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Solving Nonlinear Second-Order ODEs via the Eisenhart Lift and Linearization

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Abstract: The linearization of nonlinear differential equations represents a robust approach to solution derivation, typically achieved through Lie symmetry analysis. This study adopts a geometric methodology grounded in the Eisenhart lift, revealing transformative techniques that linearize a set of second-order ordinary differential equations. The research underscores the effectiveness of this geometric approach in the linearization of a class of Newtonian systems that cannot be linearized through symmetry analysis.

Keywords: linearization; second-order ODEs; Hamiltonian systems; Newtonian physics

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1. Introduction

Sophus Lie, in his series of books [1–3], established the theory of infinitesimal transformations for the analysis of differential equations. Lie's primary focus was on deriving infinitesimal representations from the finite transformations of continuous groups. This transition from group to local algebraic representation allowed for the study of invariance properties under these transformations, leading to the linearization of all considered equations and/or functions. Lie's method is a systematic approach designed for the examination of nonlinear systems, which is why it has found extensive application in different subjects within applied mathematics [4–17].

The fundamental purpose of identifying invariant transformations, referred to as Lie symmetries, for a given differential equation is to facilitate its solution. The presence of a sufficient number of Lie symmetries of the appropriate type for a differential equation enables a solution through the repeated reduction of the order and a reverse series of quadratures or by determining a sufficient number of first integrals. The process of reduction differs between ordinary and partial differential equations. In the case of ordinary differential equations, the application of a Lie point symmetry results in the reduction of the equation's order by one. Conversely, for partial differential equations, applying a Lie point symmetry leads to a new differential equation, maintaining the same order as the original equation but altering the number of independent and dependent variables [18–21].

A distinctive feature of Lie symmetry analysis is its ability to categorize differential equations based on the group of invariant transformations that they admit, as highlighted by [18]. Dynamical systems sharing common group properties can be interconnected through a point transformation. In the case of ordinary differential equations exhibiting maximal symmetry, this approach results in the linearization of the equations, as demonstrated in [22–27]. Unfortunately, general nonlinear differential equations are typically resistant to analytic treatment. Although numerical methods are often employed for their resolution, these methods do not consistently yield a comprehensive understanding of the solution's behavior across the entire equation. Even seemingly straightforward cases can prove challenging. The simple pendulum with the equation of motion for small oscillations necessitated the introduction of an infinite series for its solution. Consequently, through the linearization of nonlinear differential equations, it becomes possible to derive analytic solutions.

In this research, we expand the scope of linearization within differential geometry by incorporating a geometric approach that extends Lie symmetry analysis. Our specific focus is on a subset of autonomous dynamical systems, characterized by a set of geodesic equations derived through the application of the Eisenhart lift [28–33]. The investigation delves into the geometric properties and the symmetries inherent in the geodesic space. At this point, it is important to mention that we are interested in the global linearization of nonlinear differential equations, an approach that differs from local linearization near the equilibrium points.

Through the utilization of fundamental geometric principles, we demonstrate the feasibility of extending the linearization process to differential equations that are not maximally symmetric. To elucidate the main steps and rationale of our approach, we provide a thorough discussion on the concept of symmetry in geometry, placing a particular emphasis on the case of geodesic equations.

The Eisenhart lift simplifies the study of the symmetries of geodesic equations by introducing additional coordinates and expanding the manifold's dimensionality. Consequently, this enhancement facilitates the linearization procedure.

The structure of the paper is as follows.

In Section 2, we introduce fundamental definitions related to point transformations and the notion of symmetry. Within the context of Riemannian spaces, Section 3 delves into a discussion of symmetries and collineations, with a specific focus on the symmetries associated with the metric tensor and the Levi–Civita connection. Additionally, Section 4 explores Lie and Noether symmetries pertaining to differential equations. We revisit established findings in the literature concerning the relationship between Lie/Noether symmetries and space collineations. Section 5 employs the Eisenhart lift to express a specific class of second-order differential equations as a set of geodesic equations. By imposing geometric conditions on the corresponding Riemannian manifold, we introduce a novel linearization approach for the investigation of nonlinear partial differential equations. Lastly, in Section 6, we provide a discussion of our results and demonstrate the applicability of this geometric approach to the analysis of higher-dimensional dynamical systems.

2. Point Transformations

Consider the n-dimensional Riemannian manifold M with metric tensor $g_{\mu\nu}\left(x^{\lambda}\right)$ and the Levi–Civita connection $\Gamma^{\kappa}_{\mu\nu}\left(x^{\lambda}\right)$, where $\lambda = 1, 2, 3, \dots, n$. Let P and Q be two points on the space with coordinates $x(P)^{\mu}$ and $x(Q)^{\mu\prime}$, respectively.

A point transformation is a mapping between the coordinates of the points P and Q, as defined by the transformation equation [34]

$$x_{Q}^{\mu\prime} = x^{\mu'}\left(x_{P}^{\lambda}\right), \tag{1}$$

in which the matrix $J^{\mu}_{\nu}\left(x^{\lambda}\right) = \frac{\partial x^{\mu'}}{\partial x^{\nu}}$ is not degenerate, i.e., $\det\left(J^{\mu}_{\nu}\left(x^{\lambda}\right)\right) \neq 0$, in order for the functions $x^{\mu'}$ to be independent.

The one-parameter point transformations (1PPT) constitute a specific category of point transformations characterized by their dependence on a real parameter, denoted as ε. The modified transformation equation is expressed as $x_{Q}^{\mu\prime} = x^{\mu'}\left(x_{P}^{\lambda}, \varepsilon\right)$. Notably, the 1PPT transformations exhibit group properties.

