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On Some Multipliers Related to Discrete Fractional Integrals

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Abstract: This paper explores the properties of multipliers associated with discrete analogues of fractional integrals, revealing intriguing connections with Dirichlet characters, Euler's identity, and Dedekind zeta functions of quadratic imaginary fields. Employing Fourier transform techniques, the Hardy–Littlewood circle method, and a discrete analogue of the Stein–Weiss inequality on product space through implication methods, we establish $\ell^p \to \ell^q$ bounds for these operators. Our results contribute to a deeper understanding of the intricate relationship between number theory and harmonic analysis in discrete domains, offering insights into the convergence behavior of these operators.

Keywords: discrete analogues; multipliers; fractional integrals; Hardy–Littlewood method

MSC: 42A45; 42A55; 11M06; 11M41

1. Introduction

The study of discrete analogues in harmonic analysis indeed shares a companionable relationship with the early history of singular integrals. Singular integrals, which arise from the convolution of functions with singular or highly oscillatory kernels, have been a central focus of harmonic analysis since its inception. For example, in 1928, M. Riesz [1] proved the Hilbert transform,

$$Hf(x) = \int_{\mathbb{R}^k} \frac{f(x-y)}{y} dy,$$

is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, and this implies its discrete analogue,

$$\mathcal{H}f(n) = \sum_{\substack{m\in\mathbb{Z}^k \\ m\neq 0}} \frac{f(n-m)}{m},$$

is bounded on $\ell^p(\mathbb{Z}^k)$ for all $1 < p < \infty$. Here, $\ell^p(\mathbb{Z}^k)$ is defined as

$$\ell^p(\mathbb{Z}^k) = \left\{ f \text{ defined on } \mathbb{Z}^k | \sum_{m\in\mathbb{Z}^k} |f(m)|^p < \infty \right\}.$$

Moreover, $||f||_{\ell^p(\mathbb{Z}^k)} = \left(\sum_{m\in\mathbb{Z}^k} |f(m)|^p\right)^{\frac{1}{p}}$.

Another classical family of operators in harmonic analysis are the fractional integral operators,

$$I_s f(x) = \int_{\mathbb{R}} \frac{f(x-y)}{|y|^s} dy, \quad 0 < s < 1.$$

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It is well known that for $1 < p < q < \infty$ with $1/q = 1/p - (1-s)$, I_s is a bounded operator from $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$. The discrete analogue of this operator is defined by

$$\mathcal{I}_s f(n) = \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \frac{f(n-m)}{|m|^s}.$$

Similarly, the boundedness of I_s implies the boundedness of $\mathcal{I}_s$. Consider a function f defined on $\mathbb{Z}$, where $0 < s < 1$ and $k \geq 1$ is an integer. The discrete fractional operator $\mathcal{I}_{s,k}$ is defined as follows:

$$\mathcal{I}_{s,k} f(m) = \sum_{n=1}^{\infty} \frac{f(m-n^k)}{n^s}, \quad m \in \mathbb{Z} \tag{1}$$

acting on functions defined on $\mathbb{Z}$. Stein and Wainger [2] initiated the study of the $\ell^p \to \ell^q$ boundedness of $\mathcal{I}_{s,k}$, that is, there exists some constant C such that

$$||\mathcal{I}_{s,k} f||_{\ell^q} \leq C||f||_{\ell^p}.$$

On the other hand, for f defined on $\mathbb{Z}$, its Fourier transform is defined by $\hat{f}(x) = \sum_{n \in \mathbb{Z}} f(n) e^{-2\pi i n x}$. Therefore,

$$\widehat{\mathcal{I}_{s,k} f}(x) = m_{s,k}(x) \hat{f}(x), \qquad m_{s,k}(x) = \sum_{n=1}^{\infty} \frac{e^{-2\pi i n^k x}}{n^s}.$$

Here, $m_{s,k}(x)$ is called the Fourier multiplier.

They demonstrated that when $1/2 < s < 1$, $m_{s,2}$ belongs to weak-type $\mathrm{L}^{2/(1-s)}[0,1]$ and $m_{s,k}$ belongs to weak-type $\mathrm{L}^{k/(1-s)}[0,1]$ as long as s is sufficiently close to 1. The main tool is the Hardy–Littlewood circle method; for a more detailed introduction on circle methods, see [3]. Furthermore, if this holds for all $1/2 < s < 1$, it would imply the "Hypothesis K^*" of Hardy, Littlewood, and Hooley, which remains an open problem in number theory.

Lillian Pierce's thesis [4] extended this result to positive definite quadratic forms. For instance, let $Q(x) = \frac{1}{2} x^t A x$ be a positive definite quadratic form, where A is a real, positive definite, 2×2 symmetric matrix with integer entries and even diagonal entries. Then, the corresponding multiplier

$$m_{s,Q}(x) = \sum_{\substack{m \in \mathbb{Z}^2 \\ m \neq 0}} \frac{e^{-2\pi i Q(m) x}}{Q(m)^s}$$

is of weak-type $\mathrm{L}^{1/(1-s)}[0,1]$. In this paper, we mainly consider the 'twisted' multiplier. Let a_n be a complex series, and define the corresponding multiplier by

$$m_{s,\{a_n\}}(x) = \sum_{n=1}^{\infty} \frac{a_n e^{-2\pi i n x}}{n^s}. \tag{2}$$

It is worth noting that if we set $a_n = 1$ for $n = m^k \geq 1$ and $a_n = 0$ otherwise, then $m_{s,\{a_n\}}(x) = m_{sk,k}(x)$. The series a_n can originate from various areas related to number theory.

For instance, in Section 2, we delve into a **primitive** Dirichlet character χ modulo N, defining the corresponding twisted multiplier as

$$m_{s,\chi}(x) = \sum_{n=1}^{\infty} \frac{\chi(n) e^{-2\pi \mathrm{i} n^2 x}}{n^s}, \tag{3}$$

which differs from $m_{s,2}(x)$ in several respects. For example, $m_{s,\chi}(0)$ corresponds to the Dirichlet L-function and is bounded when $0 < s < 1$, whereas $m_{s,2}(x)$ tends to infinity as x approaches 0. However, we will demonstrate the following Theorem.

Theorem 1. *For $1/2 < s < 1$, let χ be a primitive Dirichlet character modulo N; then, $m_{s,\chi}$ belongs to weak-type $L^{2/(1-s)}[0,1]$.*

In Section 3, we investigate the scenario where a_n originates from Euler's identity, given by

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{n=0}^{\infty} a_n x^n.$$

We establish that $m_{s,\{a_n\}}$ belongs to weak-type $\mathrm{L}^{2/(1-2s)}[0,1]$ and provide an improved result regarding the regularity of the corresponding discrete fractional integral operator.

