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On the Maximum Likelihood Estimators' Uniqueness and Existence for Two Unitary Distributions: Analytically and Graphically, with Application

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Abstract: Unit distributions, exhibiting inherent symmetrical properties, have been extensively studied across various fields. A significant challenge in these studies, particularly evident in parameter estimations, is the existence and uniqueness of estimators. Often, it is challenging to demonstrate the existence of a unique estimator. The major issue with maximum likelihood and other estimator-finding methods that use iterative methods is that they need an initial value to reach the solution. This dependency on initial values can lead to local extremes that fail to represent the global extremities, highlighting a lack of symmetry in solution robustness. This study applies a very simple, and unique, estimation method for unit Weibull and unit Burr XII distributions that both attain the global maximum value. Therefore, we can conclude that the findings from the obtained propositions demonstrate that both the maximum likelihood and graphical methods are symmetrically similar. In addition, three real-world data applications are made to show that the method works efficiently.

Keywords: unitary distribution; graphical method; Cauchy–Schwarz inequality; simulation; data analysis

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1. Introduction

Various unit distributions have been applied to model real-world data, specifically in the unit interval, such as percentages and ratios across multiple disciplines, including biological research, mortality and recovery statistics, economics, health, risk assessments, and meteorology. The beta and Kumaraswamy distributions often come to mind when modeling and analyzing data from these areas (see [1,2]). Nevertheless, these traditional models might not always suffice, leading to significant challenges in precise data analysis. This underscores the increasing focus on unitary model studies found in the academic literature. Typically, well-known continuous distributions are modified to create unit distributions. The benefit of these unit distributions lies in their ability to enhance the flexibility of the original distribution across the unit interval without introducing additional parameters. For this purpose, many proposed unit distributions, such as unit gamma [3], log-xgamma [4], unit inverse Gaussian [5], unit Gompertz [6], unit Birnbaum–Saunders [7], unit half-normal [8], unit exponential probability [9], and unit upper truncated Weibull [10], have been proposed recently in the literature. In these studies, the performance of the unit distributions is examined based on parameter estimation and practical data applications. However, most importantly, the existence and uniqueness of the estimators is a major issue in these studies based on parameter estimations using mathematical analysis techniques, because a local or global maximum value can be discovered dependent on the initial value in iterative approaches, such as Newton–Raphson used for parameter estimations.

The maximum likelihood (ML) method is known to perform better in parameter estimation compared to other estimation methods (e.g., method of moments and least squares). The literature has addressed the existence and uniqueness of ML estimators in complete samples (refer to [11,12]). Initially, the existence and uniqueness of ML estimators for the Weibull distribution in both complete and censored samples were investigated by Balakrishnan and Kateri [13]. The existence and uniqueness of the ML parameter estimators for the general class of exponential distributions were the main topics of Ghitany et al. [14]. Nagatsuka and Balakrishnan [15] analyzed the existence and uniqueness of ML estimation for a three-parameter Weibull distribution, specifically in Type II right-censored samples. The Birnbaum–Saunders distribution's ML estimators for Type I, Type II, and hybrid censored samples were thoroughly examined by Balakrishnan and Zhu [16].

The ML estimators of the model parameters are obtained by solving likelihood equations. The likelihood equations cannot be solved generally. In this case, some numerical (iterative) methods, such as Newton–Raphson, should be applied to obtain ML estimators. The numerical methods have some difficulties in understanding the structure of equations. That is, the numerical methods do not give any information about the existence and uniqueness of the likelihood equations. However, the graphical method can be helpful to overcome this problem. Motivated by the graphical method studied by Balakrishnan and Kateri [13], we study the existence and uniqueness of ML estimators for the unit Weibull and unit Burr XII distributions. Through mathematical analysis, we investigate the conditions under which these estimators exist and are unique. Indeed, the graphical method plays a significant role in determining the ML estimator of the shape parameter for the mentioned distributions. Additionally, a numerical analysis was conducted to illustrate the existence, uniqueness, and outcomes of ML estimators. This approach not only ensures symmetry with analytical methods but also enhances the reliability of the findings. Furthermore, this work is the first to apply this method to unit distributions.

The article's remaining sections are arranged as follows: In Sections 2 and 3, parameter estimation is obtained for the unit Weibull and unit Burr XII using the ML and graphical methods. Furthermore, mathematical analysis is utilized in the proofs of obtained propositions for both methods. Section 4 presents the performances and iteration times (in seconds) of the graphical method compared to the ML method through a Monte Carlo simulation study. Section 5 demonstrates the uniqueness of the ML on three real-world datasets using the graphical method. Section 6 summarizes the paper and highlights the obtained results.