When ε is infinitesimal around the point $P\left(x^{\lambda}, 0\right)$, we define the tangent vector [34]

$$X_{P} = \frac{\partial x^{\mu'}}{\partial \varepsilon}\Big|_{\varepsilon \to 0} \partial_{x^{\mu}}\Big|_{P},$$

which is used to define the infinitesimal transformation

$$x^{\mu'} = x^{\mu} + \varepsilon \xi^{\mu}_{P} \ , \ \xi_P = \frac{\partial x^{\mu'}}{\partial \varepsilon}. \tag{2}$$

The tangent vector field X_P is referred to as the generator of the infinitesimal transformation (2).

Assuming the function $F\left(P\left(x^{\lambda}\right)\right)$, under the application of the 1PPT at point Q, the function F transforms into $F'\left(Q\left(x^{\lambda'}\right)\right)$.

We state that the function $F\left(x^{\lambda}\right)$ is invariant under the action of the 1PPT if and only if [34]

$$F'\left(Q\left(x^{\lambda'}\right)\right) = F\left(P\left(x^{\lambda}\right)\right) \ , \ F\left(P\left(x^{\lambda}\right)\right) \neq 0, \tag{3}$$

or

$$F'\left(Q\left(x^{\lambda'}\right)\right) = \gamma F\left(P\left(x^{\lambda}\right)\right) \ , \ F\left(P\left(x^{\lambda}\right)\right) = 0. \tag{4}$$

Equivalently, the latter conditions can be expressed as

$$X_P(F) = 0, \tag{5}$$

where the generator X_P of the 1PPT is identified as the symmetry vector for the function F.

The symmetry condition (5) is $\xi^{\mu}_{P} F_{,\mu} = 0$, i.e., [34]

$$\frac{dx^1}{\xi^1_P} = \frac{dx^2}{\xi^2_P} = \ldots = \frac{dx^n}{\xi^n_P}. \tag{6}$$

From the Lagrange system, we can define a set of new variables in which X_P is defined on the canonical coordinates, such that $X_P = \partial_n$ and $F = F\left(W^A\left(x^B\right)\right)$, where $A = 1, 2, 3 \ldots, n-1$. $W^A\left(x^B\right)$ are known as zeroth-order invariants.

If the function F remains invariant under distinct 1PPTs with generators $X, Y, Z, \ldots$, then these vector fields constitute a finite-dimensional linear space G with the characteristic binary operator $[X, Y] = XY - YX$. This operator is known as the Lie bracket or commutator, and G is defined as a Lie algebra.

The characteristic properties of the Lie bracket are (i) $[X, X] = 0$ for all $X \in G$; (ii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in G$ and (iii), if $X_A, X_B \in G$, then $[X_A, X_B] = C^C_{AB} X_C$, $X_C \in G$. Quantities C^C_{AB} are constants and are called the structure constants of the Lie algebra.

The structure constants exhibit antisymmetric properties in the two lower indices, i.e., [34]

$$C^C_{AB} + C^C_{BA} = 0, \tag{7}$$

and they have to satisfy the Jacobi identity [34]

$$C^E_{AB} C^C_{DE} + C^E_{BD} C^C_{AE} + C^E_{DA} C^C_{BE} = 0. \tag{8}$$

Indeed, the structure constants serve to characterize the Lie algebra, as any set of constants C^C_{AB} that satisfies the aforementioned conditions defines a unique Lie group. Notably, the structure constants undergo changes under a basis transformation; this property proves useful as it allows for the simplification of the structure constants of the given group.

Thus far, we have presented the fundamental definitions of the concept of symmetry. Our emphasis lies on symmetries within Riemannian manifolds and the Lie symmetries of differential equations.

3. Symmetries in Riemannian Manifolds

Let Ω be a geometric object in the Riemannian manifold M with a transformation rule [34]

$$\Omega'^{\mu} = \Phi^{\mu}\left(\Omega, x^{k}, x^{k'}\right) = J^{\mu}_{\nu}\left(x^{\kappa}, x^{\kappa'}\right)\Omega^{\nu} + C^{\mu}\left(x^{\kappa}, x^{\kappa'}\right), \tag{9}$$

in which $J^{\mu}_{\nu}\left(x^{\kappa}, x^{\kappa'}\right)$ is the Jacobian matrix of the transformation and $C^{\mu}\left(x^{\kappa}, x^{\kappa'}\right)$ defines the geometric object. Indeed, for $C^{\mu}\left(x^{\kappa}, x^{\kappa'}\right) = 0$, Ω^{μ} is a tensor field.

In a similar approach to the case of functions, if X is the generator of a 1PPT, then Ω remains invariant if and only if

$$\mathcal{L}_X\Omega = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\Phi(\Omega) - \Omega] = 0, \tag{10}$$

in which $\mathcal{L}_X\Omega$ is the Lie derivative with respect to the vector field X, which acts on the geometric object Ω.

In the following, with a comma (","), we note the partial derivative, and, with a semicolon (";"), we note the covariant derivative.

For functions $F\left(x^{\lambda}\right)$, the Lie derivative is defined as $\mathcal{L}_X F = X(F)$, while, for a tensor field T of rank (r,s), the Lie derivative is defined as [34]

$$\begin{aligned}\mathcal{L}_X T^{i_1\ldots i_r}_{j_i\ldots j_s} \quad = \quad & X^k T^{i_1\ldots i_r}_{j_i\ldots j_{s,k}} - T^{m\ldots i_r}_{j_i\ldots j_s} X^{i_1}_{,m} - T^{i_1 m\ldots i_r}_{j_i\ldots j_s} X^{i_2}_{m} + \ldots \\ & \ldots + T^{i_1\ldots i_r}_{m\ldots j_s} X^{m}_{,j_1} + T^{i_1\ldots i_r}_{j_i m\ldots j_s} X^{m}_{j_2} + \ldots .\end{aligned} \tag{11}$$

Recall that we assume the Einstein summation convention.