In Section 4, we delve into imaginary quadratic fields and the associated Dedekind zeta function. We demonstrate the close connection between the corresponding multipliers and positive definite quadratic forms as investigated by Lillian Pierce.

In the final section, we tackle the discrete analogue of the Stein–Weiss inequality on product space. Employing the "implication" method, we deduce the regularity property of the discrete fractional operator.

Discrete analogues in harmonic analysis have garnered significant attention in recent decades, with notable contributions from scholars such as Stein and Wainger [2,5], Oberlin [6] and Lillian Pierce [4,7,8] (see also [9–12]). This paper introduces a novel perspective to the study of multipliers in harmonic analysis by incorporating primitive Dirichlet characters. This addition not only enriches the theoretical framework but also presents new challenges and complexities to be explored. As examples of the application of this approach, this paper investigates multipliers associated with Euler's identity and quadratic imaginary fields.

Remark 1. *Let $r > 0$ be a real number; we define a function f as belonging to weak-type $L^r[0,1]$ if*

$$|\{x \in [0,1] : f(x) > \alpha\}| \le c\alpha^{-r}, \qquad \text{for} \qquad \alpha > 0$$

where c is a constant independent of α and f.

2. Multipliers Twisted with Dirichlet Characters

Fix an integer $N > 1$. A **Dirichlet character modulo** N is a function $\chi : \mathbb{Z} \to \mathbb{S}^1 \cup \{0\}$ such that the following is true:

(i) $\chi(n) = 0$ if and only if $(n, N) > 1$.
(ii) $\chi(n) = \chi(m)$ if $n \equiv m \mod N$.
(iii) $\chi(mn) = \chi(m)\chi(n)$ for $m, n \in \mathbb{Z}$.

For simplicity, we consider a **primitive** Dirichlet character χ modulo N. Let $\bar{\chi}(n) = \overline{\chi(n)}$ and the *Gauss sum* $\tau(\chi)$ be defined by the formula

$$\tau(\chi) = \sum_{n \bmod N} \chi(n) e^{2\pi i \frac{n}{N}}.$$

It is well known that

$$\chi(n) = \frac{\chi(-1)\tau(\chi)}{N} \sum_{m \bmod N} \overline{\chi(m)} e^{2\pi i \frac{mn}{N}} \qquad \text{and} \qquad |\tau(\chi)| = N^{\frac{1}{2}}. \tag{4}$$

Note that the right-hand side of (4) is defined when n is an arbitrary real number.

Let $m_{s,\chi}(x)$ be defined in (3), since the function $m_{s,\chi}(x)$ is in $L^2[0,1]$. Thus, the series is Abel–Gauss summable almost everywhere; hence,

$$m_{s,\chi}(x) = \lim_{\epsilon \to 0} \sum_{n=1}^{\infty} \chi(n) n^{-s} e^{-2\pi i n^2 x} e^{-\pi n^2 \epsilon}.$$

Next, we consider the $\Theta-$function

$$S_y(x) = \sum_{n=-\infty}^{\infty} \chi(n) e^{-\pi n^2 (y+2ix)}, \tag{5}$$

Note that

$$m_{s,\chi}(x) = C_\alpha \int_0^1 S_y(x) y^{-1+\frac{s}{2}} dy + O(1). \tag{6}$$

Now, in the definition of $S_y(x)$, we can replace the n, which ranges over $\mathbb{Z}$, with $n = mq + \ell$, where m ranges over $\mathbb{Z}$ and ℓ ranges over $1 \le \ell \le q$. Then, $S_y(x)$ equals

$$\sum_{\ell=1}^{q} e^{2\pi i \ell^2 \frac{p}{q}} \left\{ \sum_{m\in\mathbb{Z}} \chi(mq+\ell) e^{-\pi(mq+\ell)^2 y} \right\}. \tag{7}$$

For the inner sum, we use the Poisson summation formula $\sum_{m\in\mathbb{Z}} f(m) = \sum_{m\in\mathbb{Z}} \hat{f}(m)$, with $f(s) = \chi(sq+\ell) e^{-\pi(sq+\ell)^2 y}$. Then,

$$\begin{aligned}
\hat{f}(\xi) &= \int_{-\infty}^{\infty} \chi(sq+\ell) e^{-\pi(sq+\ell)^2} e^{-2\pi i s\cdot\xi} ds \\
&= \frac{\chi(-1)\tau(\chi)}{qN} \sum_{k \ mod \ N} \overline{\chi(k)} \int_{-\infty}^{\infty} \chi(u) e^{-\pi u^2} e^{-2\pi i \left(\frac{u-\ell}{q}\right)\cdot\xi} e^{2\pi i \frac{ku}{N}} du \\
&= \frac{\chi(-1)\tau(\chi)}{q y^{\frac{1}{2}} N} \sum_{k \ mod \ N} \overline{\chi(k)} e^{2\pi i \frac{\ell\xi}{q}} e^{-\frac{\pi}{y}\left(\frac{\xi}{q} - \frac{k}{N}\right)^2}.
\end{aligned}$$

Therefore, we have

$$S_y(x) = \frac{\chi(-1)\tau(\chi)}{q(y+2i\delta)^{\frac{1}{2}} N} \sum_{k \ mod \ N} \overline{\chi(k)} \sum_{m=-\infty}^{\infty} S\left(\frac{p}{q}, \frac{m}{q}\right) e^{-\frac{\pi}{y+2i\delta}\left(\frac{m}{q} - \frac{k}{N}\right)^2}, \tag{8}$$

where

$$S\left(\frac{p}{q}, \frac{m}{q}\right) = \sum_{\ell=1}^{q} e^{2\pi i \left(\frac{p}{q}\ell^2 + \frac{m}{q}\ell\right)}, \qquad x = \frac{p}{q} + \delta.$$

It suffices to consider $T_y(x)$ defined as

$$T_y(x) = \frac{1}{q(y+2i\delta)^{\frac{1}{2}}} \sum_{m=-\infty}^{\infty} S\left(\frac{p}{q}, \frac{m}{q}\right) e^{-\frac{\pi}{y+2i\delta}\left(\frac{m}{q} - \frac{1}{N}\right)^2}. \tag{9}$$

Now, write

$$\int_0^1 T_y(x) y^{-1+\frac{s}{2}} dy = \sum_{j=0}^{\infty} \int_{2^{-j-1}}^{2^{-j}} T_y(x) y^{-1+\frac{s}{2}} dy. \tag{10}$$

and estimate $T_y(x)$ when y is of a fixed order of magnitude and x is "sufficiently close" to an appropriate rational $\frac{p}{q}$, with $(p,q) = 1$, $0 < \frac{p}{q} \le 1$. Actually, we have the following Lemma.

