(z) \in \mathcal{J}\mathcal{Q}^{B}_{A}(q)$. Then,*

$$
\left|l_3 - Il_2^2\right| \leq \begin{cases} \frac{(A_q-B_q)^2(1-I)}{[2]_q^2} & \text{if } I \leq 1 - R(q) \\ \\ \frac{(A_q-B_q)^2 R(q)}{[2]_q^2} & \text{if } 1 - R(q) \leq I \leq 1 + R(q) \\ \\ \frac{(A_q-B_q)^2(I-1)}{[2]_q^2} & \text{if } 1 + R(q) \leq I, \end{cases} \tag{36}
$$

where

$$
R(q) = \frac{[2]_q^2}{(A_q - B_q)[3]_q}.
$$

Additionally, we can derive the following for $I = 1$ from Theorem 4.

Corollary 10. *Let $f(z) \in \mathcal{J}\mathcal{Q}^{B}_{\eth,A}(q)$. Then,*

$$
|l_3 - l_2^2| \leq \frac{(1-\eth)(A_q - B_q)}{[3]_q}.
$$

7. Conclusions

This research extensively investigated the second Hankel determinant for a particular new subclass of bi-univalent functions characterized by the q-generalized Janowski function, generalized derivatives, and q-derivative. This specific subcategory is extremely fascinating in various mathematical fields, including geometry function theory and complex analysis. Our findings set the upper bounds for the bi-univalent functions in this newly proposed subclass. Furthermore, the Fekete–Szegö and second Hankel determinants were obtained, and these results are all accurate. The maximum value for the group was also identified to validate all the conclusions reviewed. Our study contributes to the overall understanding of bi-univalent functions, their subcategories, and their applications in diverse mathematical contexts.

Open Problems: The research presents some unresolved issues:

$$\eth_q f(z) \prec (1-\partial)\left[\frac{1+A_q z}{1+B_q z}\right]^B + \partial \tag{37}$$

and

$$\eth_q f^{-1}(w) \prec (1-\partial)\left[\frac{1+A_q w}{1+B_q w}\right]^B + \partial, \tag{38}$$

where $B \in (0,1]$ and $0 \leq \partial < 1$.

Researchers in the field have the opportunity to investigate this problem and utilize recent established operators like [40,41] in geometric function theory on bounded turning function or other subclasses like starlike function, convex functions, and others, too, to uncover additional properties like the third Hankel determinant [39].

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