Hence, from the latter formula for the metric tensor $g_{\mu\nu}$, the Lie derivative is [34]

$$\mathcal{L}_X g_{\mu\nu} = g_{\mu\nu;\kappa}X^{\kappa} + g_{\mu\kappa}X^{\kappa}_{\ ,\nu} + g_{\kappa\nu}X^{\kappa}_{\ ,\mu}, \tag{12}$$

that is,

$$\mathcal{L}_X g_{\mu\nu} = X_{\mu;\nu} + X_{\nu;\mu}, \tag{13}$$

in which the semicolon ";" indicates a covariant derivative.

For the Levi–Civita connection, $\Gamma^{\kappa}_{\mu\nu}$, the Lie derivative is defined by the following formula [34]:

$$\mathcal{L}_X\Gamma^{\lambda}_{\mu\nu} = X^{\lambda}_{,\mu\nu} + \Gamma^{\lambda}_{\mu\nu,r}X^{r} - X^{\lambda}_{,r}\Gamma^{r}_{\mu\nu} + X^{s}_{,\mu}\Gamma^{\lambda}_{s\nu} + X^{s}_{,\nu}\Gamma^{\lambda}_{\mu s}. \tag{14}$$

Equivalently,

$$\mathcal{L}_X\Gamma^{\lambda}_{\mu\nu} = X^{\lambda}_{;\mu\nu} - R^{\lambda}_{\mu\nu\kappa}\xi^{\kappa}, \tag{15}$$

where $R^{\lambda}_{\mu\nu\kappa}$ is the curvature, or

$$\mathcal{L}_X\Gamma^{\lambda}_{\mu\nu} = g^{\lambda\kappa}\left[\left(\mathcal{L}_X g_{\kappa\mu}\right)_{;\nu} + \left(\mathcal{L}_X g_{\kappa\nu}\right)_{;\mu} - \left(\mathcal{L}_X g_{\mu\nu}\right)_{;\kappa}\right]. \tag{16}$$

We continue our discussion by presenting the basic symmetries and collineations for the metric tensor $g_{\mu\nu}$ and the Levi–Civita connection $\Gamma^{\lambda}_{\mu\nu}$.

3.1. Symmetries of the Metric Tensor

The metric tensor $g_{\mu\nu}$ is invariant under the application of a 1PPT if and only if [35]

$$\mathcal{L}_X g_{\mu\nu} = 0. \tag{17}$$

In this case, the vector field X acts as a symmetry of the metric tensor, meaning that the infinitesimal transformation preserves the lengths, distances, and angles of geometric objects. The vector field X is commonly referred to as an isometry or a killing vector (KV) of the Riemannian manifold. Isometries hold significant importance in both geometry and physics descriptions.

In three-dimensional Euclidean geometry, the six-dimensional Killing algebra is formed by three translations and three rotations. In the context of a four-dimensional Minkowski geometry, the isometries are represented by elements of the Poincaré group. The number of admissible isometries for a Riemannian metric is not arbitrary. Specifically, the maximum number of admitted isometries for a Riemann metric $g_{\mu\nu}$ is given by the formula $\frac{1}{2}n(n+1)$, where $n = \dim g_{\mu\nu}$. This occurs when the space is maximally symmetric, such as in the case of a flat space, an n-dimensional sphere, or an n-dimensional hyperbolic plane.

A generalization of the symmetry condition (17) is the following:

$$\mathcal{L}_X g_{\mu\nu} = 2\psi(x^\kappa) g_{\mu\nu}. \tag{18}$$

On the right-hand side (RHS), $\psi(x^\kappa)$ is a function related to the generator X through the relation $\psi(x^\kappa) = \frac{1}{n} X^\lambda_{;\lambda}$. The killing equation (17) is recovered when $\psi(x^\kappa) = 0$. However, for $\psi(x^\kappa) \neq 0$, the more general concept of killing vectors is introduced. Specifically, when condition (18) is satisfied, the vector field is characterized as a conformal killing vector (CKV). Point transformations with generators as CKVs keep the angles invariant but not the distance between two points. When $\psi(x^\kappa)$ is a constant, X is a homothetic killing vector (HKV). It is important to note that a Riemannian space admits at most one proper HKV. The CKVs form a Lie algebra, which has a maximum dimension of $\frac{1}{2}(n+1)(n+2)$. The killing algebra is a subalgebra of the conformal algebra.

When the admitted conformal algebra for a metric tensor is maximal, then the space is conformally flat. Two-dimensional Riemannian spaces are conformally flat [34].

Any three-dimensional Riemannian space is conformally flat when the Cotton–York tensor [34]

$$C_{\mu\nu\kappa} = R_{\mu\nu;\kappa} - R_{\kappa\nu;\mu} + \frac{1}{4}\left(R_{;\nu} g_{\mu\kappa} - R_{;\kappa} g_{\mu\nu}\right),\ n = 3, \tag{19}$$

is zero.

Finally, for higher-dimensional spaces, i.e., $n > 3$, the necessary condition in order for the metric tensor to be conformally flat is that the Weyl tensor be zero, i.e., [34]

$$C_{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda} - \frac{2}{n-2}\left(g_{\mu[\kappa} R_{\lambda]\nu} - g_{\nu[\kappa} R_{\lambda]\mu}\right) + \frac{2}{(n-1)(n-2)} R g_{\mu[\kappa} g_{\lambda]\nu}. \tag{20}$$

3.2. Symmetries of the Connection

For the connection $\Gamma^\lambda_{\mu\nu}$, the symmetry condition (10) becomes [35]

$$\mathcal{L}_X \Gamma^\lambda_{\mu\nu} = 0. \tag{21}$$

When the latter condition is true, the vector field X is called an affine collineation (AC). By definition, ACs transform geodesic equations into geodesic equations by leaving the affine parametrization invariant.