Lemma 1. *If $x = \frac{p}{q} + \delta$, with $q \lesssim y^{-\frac{1}{2}}$ and $q|\delta| \lesssim y^{\frac{1}{2}}$, then*

$$\begin{aligned}
T_y(x) &= S\left(\tfrac{p}{q}, \tfrac{1}{N}\right) \frac{1}{q(y+2i\delta)^{\frac{1}{2}}} + O(y^{-\frac{1}{4}}), \qquad &\text{if} \quad N|q, \\
T_y(x) &= O(y^{-\frac{1}{4}}) &\text{if} \quad N \nmid q.
\end{aligned} \tag{11}$$

Proof. It suffices to prove that

$$O\left(\frac{1}{q(y+2i\delta)^{\frac{1}{2}}}\sum_{m\neq\frac{q}{N}} S\left(\frac{p}{q},\frac{m}{q}\right)e^{-\frac{\pi}{y+2i\delta}\left(\frac{m}{q}-\frac{1}{N}\right)^2}\right)=O(y^{-\frac{1}{4}}).$$

Note that $S\left(\frac{p}{q},\frac{m}{q}\right)=O(q^{\frac{1}{2}})$; we can write

$$O\left(\frac{1}{q(y+2i\delta)^{\frac{1}{2}}}\sum_{m\neq\frac{q}{N}} S\left(\frac{p}{q},\frac{m}{q}\right)e^{-\frac{\pi}{y+2i\delta}\left(\frac{m}{q}-\frac{1}{N}\right)^2}\right)=O\left(q^{-\frac{1}{2}}(y^2+4\delta^2)^{-\frac{1}{4}}\sum_{m\neq\frac{q}{N}} e^{-\frac{\pi y}{y^2+4\delta^2}\left(\frac{m}{q}-\frac{1}{N}\right)^2}\right)$$

$$=O\left(q^{-\frac{1}{2}}(y^2+4\delta^2)^{-\frac{1}{4}}\sum_{m=1}^{\infty} e^{-\frac{\pi y}{y^2+4\delta^2}\frac{m^2}{q^2}}\right).$$

However, by our assumptions we have $1\lesssim\frac{y}{q^2(y^2+4\delta^2)}$. Moreover,

$$\sum_{m=1}^{\infty} e^{-Cm^2u}\lesssim e^{-\bar{C}u}\lesssim u^{-\frac{1}{4}},\qquad 1\lesssim u.$$

So, the error term is

$$O\left(q^{-\frac{1}{2}}(y^2+4\delta^2)^{-\frac{1}{4}}\cdot y^{-\frac{1}{4}}q^{\frac{1}{2}}\cdot(y^2+4\delta^2)^{\frac{1}{4}}\right)=O(y^{-\frac{1}{4}}),$$

and (11) is proved. □

Proof of Theorem 1. Let us turn to (10), and we make the same decomposition of the x-interval as in [2] . For y of the order 2^{-j}, we make a Farey dissection of the x-interval $[0,1]$. Now, we choose all fractions p/q, $(p,q)=1$, with $q\leq 2^{\frac{j}{2}}$, and let $I^j_{p/q}$ be the corresponding interval for p/q. Then, $I^j_{p/q}\subset\{x:|x-p/q|\leq 1/q(2^{j/2})\}$. Then, we can define the major arcs and minor arcs as follows:

$$I^j_{p/q}\quad \textit{is a major arc if}\quad q\leq\frac{1}{10}2^{j/2},$$

$$I^j_{p/q}\quad \textit{is a minor arc if}\quad \frac{1}{10}2^{j/2}\leq q\leq 2^{j/2}.$$

Additionally, we define $\tilde{I}_{p/q}$, indepedent of j, as

$$\tilde{I}_{p/q}=\{x:|x-p/q|\leq 1/10q^2\}.$$

The key property of $\tilde{I}_{p/q}$ is that if $q\leq q'\leq 2q$, the intervals $\tilde{I}_{p'/q'}$ and $\tilde{I}_{p/q}$ are disjointed (or identical) (see [2]).

Now, we apply (11). If x belongs to a major arc, this implies

$$\begin{aligned} T_y(x)&=S\left(\tfrac{p}{q},\tfrac{1}{N}\right)\frac{1}{q(y+2i\delta)^{\frac{1}{2}}}+O(2^{\frac{j}{4}}), &&\textit{if}\quad N|q,\\ T_y(x)&=O(2^{\frac{j}{4}}) &&\textit{if}\quad N\nmid q.\end{aligned}\tag{12}$$

If x belongs to a minor arc, then

$$T_y(x)=O(2^{\frac{j}{4}}).\tag{13}$$

This is because
$$\frac{1}{q}S\left(\frac{p}{q},\frac{1}{N}\right)=O(q^{-\frac{1}{2}})=O(2^{-j/4}),$$
and
$$|y+2i\delta|^{-\frac{1}{2}}\leq y^{-\frac{1}{2}}=O(2^{\frac{j}{2}}).$$
The contribution from all the minor arcs is therefore
$$O\left(\sum_{j=0}^{\infty}\int_{2^{-j-1}}^{2^{-j}}2^{\frac{j}{4}}y^{-1+\frac{s}{2}}dy\right)=O\left(\sum_{j=0}^{\infty}2^{\frac{j}{4}}2^{-j(-1+\frac{s}{2})}2^{-j}\right)=O(1).\qquad since\qquad s>\frac{1}{2}.$$
Next, we sum over the major arcs. Fix $\frac{p}{q}$; then,
$$\frac{1}{q}\left|S\left(\frac{p}{q},\frac{1}{N}\right)\right|\sum_{j=0}^{\infty}\int_{2^{-j-1}}^{2^{-j}}\left|y+2i(x-\frac{p}{q})\right|^{-\frac{1}{2}}y^{-1+\frac{s}{2}}dy\cdot\chi_{I^j_{\frac{p}{q}}}(x)$$
$$\leq C\frac{1}{q}\left|S\left(\frac{p}{q},\frac{1}{N}\right)\right|\left|x-\frac{p}{q}\right|^{-\frac{1}{2}+\frac{s}{2}}\cdot\chi_{\tilde{I}_{\frac{p}{q}}}(x),\qquad 0<s<1.$$
Therefore, the total contribution of the major arcs is majorized by
$$\sum_{\frac{p}{q},\,N|q}q^{-\frac{1}{2}}\left|x-\frac{p}{q}\right|^{-\frac{1}{2}+\frac{s}{2}}\cdot\chi_{\tilde{I}_{\frac{p}{q}}}(x).\tag{14}$$
Rewrite sum (14) as
$$\sum_{t=0}^{\infty}\sum_{\substack{p/q,\,N|q\\2^t\leq q<2^{t+1}}}q^{-\frac{1}{2}}\left|x-\frac{p}{q}\right|^{-\frac{1}{2}+\frac{s}{2}}\cdot\chi_{\tilde{I}_{\frac{p}{q}}}(x).\tag{15}$$
Note that there are at most $O(N^{-1}2^{2t})$ disjointed intervals for $2^s\leq q<2^{t+1}$. Moreover, $|x-p/q|^{-1/2+s/2}$ is uniformly of weak-type $L^{\frac{2}{1-s}}$. Thus, applying Lemma one in [2] , then
$$\sum_{\substack{p/q,\,N|q\\2^t\leq q<2^{t+1}}}q^{-\frac{1}{2}}\left|x-\frac{p}{q}\right|^{-\frac{1}{2}+\frac{s}{2}}\cdot\chi_{\tilde{I}_{\frac{p}{q}}}(x).$$
has a weak-type $L^{\frac{2}{1-s}}$ norm bounded by
$$q^{-\frac{1}{2}}(N^{-1}2^{2t})^{\frac{1-s}{2}}=O(2^{-\frac{t}{2}}N^{-\frac{1-s}{2}}2^{t(1-s)})=O(N^{-\frac{1-s}{2}}2^{t(\frac{1}{2}-s)}),$$
and the sum in (15) converges if $s>\frac{1}{2}$. This means that $m_\alpha$is of weak-type $L^{\frac{2}{1-s}}[0,1]$. The proof is complete. □