2. Estimation of Parameters for the Unit Weibull Distribution

2.1. The Maximum Likelihood Method

Recently, Mazucheli et al. [17] introduced a probability distribution known as the unit Weibull (UW) distribution, which is supported on the unit interval. The UW distribution's probability density function (pdf) and cumulative distribution function (cdf) are provided, respectively, by

$$F(x;\alpha,\beta) = \exp\Big[(-\alpha(-\log(x))^{\beta}\Big], \tag{1}$$

$$f(x;\alpha,\beta) = \frac{1}{x}\alpha\beta(-\log(x))^{\beta-1}\exp\Big[(-\alpha(-\log(x))^{\beta}\Big], \tag{2}$$

$0 < x < 1$, $\alpha > 0$, $\beta > 0$, where α and β are the scale and shape parameters, respectively. Consider n independent and identically random variables $(X_1, \ldots, X_n)$ from the UW distribution, with $x_1, \ldots, x_n$ representing their observations. Then, thanks to Equation (2), the corresponding log-likelihood function is obtained as

$$\ell(\alpha,\beta) = n\log(\alpha) + n\log(\beta) - \sum_{i=1}^{n}\log(x_i) + (\beta-1)\sum_{i=1}^{n}\log(-\log(x_i)) - \alpha\sum_{i=1}^{n}(-\log(x_i))^{\beta}.$$

For the complete sample case, the score equations are

$$\frac{\partial \ell(\alpha,\beta)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} (-\log(x_i))^{\beta}, \tag{3}$$

and

$$\frac{\partial \ell(\alpha,\beta)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log(-\log(x_i)) - \alpha \sum_{i=1}^{n} (-\log(x_i))^{\beta} \log(-\log(x_i)). \tag{4}$$

The parameters α and β can be estimated by numerically solving Equations (3) and (4). These equations require a numerical method for their vectorial solution. For example, the Newton–Raphson method can be utilized via the "nlm package" in R programming to drive the ML estimators. Considering the other parameters are known, we concentrate on the presence and uniqueness of each ML estimator of the UW distribution parameters in the following results; for more information, refer to Popović et al. [18].

Proposition 1. *From Equation (3), let $f_1(\alpha,\beta) = \partial \ell(\alpha,\beta)/\partial \alpha$. Then, there exists a solution to solve $f_1(\alpha,\beta) = 0$ for $\alpha \in (0,\infty)$, and the solution is unique.*

Proof. We have

$$f_1(\alpha,\beta) = \frac{n}{\alpha} - \sum_{i=1}^{n} (-\log(x_i))^{\beta}.$$

The limiting values of $f_1(\alpha,\beta)$ as $\alpha \to 0$ and $\alpha \to \infty$ are attained in the manner described as follows: $\lim_{\alpha\to 0} f_1(\alpha,\beta) = \infty$ and $\lim_{\alpha\to\infty} f_1(\alpha,\beta) = -\infty < 0$. Consequently, at least one root exists, say $\hat{\alpha} \in (0,\infty)$. To illustrate the uniqueness, we must demonstrate that $\partial f_1(\alpha,\beta)/\partial\alpha < 0$; that is, $\partial f_1(\alpha,\beta)/\partial\alpha = -n/\alpha^2 < 0$. Because of this, there exists a solution for $f_1(\alpha,\beta) = 0$, and the root $\hat{\alpha}$ is unique. □

Proposition 2. *From Equation (4), let $f_2(\alpha,\beta) = \partial \ell(\alpha,\beta)/\partial\beta$ and $S_1 = -\frac{n}{\beta^2} - \alpha \sum_{i=1}^{n} (-\log (x_i))^{\beta} \log(-\log(x_i))^2$. Consequently, there exists a solution to $f_2(\alpha,\beta) = 0$ for $\beta \in (0,\infty)$, and the solution is unique when $x > 1/e$.*

Proof. We have

$$f_2(\alpha,\beta) = \frac{n}{\beta} + \sum_{i=1}^{n} \log(-\log(x_i)) - \alpha \sum_{i=1}^{n} (-\log(x_i))^{\beta} \log(-\log(x_i)).$$