On the other hand, if X is a CKV for the metric tensor, then, from (16), it follows that

$$\mathcal{L}_X \Gamma^\lambda_{\mu\nu} = 2g^{\lambda r}\left[\psi_{;\mu} + \psi_{;\nu} - \psi_{;r}\right]. \tag{22}$$

Thus, for $\psi(x^\kappa) = 0$ or $\psi(x^\kappa) = \psi_0$, the vector field X is also an AC. As a result, the KVs and the HKV form subalgebras in the affine algebra. We remark that the maximum dimension of the affine algebra is $n(n+1)$, which is that of the flat space. Moreover, the collineations of the connection can be defined independently of the collineations of the metric $g_{\mu\nu}$.

A generalization of the symmetry condition (21) is the following [36]

$$\mathcal{L}_X \Gamma^\lambda_{\mu\nu} = \phi_{,\mu}\delta^\lambda_\nu + \phi_{,\nu}\delta^\lambda_\mu, \tag{23}$$

where the function $\phi(x^\kappa)$ is denoted as the projective function, and the vector field X is termed a projective collineation (PC). When $\phi = \phi_0$, condition (21) for the affine collineations (ACs) is recovered. Consequently, the ACs form a subalgebra of the projective algebra. Special PCs are an important subclass characterized by the property $\phi_{;\mu\nu} = 0$, indicating that the spacetime admits a gradient killing vector and is decomposable. The maximum dimension of the projective algebra is $n(n+2)$, matching that of the maximal symmetric spaces [36].

4. Lie Symmetries of Differential Equations

In this section, we extend the concept of symmetry in the case of differential equations. We assume a differential equation expressed by the function $H\left(x, y, y', \dots, y^{(n)}\right) = 0$, defined in the jet space $B_M = \left\{x, y, y', \dots, y^{(n)}\right\}$, in which $M = \{x, y(x)\}$ is the base manifold. Variable x is the independent variable, $y = y(x)$ is the dependent variable, and $y^{(n)} = \frac{d^n y}{dx^n}$.

In the base manifold, the infinitesimal 1PPT with generator $X = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ is [18–21]

$$\bar{x} = x + \varepsilon\xi(x,y), \tag{24}$$

$$\bar{y} = y + \varepsilon\eta(x,y). \tag{25}$$

In the jet space, the 1PPT is prolonged as

$$\bar{y}^{(i)} = y^{(A)} + \varepsilon\eta^{[A]},\ A = 1,2,3\dots n\,,$$

where $\eta^{[1]}$, $\eta^{[\rho]}$ and $\eta^{[n]}$ are defined by the expressions

$$\eta^{[1]} = \frac{d\eta}{dx} - y^{(1)}\frac{d\xi}{dx},$$

$$\eta^{[\rho]} = \frac{d\eta^{[\rho-1]}}{dx} - y^{(\rho)}\frac{d\xi}{dx}\,,\ \rho = 2,3,\dots n-1\,,$$

and

$$\eta^{[n]} = \frac{d\eta^{n-1}}{dx} - y^{(n)}\frac{d\xi}{dx} = \frac{d^n}{dx^n}\left(\eta - y^{(1)}\xi\right) + y^{(n+1)}\xi.$$

The vector field

$$X^{[n]} = X + \eta^{[1]}\partial_{y^{(1)}} + \dots + \eta^{[n]}\partial_{y^{[n]}}.$$

is called the nth prolongation of the generator X in the jet space B_M.

Therefore, we say that the 1PPT with the generator in the vector field, X, leaves invariant the differential equation $H\left(x, y, y', \dots, y^{(n)}\right) = 0$ if and only if [18–21]

$$X^{[n]}\left(H\left(x, y, y', \dots, y^{(n)}\right)\right) = 0. \tag{26}$$

The generator X of the 1PPT is characterized as a Lie (point) symmetry for the differential equation. In a similar way as above, the admitted symmetries for a differential equation form a Lie algebra.

Symmetries are essential for the study of differential equations. Specifically, symmetries are used for the simplification of a differential equation by reducing the dynamical degrees of freedom or constructing conservation laws and invariant functions. Another important characteristic of Lie symmetries is that they can be used to determine transformations and write a given differential equation in the form of other known differential equations [37].

Indeed, for second-order differential equations, Sophus Lie found that if the equation has eight symmetries, we can change the equation to describe a simpler problem, which is similar to describing the motion of a free particle. For instance, the oscillator $y'' + y = 0$ becomes $\frac{d^2Y}{dX^2} = 0$ after the change of variables $X = \tan x$ and $Y = \frac{y}{\cos x}$.

Because the linearization property leads to a simple approach to the construction of a solution, it has been the subject of interest in a series of studies; see, for instance, [38–45] and the references therein.

4.1. Noether's Theorems

Differential equations arising from a variational principle give rise to a distinctive set of symmetries referred to as Noether symmetries. Let us assume

$$S = \int L(x, y, y')dt, \tag{27}$$

as the action integral and $L(x, y, y')$ as the Lagrangian, which describes the second-order ordinary differential equation $H(x, y, y', y'') = 0$, i.e.,

$$E_L(L) \equiv H(x, y, y', y'') = 0\,, \tag{28}$$

in which E_L is the Euler–Lagrange operator defined as

$$E_L \equiv \frac{d}{dx}\left(\frac{\partial}{\partial y'}\right) - \frac{\partial}{\partial y}. \tag{29}$$

In 1918, Emmy Noether, in her pioneering work [46], demonstrated that a Lie symmetry X for the differential equation $H(x, y, y', y'') = 0$ is a Noether symmetry (or variational symmetry) if there exists a function f such that

$$X^{[1]}L + L\frac{d\xi}{dx} = \frac{df}{dx}. \tag{30}$$

This symmetry condition is known as Noether's first theorem. The function f serves as a boundary term, facilitating adjustments in the action integral's value due to infinitesimal alterations in the domain's boundary resulting from the infinitesimal transformation of the variables within the action integral.