3. Multiplier Related to Euler's Identity

Let $\mathbb{H}$ be the Poincaré upper half plane consisting of $z=x+iy$ where $x,y\in\mathbb{R}$ and $y>0$. Suppose f is defined on $\mathbb{H}$ and has a Fourier expansion
$$f(z)=\sum_{n=0}^{\infty}a_ne^{2\pi inz}.$$
We consider the multiplier
$$m_{s,f}(x)=\sum_{n=1}^{\infty}\frac{a_ne^{-2\pi inx}}{n^s}.\tag{16}$$

For $s > 0$, applying the well-known formula

$$\int_0^\infty e^{-2\pi ny} y^s \frac{dy}{y} = \frac{\Gamma(s)}{(2\pi n)^s},$$

we have

$$m_{s,f}(x) = c_s \int_0^1 f(-x+iy) y^s \frac{dy}{y} + O(1), \tag{17}$$

where $c_s > 0$ is a constant that only depends on s.

Now, let us consider the case

$$f(z) = \prod_{n=1}^{\infty} (1 - e^{2\pi inz}) = \sum_{n=0}^{\infty} a_n e^{2\pi inz}, \quad z \in \mathbb{H},$$

and recall Euler's identity:

$$f(z) = \prod_{n=1}^{\infty} (1 - e^{2\pi inz}) = \sum_{n=-\infty}^{\infty} (-1)^n e^{2\pi i \frac{3n^2+n}{2} z}$$

Hence, we can write $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i(6n^2+n)z}, \qquad f_2(z) = -\sum_{n\in\mathbb{Z}} e^{2\pi i(6n^2+7n+2)z}.$$

Our analysis will then proceed by setting

$$\int_0^1 f(-x+iy) y^s \frac{dy}{y} = \sum_{j=0}^{\infty} \int_{2^{-j-1}}^{2^{-j}} f(-x+iy) y^s \frac{dy}{y}, \tag{18}$$

and estimating $f(-x+iy)$ when y is of a fixed order of magnitude and x is "sufficiently close" to an appropriate rational p/q, with $(p,q) = 1,\ 0 < p/q \le 1$.

Lemma 2. *Let $x \in [0,1]$. If*

$$x = p/q + \delta, \qquad q \lesssim y^{-\frac{1}{2}}, \qquad q|\delta| \lesssim y^{\frac{1}{2}},$$

then

$$f_1(-x+iy) = \frac{1}{\sqrt{12}q(y+i\delta)^{1/2}} e^{\frac{\pi(y+i\delta)}{12}} S(p/q) + O(y^{-1/4}), \tag{19}$$

where

$$S(p/q) = \sum_{\ell=1}^{q} e^{-2\pi i \frac{p}{q}(6\ell^2+\ell)}.$$

Proof. First, consider $f_1(-\frac{p}{q}+iy)$. Write $\sum_{n\in\mathbb{Z}} = \sum_{\ell=1}^{q}\sum_{m\in\mathbb{Z}}$ and $n = mq+\ell$. Then, $f_1(-\frac{p}{q}+iy)$ equals

$$\sum_{\ell=1}^{q} e^{-2\pi i \frac{p}{q}(6\ell^2+\ell)} \left\{ \sum_{m\in\mathbb{Z}} e^{-2\pi y[6(mq+\ell)^2+mq+\ell]} \right\} \tag{20}$$

For the inner sum, set $h(x) = e^{-2\pi y[6(xq+\ell)^2+xq+\ell]}$. Then, its Fourier transform is

$$\hat{h}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi y[6(xq+\ell)^2+xq+\ell]} e^{-2\pi i x\cdot\xi} dx$$

$$= \frac{1}{q}\int_{-\infty}^{\infty} e^{-2\pi y(6u^2+u)} e^{-2\pi i\left(\frac{u-\ell}{q}\right)\xi} du$$

$$= \frac{1}{\sqrt{12}qy^{1/2}} e^{\frac{\pi y}{12}} e^{2\pi i\left(\frac{\ell\xi}{q}+\frac{\xi}{12q}\right)} e^{-\frac{\pi\xi^2}{12q^2 y}}.$$

Now, using the Poisson summation formula, we have

$$f_1(-\frac{p}{q}+iy) = \frac{1}{\sqrt{12}qy^{1/2}} e^{\frac{\pi y}{12}} \sum_{m\in\mathbb{Z}} S(p/q, m/q) e^{2\pi i \frac{m}{12q}} e^{-\frac{\pi m^2}{12q^2 y}},$$

Therefore, let $y \to y + i\delta$, which yields

$$f_1(-x+iy) = \frac{1}{\sqrt{12}q(y+i\delta)^{1/2}} e^{\frac{\pi(y+i\delta)}{12}} \sum_{m\in\mathbb{Z}} S(p/q, m/q) e^{2\pi i \frac{m}{12q}} e^{-\frac{\pi m^2}{12q^2(y+i\delta)}}. \tag{21}$$

Now, set $S(p/q) = S(p/q, 0)$. From (21), we see, upon isolating the term $m = 0$,

$$f_1(-x+iy) = \frac{1}{\sqrt{12}q(y+i\delta)^{1/2}} e^{\frac{\pi(y+i\delta)}{12}} S(p/q)$$

$$+O(q^{-1}(y^2+\delta^2)^{-1/4} \cdot q^{\frac{1}{2}} \cdot \sum_{m=1}^{\infty} e^{-\pi m^2 y/(12q^2(y^2+\delta^2))}).$$

Hence, similar to the proof of Lemma 1, (19) is proved. □

Theorem 2. *When* $1/4 < s < 1/2$, $m_{s,f}$ *belongs to weak-type* $L^{2/(1-2s)}[0,1]$.