The limiting values of $f_2(\alpha,\beta)$ as $\beta \to 0$ and $\beta \to \infty$ are obtained as follows: $\lim_{\beta\to 0} f_2(\alpha,\beta) = \infty$ and $\lim_{\beta\to\infty} f_2(\alpha,\beta) = \log(-\log(x)) < 0$ for $x > 1/e$. Consequently, at least one root exists, say $\hat{\beta} \in (0,\infty)$. To illustrate the uniqueness, we must demonstrate that $\partial f_2(\alpha,\beta)/\partial\beta < 0$; that is,

$$-\frac{n}{\beta^2} - \alpha \sum_{i=1}^{n} \overbrace{(-\log(x_i))^{\beta}}^{>0} \overbrace{\log(-\log(x_i))^2}^{>0} < 0,$$

hence, $S_1 < 0$. For that reason, there exists a solution for $f_2(\alpha,\beta) = 0$, and the root $\hat{\beta}$ is unique. □

2.2. The Graphical Method

The numerical methods do not give any information about the existence and uniqueness of the likelihood equations. However, the graphical method can be helpful to overcome this problem. The graphical method studied by Balakrishnan and Kateri [13] is used to obtain ML estimators and their existing uniqueness for the UW distribution. However, since there is uncertainty about the uniqueness of this solution, the estimation of the α parameter ($\hat{\alpha}$) with the help of Equation (3) is given below depending on the β parameter:

$$\widehat{\alpha} = \frac{n}{\sum_{i=1}^{n}(-\log(x_i))^{\beta}} \tag{5}$$

and if Equation (5) is substituted in Equation (4), the following Equation (6) is obtained:

$$\frac{n}{\beta} - \left(\frac{n}{\sum_{i=1}^{n}(-\log(x_i))^{\beta}}\right)\sum_{i=1}^{n}(-\log(x_i))^{\beta}\log(-\log(x_i)) + \sum_{i=1}^{n}\log(-\log(x_i)) = 0. \tag{6}$$

If Equation (6) is rearranged, we obtain

$$\frac{1}{\beta} = \left(\frac{1}{\sum_{i=1}^{n}(-\log(x_i))^{\beta}}\right)\sum_{i=1}^{n}(-\log(x_i))^{\beta}\log(-\log(x_i)) - \left(\frac{1}{n}\right)\sum_{i=1}^{n}\log(-\log(x_i)), \tag{7}$$

in which the left side $\frac{1}{\beta}$ is a decreasing function of β. Accordingly, denoting the right side must be an increasing function of β. The first derivative of Equation (7) with respect to β is as follows:

$$\begin{aligned} G(\beta, x) \quad = \quad & \frac{\sum_{i=1}^{n}(-\log(x_i))^{\beta}\sum_{i=1}^{n}(-\log(x_i))^{\beta}\log(-\log(x_i))^2}{\left[\sum_{i=1}^{n}(-\log(x_i))^{\beta}\right]^2} \\ & - \frac{\left[\sum_{i=1}^{n}(-\log(x_i))^{\beta}\log(-\log(x_i))^2\right]^2}{\left[\sum_{i=1}^{n}(-\log(x_i))^{\beta}\right]^2}. \end{aligned} \tag{8}$$

We indicate that $G(\beta, x)$ is greater than zero for a specific sample x. $G(\beta, x)$ is a monotone increasing function of β and approaches a finite, positive limit as $\beta \to \infty$. Since $\frac{1}{\beta}$ is strictly decreasing with the right limit $+\infty$ at 0, it follows that the plots of $\frac{1}{\beta}$ and $G(\beta, x)$ will intersect exactly once, providing the ML estimate of β. It is evident that the denominator of the $G(\beta, x)$ is greater than zero. $G^*(\beta; x)$ is the equivalent of demonstrating that the numerator of $G(\beta, x)$ is greater than zero, where

$$\begin{aligned} G^*(\beta; x) \quad = \quad & \sum_{i=1}^{n}(-\log(x_i))^{\beta}\sum_{i=1}^{n}(-\log(x_i))^{\beta}\log - \log(x_i))^2 \\ & - \left[\sum_{i=1}^{n}(-\log(x_i))^{\beta}\log(-\log(x_i))^2\right]^2. \end{aligned} \tag{9}$$