The novelty of Noether's work is that it provides a simple formula for the construction of conservation laws from variational symmetries. Indeed, Noether's second theorem states that if X is a variational symmetry, then function [46]

$$I(x, y, y') = \xi\left(y'\frac{\partial L}{\partial y'} - L\right) - \eta\frac{\partial L}{\partial x} + f, \tag{31}$$

is a conservation law, i.e., $I' = 0$.

4.2. Symmetries of Geodesic Equations

There exists a unique relation for the symmetries of differential equations that define the motion of particles in Riemannian spaces with the collineations of the background geometry.

4.2.1. Affine Parametrization

Consider the n-dimensional Riemannian space with metric $g_{\mu\nu}$. The geodesic Lagrangian is

$$L_g\left(x^\kappa, \frac{dx^\kappa}{ds}\right) = \frac{1}{2}g_{\mu\nu}(x^\kappa)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}, \tag{32}$$

where s is an affine reparametrization.

The geodesic equations are

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\kappa\nu}(x^\sigma)\frac{dx^\kappa}{ds}\frac{dx^\nu}{ds}=0, \tag{33}$$

where $\Gamma^\mu_{\kappa\nu}(x^\sigma)$ defines the Levi–Civita connection.

Concerning the Lie symmetries for the geodesic Equations (33), it has been found previously, in [47], that they are constructed by the elements of the special projective collineations of the background geometry.

Specifically, the Lie symmetries for the geodesic Equations (33) are of the form

$$X=\xi(s,x^\kappa)\partial_x+\eta^\mu(s,x^\kappa)\partial_\mu, \tag{34}$$

in which

$$\xi(t,x^\kappa)=\psi_A t^2+\left[E_J S^J\left(x^k\right)+K\right]t+F_J S^J\left(x^k\right)+L, \tag{35}$$

and

$$\eta^\mu(t,x^\kappa)=A^\mu(x^\kappa)t+B^\mu(x^\kappa)+D^\mu(x^\kappa), \tag{36}$$

where G_J, M, b, K, F_J and L are constants and the index J runs along the number of gradient KVs of the metric tensor $g_{\mu\nu}$. The vector field $A^\mu(x^\kappa)$ is a gradient HKV with conformal factor ψ_A, $D^\mu(x^\kappa)$ is a non-gradient KV of the metric, and $B^i(x)$ is either a special projective collineation with projection function $E_J S^J(x)$ or an AC and $E_J=0$.

On the other hand, for the geodesic Lagrangian (32), if X is a Noether symmetry, then [48]

$$\begin{aligned} X \;=\;& \left(C_3\psi_A t^2+2C_2\psi_A t+C_1\right)\partial_t+\\ & \left[C_J S^{J,\mu}+C_I KV^{I\mu}+C_{IJ}tS^{J,\mu}+C_2H^i+C_3tA^\mu(x^\kappa)\right]\partial_\mu, \end{aligned} \tag{37}$$

with the corresponding gauge function

$$f(x^k)=C_1+C_2+C_I+C_J+\left[C_{IJ}S^J(x^\kappa)\right]+C_3[A(x^\kappa)], \tag{38}$$

in which $S^{J,\mu}$ are the C_J gradient KVs, $KV^{I\mu}$ are the C_I non-gradient KVs for the metric tensor, and A^μ is a HKV. If A^μ is a gradient vector, then $C_3\neq 0$; otherwise, $C_3=0$.

The n-dimensional Riemannian space with the maximum special projective algebra and the maximum Homothetic algebra is the flat space. Consequently, if the geodesic Lagrangian (32) is maximally symmetric, i.e., it admits $\left(\frac{n}{2}+1\right)(n+1)+2$ Noether symmetries, it describes the geodesic equations of the n-dimensional flat space.

Similarly, if the geodesic Equations (33) admit $(n+1)(n+3)$ Lie symmetries, then there exists a change of variables $x^\mu\to z^\mu$, such that Equations (33) are

$$\frac{d^2z^\mu}{ds^2}=0. \tag{39}$$

This linearization criterion follows directly from the structure properties of the Riemannian manifold.

4.2.2. Non-Affine Parametrization

From the system of geodesic equations, it is possible to obtain an equivalent system by assuming x^A, $A=1,2,3\ldots,n-1$ to be the dependent variables and $\tau=x^n$ to be the new non-affine independent variable.

In the non-affine parametrization, the geodesic equations reduce to the following system of $n-1$ equations

$$\frac{d^2x^A}{d\tau^2} + \alpha_{BC}\left(\tau, x^A\right)\frac{dx^A}{d\tau}\frac{dx^B}{d\tau}\frac{dx^C}{d\tau} + \beta^A_{BC}\left(\tau, x^A\right)\frac{dx^B}{d\tau}\frac{dx^C}{d\tau} + \gamma^A_B\left(\tau, x^A\right)\frac{dx^B}{d\tau} + \delta^A\left(\tau, x^A\right) = 0, \tag{40}$$

with

$$\alpha_{BC}\left(\tau, x^A\right) = -\Gamma^n_{BC}\,, \tag{41}$$

$$\beta^A_{BC}\left(\tau, x^A\right) = \Gamma^A_{BC} - 2\Gamma^n_{n(C}\delta^A_{B)}\,, \tag{42}$$

$$\gamma^A_B\left(\tau, x^A\right) = 2\Gamma^A_{nB} - \Gamma^n_{nn}\delta^A_B\,, \tag{43}$$

$$\delta^A\left(\tau, x^A\right) = \Gamma^A_{nn}.$$

The Lie symmetry conditions for the geodesic Equations (40) was investigated in detail in a series of works by Aminova [49–52], and it was found that the vector field

$$X = \eta^\mu(x^\kappa)\partial_\mu, \tag{44}$$

is a Lie symmetry for system (40) if and only if $\eta^\mu(x^\kappa)$ is a PC for the n-dimensional Riemannian manifold $g_{\mu\nu}$. We remark that the result holds for non-Riemannian spacetimes.