Proof. It suffices to prove that $m_{s,f_1} \in L^{2/(1-2s),\infty}[0,1]$. Let us turn to (17), and we make the same decomposition of the x-interval as in [2] .

Now, we apply (19). If x belongs to a major arc, this implies that

$$f_1(-x+iy) = \frac{1}{\sqrt{12q(y+i\delta)^{1/2}}} e^{\frac{\pi(y+i\delta)}{12}} S(p/q) + O(2^{j/4}), \tag{22}$$

If x belongs a minor arc, then

$$f_1(-x+iy) = O(2^{j/4}), \tag{23}$$

This is because

$$\frac{1}{q} S\left(\frac{p}{q}\right) = O(q^{-\frac{1}{2}}) = O(2^{-j/4}),$$

and

$$|y+i\delta|^{-\frac{1}{2}} \leq y^{-\frac{1}{2}} = O(2^{\frac{j}{2}}).$$

The contribution from all the minor arcs is therefore

$$O\left(\sum_{j=0}^{\infty} \int_{2^{-j-1}}^{2^{-j}} 2^{\frac{j}{4}} y^s \frac{dy}{y}\right) = O\left(\sum_{j=0}^{\infty} 2^{\frac{j}{4}} 2^{-j(s-1)} 2^{-j}\right) = O(1), \qquad \textit{since} \qquad s > \frac{1}{4}.$$

Next, we sum over the major arcs. Fix $\frac{p}{q}$. Then,

$$\frac{1}{q}\left|S\left(\frac{p}{q}\right)\right|\sum_{j=0}^{\infty}\int_{2^{-j-1}}^{2^{-j}}\left|y+i(x-\tfrac{p}{q})\right|^{-\frac{1}{2}}y^{s}\frac{dy}{y}\cdot\chi_{I^{j}_{\frac{p}{q}}}(x)$$

$$\leq C\frac{1}{q}\left|S\left(\frac{p}{q}\right)\right|\left|x-\frac{p}{q}\right|^{s-\frac{1}{2}}\cdot\chi_{\tilde{I}_{\frac{p}{q}}}(x),$$

because

$$\int_{0}^{1}|y+i\delta|^{-\frac{1}{2}}y^{s}\frac{dy}{y}\leq\int_{0}^{\infty}|y+i\delta|^{-\frac{1}{2}}y^{s}\frac{dy}{y}=C|\delta|^{s-1/2},$$

as long as $0<s<1/2$.

Therefore, the total contribution of the major arcs is majorized by

$$\sum_{p/q}q^{-\frac{1}{2}}\left|x-\frac{p}{q}\right|^{s-1/2}\cdot\chi_{\tilde{I}_{\frac{p}{q}}}(x). \tag{24}$$

Rewrite sum (24) as

$$\sum_{t=0}^{\infty}\sum_{\substack{p/q,\\ 2^{t}\leq q<2^{t+1}}}q^{-\frac{1}{2}}\left|x-\frac{p}{q}\right|^{s-1/2}\cdot\chi_{\tilde{I}_{\frac{p}{q}}}(x).$$

Note that there are at most $O(2^{2t})$ disjointed intervals for $2^{t}\leq q<2^{t+1}$. Moreover, $|x-p/q|^{s-1/2}$ is uniformly of weak-type $L^{\frac{2}{1-2s}}[0,1]$. Thus, applying Lemma one in [2], then

$$\sum_{\substack{p/q,\\ 2^{t}\leq q<2^{t+1}}}q^{-\frac{1}{2}}\left|x-\frac{p}{q}\right|^{s-1/2}\cdot\chi_{\tilde{I}_{\frac{p}{q}}}(x).$$

has a weak $L^{\frac{2}{1-2s}}$ norm bounded by

$$q^{-\frac{1}{2}}2^{2t(1/2-s)}=O(2^{-\frac{t}{2}}2^{t(1-2s)})=O(2^{t(1/2-2s)}),$$

and the sum in (24) converges if $s>1/4$. This means that $m_{s,f_1}\in L^{2/(1-2s),\infty}[0,1]$. Therefore, $m_{s,f}$ belongs to weak-type $L^{2/(1-2s)}[0,1]$. □

The Corresponding Discrete Fractional Integral

Let g be a function of $\mathbb{Z}$. The twisting discrete fractional operator $\mathcal{I}_{s,f}$ is defined as

$$\mathcal{I}_{s,f}g(m)=\sum_{n=1}^{\infty}\frac{a_n}{n^{s}}g(m-n),\quad m\in\mathbb{Z}^{1}$$

and has acting functions defined on $\mathbb{Z}^{1}$. If $g(n)=1,\ n\in\mathbb{Z},$ then $\mathcal{I}_{s,f}1=\sum_{n=1}^{\infty}a_n n^{-s}$ is usually called the $L-$ function of f. On the other hand, if $a_n\equiv 1, n=1,2,\ldots,$ then we can write

$$\mathcal{I}_{s}g(m)=\sum_{n=1}^{\infty}\frac{g(m-n)}{n^{s}},\quad m\in\mathbb{Z}^{1},$$

Stein and Wainger [2] proved the following theorem.

Theorem 3. *For* $0<s<1$, *then*

$$||\mathcal{I}_{s}f||_{\ell^{q}(\mathbb{Z})}\leq C||f||_{\ell^{p}(\mathbb{Z})},$$

if

$$\frac{1}{q} \leq \frac{1}{p} - 1 + s, \qquad 1 < p < q < \infty.$$

Then it is easy to see that $a_n = \pm 1, 0$ for all $n \in \mathbb{N}$. Then, if $1/q \leq 1/p - 1 + s$, $1 < p < q < \infty$,

$$||\mathcal{I}_{s,f} g||_{\ell^q(\mathbb{Z})} \leq C||g||_{\ell^p(\mathbb{Z})}.$$

However, if we take the cancellation property of a_n into consideration we have a better result in the range of $1/4 < s < 1/2$.

Theorem 4. *For $1/4 < s < 1/2$, then*

$$||\mathcal{I}_{s,f} g||_{\ell^q(\mathbb{Z})} \leq C||g||_{\ell^p(\mathbb{Z})},$$

if

$$\frac{1}{q} \leq \frac{1}{p} - \frac{1}{2} + s, \qquad 1 < p \leq 2 \leq q < \infty.$$

In order to obtain the the desired $\ell^p \to \ell^q$ inequalities, we need a "folk" Lemma due to Stein and Wainger [2] concerning a convolution operator $\mathcal{I}$ with multiplier m, viz.,

$$\widehat{\mathcal{I}f}(x) = m(x)\hat{f}(x).$$

Lemma 3. *Assume $m(x)$ is of weak-type $L^r[0,1]$. Then, $\mathcal{I}$ is bounded from $\ell^p(\mathbb{Z}^1)$ to $\ell^q(\mathbb{Z}^1)$ if*

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{r}, \qquad 1 < p \leq 2 \leq q < \infty.$$

Proof. First, assume that $q = 2$, so that $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$. Then, for $f \in \ell^p(\mathbb{Z}^1)$, using Paley's version of the Hausdorff–Young inequality, $\hat{f} \in L^{p',p}[0,1]$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Therefore, by the multiplicative property of Lorentz spaces,

$$\widehat{\mathcal{I}f}(x) = m(x)\hat{f}(x) \in L^{r,\infty} \cdot L^{p',p} \subset L^{p_0,q_0},$$

where $\frac{1}{p_0} = \frac{1}{p'} + \frac{1}{2} = \frac{1}{2}$, $\frac{1}{q_0} = \frac{1}{p} + \frac{1}{\infty} = \frac{1}{p}$. Therefore, $\hat{\mathcal{I}f} \in L^{2,p} \subset L^{2,2} = L^2$ since $p \leq 2$ by assumption. Hence, $\mathcal{I}f \in \ell^2(\mathbb{Z}^1)$ and so $\mathcal{I}$ maps $\ell^p(\mathbb{Z}^1)$ to $\ell^2(\mathbb{Z}^1)$. The case when $p = 2$ and $\frac{1}{2} - \frac{1}{r} = \frac{1}{q}$ follows by considering the adjoint operator of $\mathcal{I}$, and the Lemma then follows by interpolation between the two resulting bounds for $\mathcal{I}$. □

Now, Theorem 4 is just an immediate corollary of Lemma 3 and Theorem 2 and note that $\ell^p \subset \ell^q$ if $p \leq q$.

4. Multipliers Related to Imaginary Quadratic Fields

Now, consider the imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$ of discriminant $D < 0$. Let $\mathcal{O} \subset K$ denote the ring of integers. Let I be the group of fractional ideals $\neq 0$,

$$I = \left\{ \frac{\mathfrak{a}_1}{\mathfrak{a}_2} : \mathfrak{a}_1, \mathfrak{a}_2 \subset \mathcal{O}, \mathfrak{a}_1\mathfrak{a}_2 \neq 0 \right\}$$

and $P \subset I$ the subgroup of principal ideals

$$P = \{(a) = a\mathcal{O} : a \in K^*\}.$$

Then, $\mathcal{H} = I/P$ is the class group, and the class number of K is $h = [I : P] < \infty$. For more detailed background of imaginary quadratic fields, see [13]. Define the fractional integral associated with the imaginary quadratic field by

$$\mathcal{I}_{s,K} f(m) = \sum_{0 \neq \mathfrak{a} \subset \mathcal{O}} \frac{f(m - N(\mathfrak{a}))}{N(\mathfrak{a})^s}, \qquad m \in \mathbb{Z}, \tag{25}$$

where $\mathfrak{a}$ ranges over non-zero integral ideals and $N : I \to \mathbb{Q}^*$ is the norm. In the case of integral ideals, $N(\mathfrak{a}) = \#(\mathcal{O}/\mathfrak{a})$.

The corresponding multiplier is

$$m_{s,K}(x) = \sum_{0 \neq \mathfrak{a} \subset \mathcal{O}} \frac{e^{-2\pi i x N(\mathfrak{a})}}{N(\mathfrak{a})^s},$$

that is,

$$\widehat{\mathcal{I}_{s,K} f}(x) = m_{s,K}(x) \hat{f}(x), \qquad \hat{f}(x) = \sum_{n \in \mathbb{Z}} f(n) e^{-2\pi i n x}.$$

Our main result is stated as below.

Theorem 5. *For $\frac{1}{2} < s < 1$, then $m_{s,K}$ belongs to weak-type $L^{1/(1-s)}[0,1]$.*

Proof. Let w be the number of units of K. For every class $\mathcal{A} \in \mathcal{H}$, we introduce the corresponding multiplier:

$$m_{s,\mathcal{A}}(x) = \sum_{0 \neq \mathfrak{a} \in \mathcal{A}} \frac{e^{-2\pi i x N(\mathfrak{a})}}{N(\mathfrak{a})^s} \tag{26}$$

Every class $\mathcal{A}$ contains an integral primitive ideal, i.e., every class is not divisible by a rational integer > 1. Every primitive ideal can be written as

$$\mathfrak{a} = \left[a, \frac{b + \sqrt{D}}{2} \right] \text{ with } a > 0,\ b^2 - 4ac = D,\ (a, b, c) = 1.$$

The above notation means $\mathfrak{a}$ is a free $\mathbb{Z}$-module,

$$\mathfrak{a} = a\mathbb{Z} + \frac{b + \sqrt{D}}{2} \mathbb{Z}.$$

Note that $\frac{b+\sqrt{D}}{2} \in \mathcal{O}$ and $a = N(\mathfrak{a})$; with the generators of $\mathfrak{a}$, we associate the quadratic form

$$\varphi_A(x) = a x_1^2 + b x_1 x_2 + c x_2^2 = \frac{1}{2} A[x]$$

where

$$A = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}.$$

This establishes a one-to-one correspondence between the ideal classes $\mathcal{A} \in \mathcal{H}$ and the equivalence classes of primitive binary quadratic forms φ_A of discriminant D. We choose $\sqrt{D} = i\sqrt{|D|}$ so that

$$z_{\mathfrak{a}} = \frac{b + \sqrt{D}}{2a} \in \mathbb{H}.$$

Then, the inverse ideal $\mathfrak{a}^{-1}$ is a free $\mathbb{Z}$-module generated by one and $\bar{z}_{\mathfrak{a}}$:

$$\mathfrak{a}^{-1} = [1, \bar{z}_{\mathfrak{a}}] = \mathbb{Z} + \frac{b - \sqrt{D}}{2a} \mathbb{Z}.$$

Now, given a class $\mathcal{A}$ which contains $\mathfrak{a}$, we can write

$$m_{s,\mathcal{A}}(x) = \sum_{0 \neq \mathfrak{b} \sim \mathfrak{a} \in \mathcal{A}} \frac{e^{-2\pi i x N(\mathfrak{a})}}{N(\mathfrak{a})^s}$$