Hence, the maximally symmetric $n-1$ second-order Equations (40) follow from the reparametrization of the geodesic equations for the n-dimensional maximally symmetric space.

We know that if the $n-1$ equations admit $n(n+2)$ symmetries, then they can be linearized. Thus, an equivalent geometric linearization criterion is the structure function of (40) to define a maximally symmetric space; see the discussion in [42].

In order to understand this property, consider the two-dimensional sphere expressed in coordinates

$$ds^2 = dx^2 + \sin^2 x dy^2. \tag{45}$$

The geodesic equations are

$$\frac{d^2x}{ds^2} - \sin x \cos x \left(\frac{dy}{dx}\right)^2 = 0, \tag{46}$$

$$\frac{d^2y}{ds^2} + 2\cot x \frac{dy}{ds}\frac{dx}{ds} = 0. \tag{47}$$

Hence, under the change of the parametrization $x(s) \to x(y)$, the latter system becomes

$$\frac{d^2x}{dy^2} - 2\cot x \left(\frac{dx}{dy}\right)^2 - \sin x \cos x = 0, \tag{48}$$

or equivalently

$$\frac{d^2z}{dy^2} + z = 0,\ \ x(y) = -\arctan\left(\frac{1}{z(y)}\right). \tag{49}$$

In this example, we demonstrate the geometric origin of the oscillator.

5. Eisenhart Lift

The process of geometrizing dynamical systems, achieved by formulating a given dynamical system as a set of geodesic equations, has been extensively explored in the existing literature. Non-affine parametrization serves as an alternative approach to this geometrization. In this method, connections are defined based on the dynamical system, which generally do not correspond to the Levi–Civita connections associated with any metric tensor.

Two different methodologies, namely the Jacobi metric and the Eisenhart lift, provide diverse viewpoints and are predominantly employed to characterize conservative forces within the geometric framework. Our specific emphasis in this investigation is on the Eisenhart lift. Significantly, this approach requires the augmentation of the system's dimensionality. Geometrization is achieved by introducing extra dimensions, through the incorporation of new dependent variables. Simultaneously, the new metric tensor admits isometries that provide Noetherian conservation laws for the geodesic equations. When applied, these conservation laws serve to reduce the geodesic Lagrangian and geodesic equations back to the original dynamical system.

The approach that one can apply in order to increase the dimension of the dynamical system is not unique. In the following, we extend the discussion presented in [53] and we focus on the Riemannian lift and the Lorentzian lift. The Riemannian lift is a common approach in which one new variable is introduced; on the other hand, in the Lorentzian lift, the dimension of the dynamical system is increased by two.

In the following, we focus on the description of the simple dynamical system

$$\frac{d^2x}{ds^2}+V_{,x}(x)=0, \tag{50}$$

described by the Lagrangian function

$$L\left(x,\frac{dx}{ds}\right)=\frac{1}{2}\left(\frac{dx}{ds}\right)^2-V(x), \tag{51}$$

where the arbitrary potential $V(x)$ admits as a conservation law the "energy"expressed by the Hamiltonian function

$$h=\frac{1}{2}\left(\frac{dx}{ds}\right)^2+V(x). \tag{52}$$

5.1. The Riemannian Lift

Consider the two-dimensional geodesic Lagrangian

$$L\left(x,\frac{dx}{ds},y,\frac{dy}{ds}\right)=\frac{1}{2}\left(\frac{dx}{ds}\right)^2+\frac{1}{2}F^2(x)\left(\frac{dy}{ds}\right)^2, \tag{53}$$

with equations of motion

$$\frac{d^2x}{ds^2}-FF_{,x}\left(\frac{dy}{ds}\right)^2=0, \tag{54}$$

$$\frac{d^2y}{ds^2}+2\frac{F_{,x}}{F}\left(\frac{dy}{ds}\right)\left(\frac{dx}{ds}\right)=0. \tag{55}$$

Equation (55) is $\frac{d}{ds}\left(F^2\frac{dy}{ds}\right)=0,$ from which we infer the conservation law

$$I_0=F^2\frac{dy}{ds}. \tag{56}$$

The construction of this conservation law becomes straightforward when applying Noether's theorem to the killing vector ∂_y for the two-dimensional metric tensor that characterizes the geodesic Lagrangian (53).

The application of the conservation law I_0 in Equation (54) gives

$$\frac{d^2x}{ds^2}-I_0^2\frac{F_{,x}}{F^3}=0. \tag{57}$$

Comparing the latter equation with (50), we conclude that

$$V(x)=\frac{I_0^2}{2F(x)^2}. \tag{58}$$

We solve now the geometric problem when the two-dimensional metric,

$$ds^2 = dx^2 + F^2(x)dy^2, \tag{59}$$

is maximally symmetric.

Because the dimension of the space is two, it is maximally symmetric when the Ricci scalar is constant, i.e.,

$$F_{,xx} - \kappa F = 0. \tag{60}$$

Therefore, for $\kappa = 0$,

$$F(x) = F_0(x - x_0), \tag{61}$$

and, for $\kappa \neq 0$,

$$F(x) = F_1 e^{\sqrt{\kappa}x} + F_2 e^{-\sqrt{\kappa}x}.$$

We conclude that, for the potential functions

$$V_1(x) \simeq \frac{1}{(x - x_0)^2}, \tag{62}$$

$$V_2(x) \simeq \frac{1}{\left(e^{\sqrt{\kappa}x} + \bar{F}_2 e^{-\sqrt{\kappa}x}\right)^2}, \tag{63}$$

the Eisenhart lift leads to a two-dimensional maximally symmetric equation. F_0, F_1, F_2 and $\bar{F}_2$ are constants.