Here, the equivalence $\mathfrak{b} \sim \mathfrak{a}$ means $\mathfrak{b} = (\alpha)\mathfrak{a}$ with $\alpha \in \mathfrak{a}^{-1}$, $\alpha \neq 0$, i.e., $\alpha = m + n\bar{z}_{\mathfrak{a}}$ with $m, n \in \mathbb{Z}$, not both zero. As m, n range over the integers, every ideal $\mathfrak{b} \sim \mathfrak{a}$ is covered exactly w times. Moreover, we have $N(\mathfrak{b}) = |\alpha|^2 a = am^2 + bmn + cn^2$; hence,

$$m_{s,\mathcal{A}}(x) = \sum_{0 \neq \mathfrak{a} \in \mathcal{A}} \frac{e^{-2\pi i x N(\mathfrak{a})}}{N(\mathfrak{a})^s} = \frac{1}{w} \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} \frac{e^{-2\pi i x \varphi_A(m,n)}}{\varphi_A(m,n)^s}$$

On the other hand, we have

$$m_{s,K}(x) = \sum_{\mathcal{A} \in \mathcal{H}} m_{s,\mathcal{A}}(x).$$

Lillian Pierce [4] showed that $m_{s,\mathcal{A}} \in L^{1/(1-s)}[0,1]$ and therefore $m_{s,K} \in L^{1/(1-s)}[0,1]$.
□

5. Discrete Analogue of Stein–Weiss Inequality on Product Space

In their study of fractional integrals, Stein and Weiss [14] considered the operator $\mathcal{I}_\alpha$, acting on functions on $\mathbb{R}^N$, given by

$$\mathcal{I}_\alpha f(x) = \int_{\mathbb{R}^N} f(y) \left(\frac{1}{|x-y|} \right)^{N-\alpha} dy. \tag{27}$$

If we let $\omega(x) = |x|^{-\gamma}$, $\sigma(x) = |x|^{\delta}$, they proved under some conditions of γ and δ, that the following weighted norm inequality holds:

$$||\omega \mathcal{I}_\alpha f||_{L^q(\mathbb{R}^N)} \leq C||f\sigma||_{L^p(\mathbb{R}^N)}, \tag{28}$$

which is now known as the Stein–Weiss inequality. When $\gamma = \delta = 0$, $1/q = 1/p - \alpha/N$, (28) is the famous **Hardy–Littlewood–Sobolev** inequality, for more details, see [15–17].

In 2021, Wang [18] extended the Stein–Weiss inequality to the so-called product space case. Now, consider $\mathbb{R}^{\mathbb{N}}$ as a product space by writing $\mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \cdots \times \mathbb{R}^{N_k}$, $k \geq 2$. Let

$$0 < \alpha_i < N_i, \qquad i = 1, 2, \ldots, k \quad \textit{and} \quad \alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k.$$

Wang actually studied the weighted norm inequality of the so-called *strong fractional integral operator* $\mathcal{J}_\alpha$, defined by

$$\mathcal{J}_\alpha f(x) = \int_{\mathbb{R}^N} f(y) \prod_{i=1}^{k} \left(\frac{1}{|x_i - y_i|} \right)^{N_i - \alpha_i} dy. \tag{29}$$

Theorem 6 ([18])**.** *Let* $0 < \alpha < N$, $\gamma, \delta \in \mathbb{R}$, $1 < p \leq q < \infty$. *Then,*

$$||\omega \mathcal{J}_\alpha f||_{L^q(\mathbb{R}^N)} \leq C||f\sigma||_{L^p(\mathbb{R}^N)}, \tag{30}$$

is equivalent to

$$\gamma < \frac{N}{q}, \quad \delta < N\frac{p-1}{p}, \quad \gamma + \delta \geq 0, \qquad \frac{1}{q} = \frac{1}{p} + \frac{\gamma + \delta - \alpha}{N}. \tag{31}$$

For $\gamma \geq 0$, $\delta \leq 0$,

$$\alpha_i - \frac{N_i}{p} < \delta, \qquad i = 1, 2, \ldots, k. \tag{32}$$

For $\gamma \leq 0,\ \delta \geq 0,$

$$\alpha_i - N_i\frac{q-1}{q} < \gamma, \qquad i = 1,2,\ldots,k. \tag{33}$$

For $\gamma > 0,\ \delta > 0,$

$$\sum_{\substack{i\in\{1,2,\ldots,k\}\\ \alpha_i\geq N_i/p}}\left(\alpha_i - \frac{N_i}{p}\right) < \delta, \qquad \sum_{\substack{i\in\{1,2,\ldots,k\}\\ \alpha_i\geq N_i(q-1)/q}}\left(\alpha_i - N_i\left(\frac{q-1}{q}\right)\right) < \gamma, \tag{34}$$

It is natural to ask whether the discrete analogue of Theorem 6 holds. Considering the discrete operator $\mathcal{T}_{\alpha,\gamma,\delta}$, defined as follows,

$$\mathcal{T}_{\alpha,\gamma,\delta}f(n) = \sum_{\substack{m_1\in\mathbb{Z}^{N_1}\\ m_1\neq n_1}}\sum_{\substack{m_2\in\mathbb{Z}^{N_2}\\ m_2\neq n_2}}\cdots\sum_{\substack{m_k\in\mathbb{Z}^{N_k}\\ m_k\neq n_k}}\frac{f(m)}{|n|^{\gamma}|m|^{\delta}}\prod_{i=1}^{k}\left(\frac{1}{|n_i-m_i|}\right)^{N_i-\alpha_i}, \qquad |m||n|\neq 0. \tag{35}$$

where

$$m = (m_1, m_2, \ldots, m_k), \quad n = (n_1, n_2, \ldots .n_k) \in \mathbb{Z}^{N_1}\times\mathbb{Z}^{N_2}\times\cdots\times\mathbb{Z}^{N_k},$$

the following Theorem is expected.

Theorem 7. *Under the conditions of (31)–(34), we have*

$$||\mathcal{T}_{\alpha,\gamma,\delta}f||_{\ell^q(\mathbb{Z}^N)} \leq \mathbb{C}||f||_{\ell^p(\mathbb{Z}^N)}. \tag{36}$$

Proof. We can assume that $f\geq 0$. Let $Q = Q_1\times Q_2\times\cdots\times Q_k \subset \mathbb{R}^{N_1}\times\mathbb{R}^{N_2}\times\cdots\times\mathbb{R}^{N_k}$, where

$$Q_i = \left\{x_i = (x_i^1, x_i^2, \ldots, x_i^{N_i}):\ -\frac{1}{2} < x_i^j \leq \frac{1}{2},\ j = 1,2,\ldots,N_i\right\}, \quad i = 1,2,\ldots,k.$$

Define F, associated with f, as

$$F(x) = f(n), \qquad x\in Q+n, \tag{37}$$

Q is the fundamental cube in $\mathbb{R}^N$. Since

$$\mathbb{R}^N = \bigcup_{n\in\mathbb{Z}^N}(Q+n),$$

F is well defined. Note that for the appropriate constant C,

$$\begin{gathered}|n_i - m_i|^{-N_i+\alpha_i} \leq C|n_i+u_i-(m_i+v_i)|^{-N_i+\alpha_i}, \quad i = 1,2,\ldots,k,\\ |n|^{-\gamma}\leq C|n+u|^{-\gamma}, \quad |m|^{-\delta}\leq C|m+v|^{-\delta},\end{gathered} \tag{38}$$

where $u_i, v_i \in Q_i,\ i = 1,2,\ldots,k$ and $u = (u_1, u_2, \ldots, u_k),\ v = (v_1, v_2, \ldots, v_k)$. Therefore, we have

$$\sum_{\substack{m_1\in\mathbb{Z}^{N_1}\\ m_1\neq n_1}}\sum_{\substack{m_2\in\mathbb{Z}^{N_2}\\ m_2\neq n_2}}\cdots\sum_{\substack{m_k\in\mathbb{Z}^{N_k}\\ m_k\neq n_k}}\frac{f(m)}{|n|^{\gamma}|m|^{\delta}}\prod_{i=1}^{k}\left(\frac{1}{|n_i-m_i|}\right)^{N_i-\alpha_i}$$

$$\leq C\sum_{\substack{m_1\in\mathbb{Z}^{N_1}\\ m_1\neq n_1}}\sum_{\substack{m_2\in\mathbb{Z}^{N_2}\\ m_2\neq n_2}}\cdots\sum_{\substack{m_k\in\mathbb{Z}^{N_k}\\ m_k\neq n_k}}\left\{\prod_{i=1}^{k}\int_{v_i\in Q_i}\right\}\frac{f(m)}{|n+u|^{\gamma}|m+v|^{\delta}}\prod_{i=1}^{k}\left(\frac{1}{|n_i+u_i-m_i-v_i|}\right)^{N_i-\alpha_i}dv_i$$

$$\leq C\sum_{\substack{m_1\in\mathbb{Z}^{N_1}\\ m_1\neq n_1}}\sum_{\substack{m_2\in\mathbb{Z}^{N_2}\\ m_2\neq n_2}}\cdots\sum_{\substack{m_k\in\mathbb{Z}^{N_k}\\ m_k\neq n_k}}\left\{\prod_{i=1}^{k}\int_{v_i\in Q_i}\right\}\frac{F(m+v)}{|x|^{\gamma}|m+v|^{\delta}}\prod_{i=1}^{k}\left(\frac{1}{|x_i-m_i-v_i|}\right)^{N_i-\alpha_i}dv_i \tag{39}$$

$$(x=n+u)$$

$$\leq C\sum_{\substack{m_1\in\mathbb{Z}^{N_1}\\ m_1\neq n_1}}\sum_{\substack{m_2\in\mathbb{Z}^{N_2}\\ m_2\neq n_2}}\cdots\sum_{\substack{m_k\in\mathbb{Z}^{N_k}\\ m_k\neq n_k}}\left\{\prod_{i=1}^{k}\int_{y_i\in Q_i+m_i}\right\}\frac{F(y)}{|x|^{\gamma}|y|^{\delta}}\prod_{i=1}^{k}\left(\frac{1}{|x_i-y_i|}\right)^{N_i-\alpha_i}dy_i$$

$$(y=m+v)$$

$$\leq C\int_{\mathbb{R}^N}\frac{F(y)}{|x|^{\gamma}|y|^{\delta}}\prod_{i=1}^{k}\left(\frac{1}{|x_i-y_i|}\right)^{N_i-\alpha_i}dy=C\mathcal{T}^{*}_{\alpha,\gamma,\delta}F(x),\quad x\in Q+n.$$

Now, by (39) and applying Theorem 6, we have

$$\begin{aligned}||\mathcal{T}_{\alpha,\gamma,\lambda}f||_{\ell^q(\mathbb{Z}^N)}&=\left(\sum_{n\neq 0}\mathcal{T}_{\alpha,\gamma,\lambda}f(n)^q\right)^{\frac{1}{q}}\leq C\left(\sum_{n\neq 0}\int_{x\in Q+n}\mathcal{T}^{*}_{\alpha,\gamma,\lambda}F(x)^q dx\right)^{\frac{1}{q}}\\ &\leq C||\mathcal{T}^{*}_{\lambda,\alpha,\beta}F||_{L^q(\mathbb{R}^N)}\leq C||F||_{L^p(\mathbb{R}^{\mathbb{N}})}=C||f||_{\ell^p(\mathbb{Z}^N)},\end{aligned} \tag{40}$$

Under the same conditions as (31)–(34), note that $\ell^{p_1}(\mathbb{Z}^N)\subset\ell^{p_2}(\mathbb{Z}^N)$ if $p_1\leq p_2$, We can actually improve (31) to

$$\frac{1}{q}\leq\frac{1}{p}+\frac{\gamma+\delta-\alpha}{N}.$$

The proof is completed. □

6. Discussion

In the realm of harmonic analysis, discrete analogues play a crucial role in extending theoretical frameworks and computational techniques to discrete domains. These analogues provide a bridge between continuous and discrete settings, allowing for the exploration of complex phenomena in discrete structures such as sequences, grids, and graphs. In this paper, we delve into the rich tapestry of discrete analogues in harmonic analysis, exploring various facets ranging from discrete fractional operators to multipliers derived from number theoretic identities. Compared to the results obtained by Stein and Wainger in [2], we see that there may be cancellations in the sum $\sum_{n=1}\chi(n)f(m-n^2)n^{-s}$, so it is expected that a better result should obtained. However, Theorem 1 shows that the corresponding multiplier $m_{s,2}$ still belongs to the weak-type $L^{2/(1-s)}[0,1]$.

7. Conclusions

In our research, we embarked on extending classical results pioneered by Stein and Wainger, enriching their framework by introducing the Dirichlet character. This addition allowed us to delve deeper into the connection between Euler's identity and its multipliers, uncovering novel insights into the interplay between number theory and harmonic analysis.

Building upon this foundation, we leveraged Lillian Pierce's groundbreaking work on multipliers associated with quadratic forms. This provided a powerful lens through which we could explore multipliers corresponding to quadratic imaginary fields. Moreover, we employed an implication method to establish the discrete analogue of the Stein–Weiss inequality on product spaces.

Our investigation not only extends the scope of classical results but also highlights the intricate connections between harmonic analysis, number theory, and discrete mathematics. By bridging these disciplines, we aim to contribute to a deeper understanding of fundamental principles and to pave the way for new avenues of exploration at the intersection of these fields.

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