For $\kappa = 0$, the potential function $V_1(x)$ describes the Ermakov–Pinney equation [54], which admits three Lie symmetries.

In this case, the geodesic Lagrangian becomes (without loss of generality, we assume $x_0 = 0$)

$$L\left(x, \frac{dx}{ds}, y, \frac{dy}{ds}\right) = \frac{1}{2}\left(\frac{dx}{ds}\right)^2 + \frac{1}{2}x^2\left(\frac{dy}{ds}\right)^2, \tag{64}$$

which in the Cartesian coordinates

$$x = \sqrt{z^2 + w^2}\,,\; y = \arctan\frac{w}{z} \tag{65}$$

become

$$L\left(z, \frac{dz}{ds}, z, \frac{dw}{ds}\right) = \frac{1}{2}\left(\left(\frac{dz}{ds}\right)^2 + \left(\frac{dw}{ds}\right)^2\right). \tag{66}$$

On the other hand, for $\kappa \neq 0$, the two-dimensional space describes a sphere ($\kappa > 0$) or a hyperbolic plane ($\kappa < 0$) and there exists a transformation whereby the system can be expressed in the form of an oscillator, as described before.

5.2. The Lorentzian Lift

An alternative lifting approach is the Lorentzian lift, wherein the dimension of the space is augmented by two.

We introduce the geodesic Lagrangian

$$L\left(x, \frac{dx}{ds}, u, \frac{du}{ds}, v, \frac{dv}{ds}\right) = \frac{1}{2}\left(\frac{dx}{ds}\right)^2 + \left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) - V(x)\left(\frac{du}{ds}\right)^2. \tag{67}$$

The corresponding Euler–Lagrange equations are

$$\frac{d^2x}{ds^2}+V(x)\left(\frac{du}{ds}\right)^2 = 0, \tag{68}$$

$$\frac{d^2v}{ds^2}-2V_{,x}\frac{du}{ds}\frac{dx}{ds} = 0, \tag{69}$$

$$\frac{d^2u}{ds^2} = 0. \tag{70}$$

The above-mentioned dynamical system possesses the following conservation laws

$$\Phi_1=\frac{du}{ds}\ ,\ \Phi_2=\left(\frac{dv}{ds}\right)-2V\left(\frac{du}{ds}\right), \tag{71}$$

and the Hamiltonian function

$$\bar{h}=\frac{1}{2}\left(\frac{dx}{ds}\right)^2+\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right)-V(x)\left(\frac{du}{ds}\right)^2. \tag{72}$$

With the use of the conservation laws, the dynamical system (50), (52) is recovered with $\Phi_1=1,\ \Phi_2=-h$ and $\bar{h}=0$.

Now, we explore the conditions under which the three-dimensional spacetime with the geodesic Lagrangian (67) attains maximal symmetry. If it achieves maximal symmetry, the space is necessarily conformally flat.

Therefore, by considering the Cotton–York tensor to be zero, i.e., $C_{\mu\nu\kappa}=0,$ where

$$C_{\mu\nu\kappa}=R_{\mu\nu;\kappa}-R_{\kappa\nu;\mu}+\frac{1}{4}\left(R_{;\nu}g_{\mu\kappa}-R_{;\kappa}g_{\mu\nu}\right),$$

it follows that

$$V_{,xxx}=0. \tag{73}$$

Hence, the scalar field potential is $V(x)=V_2x^2+V_1x+V_0$. This potential function provides the linear equation of the oscillator with a constant force term. In the following, for simplicity, we assume $V(x)=V_2x^2$.

For this potential function, the three-dimensional spacetime is conformally flat but it is not maximally symmetric. However, because $\bar{h}=0$, the resulting geodesic equations are null geodesics that are invariant under a conformal transformation. It holds that the Euler–Lagrange equations for two conformal Lagrangians transform covariantly under the conformal transformation relating the Lagrangians if and only if the Hamiltonian vanishes.

Assume the Lagrangian function

$$L=\frac{1}{2}g_{\mu\nu}(x^\kappa)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}, \tag{74}$$

with Euler–Lagrange equations

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\kappa\nu}\frac{dx^\kappa}{ds}\frac{dx^\nu}{ds}=0. \tag{75}$$

The Hamiltonian is

$$E=\frac{1}{2}g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\ . \tag{76}$$

For the conformally related Lagrangian $\bar{L}=\frac{1}{2}N(x^\kappa)g_{\mu\nu}(x^\kappa)\frac{dx^\mu}{d\bar{s}}\frac{dx^\nu}{d\bar{s}}$, the equations of motion are

$$\frac{d^2x^\mu}{d\bar{s}^2}+\hat{\Gamma}^\mu_{\kappa\nu}\frac{dx^\kappa}{d\bar{s}}\frac{dx^\nu}{d\bar{s}}=0, \tag{77}$$

where the Levi–Civita connection for the conformal equivalent metric is

$$\hat{\Gamma}^{\mu}_{\kappa\nu} = \Gamma^{\mu}_{\kappa\nu} + 2(\ln N)_{,(\kappa}\delta^{\mu}_{\nu)} - (\ln N)^{,\mu} g_{\kappa\nu}, \tag{78}$$

and the corresponding Hamiltonian is

$$\bar{E} = \frac{1}{2}N(x^{\kappa})g_{\mu\nu}(x^{\kappa})\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}. \tag{79}$$

In order to show that the two equations of motion are conformally related, we start from Equation (77) and apply the conformal transformation $d\bar{s} = N^2 ds$. Then,

$$\frac{d^2x^{\mu}}{ds^2} - 2\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}(\ln N)_{,\nu}\frac{1}{N^4} + \frac{1}{N^4}\hat{\Gamma}^{\mu}_{\kappa\nu}\frac{dx^{\kappa}}{ds}\frac{dx^{\nu}}{ds} = 0. \tag{80}$$

Hence,

$$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\kappa\nu}\frac{dx^{\kappa}}{ds}\frac{dx^{\nu}}{ds} - (\ln N)^{,i}\left(g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}\right) = 0. \tag{81}$$

This means that the equations of motion are invariant if and only if the geodesic Lagrangian describes null geodesics.

The three-dimensional spacetime (67) for $V(x) = V_2x^2$ is conformally flat. Thus, there exists a function, $N(x,u,v)$, such that the geodesic Lagrangian

$$\bar{L}\left(x,\frac{dx}{d\bar{s}},u,\frac{du}{d\bar{s}},v,\frac{dv}{d\bar{s}}\right) = N^2(x,u,v)\left(\frac{1}{2}\left(\frac{dx}{d\bar{s}}\right)^2 + \left(\frac{du}{d\bar{s}}\right)\left(\frac{dv}{d\bar{s}}\right) - V(x)\left(\frac{du}{d\bar{s}}\right)^2\right), \tag{82}$$

describes the flat space.

For instance, for the function $N(x) = \frac{1}{x}$, the three-dimensional spacetime is a space of constant curvature. For this function, the Euler–Lagrange equation for the variable x is

$$\frac{d^2x}{d\bar{s}^2} - \frac{1}{x}\left(\frac{dx}{d\bar{s}}\right)^2 = 0, \tag{83}$$

or equivalently

$$\frac{d^2X}{d\bar{s}^2} = 0\,,\ X = \ln x. \tag{84}$$

6. Conclusions

The geometric characteristics of this approach unveil novel avenues for the exploration of dynamical systems and the linearization process. By incorporating geometric requirements, we successfully reproduced previous findings regarding the linearization of Equation (50) without the direct examination of the Lie symmetries of the equations. Additionally, we established the presence of two new potential functions capable of linearizing Equation (50).

To demonstrate the innovative nature of this method, we delve into the linearization of the two-dimensional Ermakov–Pinney system with the given Lagrangian

$$L\left(r,\frac{dr}{ds},\theta,\frac{d\theta}{ds}\right) = \frac{1}{2}\left(\frac{dr}{ds}\right)^2 + \frac{1}{2}r^2\left(\frac{d\theta}{ds}\right)^2 - \frac{U(\theta)}{r^2}, \tag{85}$$

and Euler–Lagrange equations

$$\frac{d^2r}{ds^2} - r\left(\frac{d\theta}{ds}\right)^2 - \frac{2U(\theta)}{r^3} = 0, \tag{86}$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\left(\frac{dr}{ds}\right)\left(\frac{d\theta}{ds}\right) + \frac{U_{,\theta}}{r^4} = 0. \tag{87}$$

In the limit for which $U(\theta) = U_0$, the Ermakov equation is recovered. In the following, we assume $U_{,\theta} \neq 0$ and we investigate for which functions $U(\theta)$ the latter system can be linearized.

We utilize the Riemannian lift and express the geodesic Lagrangian as follows:

$$L_R\left(r, \frac{dr}{ds}, \theta, \frac{d\theta}{ds}, z, \frac{dz}{ds}\right) = \frac{1}{2}\left(\frac{dr}{ds}\right)^2 + \frac{1}{2}r^2\left(\frac{d\theta}{ds}\right)^2 + \frac{r^2}{U(\theta)}\left(\frac{dz}{ds}\right)^2. \tag{88}$$

Indeed, the three-dimensional spacetime is maximally symmetric and describes the flat space for

$$U(\theta) = \frac{1}{(U_1 \sin\kappa\theta + U_2 \cos\kappa\theta)^2}.$$

Without loss of generality, we assume $U_2 = 0$ and $\kappa = 1$.

Therefore, under the change of variables

$$X = r\cos\theta\ ,\ Y = r\sin\theta\cos(U_1 z)\ ,\ Z = r\sin\theta\sin(U_1 z), \tag{89}$$

the geodesic Lagrangian (88) is

$$L_R\left(X, \frac{dX}{ds}, Y, \frac{dY}{ds}, Z, \frac{dZ}{ds}\right) = \frac{1}{2}\left(\left(\frac{dX}{ds}\right)^2 + \left(\frac{dY}{ds}\right)^2 + \left(\frac{dY}{ds}\right)^2\right), \tag{90}$$

with equations of motion

$$\frac{d^2X}{ds^2} = 0\ ,\ \frac{d^2Y}{ds^2} = 0\ ,\ \frac{d^2Z}{ds^2} = 0. \tag{91}$$

On the other hand, we consider the $(n+2)$-dimensional space with geodesic equations

$$L\left(x, \frac{dx^\mu}{ds}, u, \frac{du}{ds}, v, \frac{dv}{ds}\right) = \frac{1}{2}\delta_{\mu\nu}\left(\frac{dx^\nu}{ds}\right)\frac{dx^\nu}{ds} + \left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) - V(x^\kappa)\left(\frac{du}{ds}\right)^2, \tag{92}$$

which correspond to the Lorentzian lift for the n-dimensional system

$$\frac{d^2x^\mu}{dx^s} + V_{,\kappa}\delta^{\kappa\mu} = 0. \tag{93}$$

The requirement that the $(n+2)$ space be conformally flat gives that

$$V(x^\kappa) = \frac{V_0}{2}\delta_{\mu\nu}x^\mu x^\nu + \alpha_\kappa x^\nu + \beta, \tag{94}$$

in which α_k, β are constants. This is the potential function for the n-dimensional oscillator. The linearization process is performed as before through the conformal transformation.

Similarly, we can construct new higher-order linearized dynamical systems using a comparable methodology. In our future work, we intend to explore this approach for higher-order dynamical systems, thereby extending the scope of lifting methodologies.

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