Setting $\alpha_i = (-\log((x_i)))^{\frac{\beta}{2}}$ and $b_i = \left((-\log(x_i))^{\beta}\log(-\log(x_i))\right)^{\frac{\beta}{2}}$, $i = 1, \ldots, n$, hence, Equation (9) becomes

$$G^*(\beta; y) = \sum_{i=1}^{n}\alpha_i^2\sum_{i=1}^{n}b_i^2 - \left(\sum_{i=1}^{n}\alpha_i b_i\right)^2$$

and the necessary condition that $G(\beta; y)$ is a monotone growing function of β is established by the Cauchy–Schwarz inequality, which implies $G^*(\beta; y) \geq 0$. It is important to note that there is a finite upper bound for $G(\beta; y)$, with $y = log(x)$ transformation, namely

$$G(\beta; y) = \frac{\sum_{i=1}^{n} y_i^{\beta}\sum_{i=1}^{n} {y_i}^{\beta}\log(y_i)^2 - \left[\sum_{i=1}^{n}({y_i}^{\beta}\log(y_i)^2\right]^2}{\left[\sum_{i=1}^{n} y_i^{\beta}\right]^2}$$

and

$$\lim_{\beta\to\infty} G(\beta; y) = \log(y_{(n)}) - \frac{1}{n}\sum_{i=1}^{n}\log(y_{(i)}) = \frac{1}{n}\sum_{i=1}^{n-1}\log\left(\frac{(y_{(n)})}{y(i)}\right) > 0.$$

Thus, a plot of the $\frac{1}{\beta}$ and the $G(\beta, x)$ provides an easy graphical way of determining the estimator of the shape parameter β.

3. Estimation of Parameters for the Unit Burr XII Distribution

3.1. The Maximum Likelihood Method

In this subsection, let us recall a probability distribution called the unit Burr XII (UB) distribution proposed by Korkmaz et al. [19]. The cdf and pdf of the UB distribution are given, respectively, by

$$F(x, \alpha, \beta) = \left(1 + (-\log(x))^{\beta}\right)^{-\alpha} \tag{10}$$

and

$$f(x, \alpha, \beta) = \frac{\alpha\beta}{x}(-\log(x))^{\beta-1}(1 + \left(-\log(x))^{\beta}\right)^{-\alpha-1}, \tag{11}$$

where $x \in (0, 1)$ and $\alpha, \beta > 0$ are the shape parameters. Let $X_{1,}X_{2,...,}X_n$ represent a size-n random sample drawn from the UB distribution with observed values $x_{1,}x_{2,}\ldots, x_n$. Next, the log-likelihood function is ascertained using the formula

$$\begin{aligned} \ell(\alpha, \beta) \quad = \quad & n\log(\alpha) + n\log(\beta) - \sum_{i=1}^{n}\log(x_i) + (\beta - 1)\sum_{i=1}^{n}\log(-\log(x_i)) \\ & -(\alpha + 1)\sum_{i=1}^{n}\log(1 + (-\log(x_i))^{\beta}). \end{aligned}$$

For the complete sample case, the score equations are

$$\frac{\partial\ell(\alpha, \beta)}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\log\left(1 + (-\log(x_i))^{\beta}\right) = 0, \tag{12}$$

$$\frac{\partial\ell(\alpha, \beta)}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log(-\log(x_i)) - (\alpha + 1)\sum_{i=1}^{n}\frac{\log(-\log(x_i))(-\log(x_i))^{\beta}}{1 + (-\log(x_i))^{\beta}} = 0. \tag{13}$$

The parameters α and β can be estimated through numerical solutions of Equations (12) and (13). A numerical solution is required for the vectorial solution of the two equations above. For instance, the nlm package in R programming can be used to apply the Newton–Raphson technique to obtain the ML estimators. In the next results, considering that the other parameters are known, we concentrate on the existence and uniqueness of each ML estimator of the UB distribution parameters; for more information on this technique, see the approach of Popović et al. [18].

Proposition 3. *From Equation (12), let $f_3(\alpha, \beta) = \partial\ell(\alpha, \beta)/\partial\alpha$. Then, there exists a solution for $f_3(\alpha, \beta) = 0$ for $\alpha \in (0, \infty)$, and the solution is unique.*

Proof. We have

$$f_3(\alpha, \beta) = \frac{n}{\alpha} - \sum_{i=1}^{n}\log\left(1 + (-\log(x_i))^{\beta}\right).$$

The limiting values of $f_3(\alpha, \beta)$ as $\alpha \to 0$ and $\alpha \to \infty$ are obtained as follows: $\lim_{\alpha\to0}f_3$ $(\alpha, \beta) = \infty$ and $\lim_{\alpha\to\infty}f_3(\alpha, \beta) = -\log\left(1 + (-\log(x))^{\beta}\right) < 0$. Consequently, at least one root exists, say $\hat{\alpha} \in (0, \infty)$. In order to establish the uniqueness, we must demonstrate that $\partial f_3(\alpha, \beta)/\partial\alpha < 0$; that is, $\partial f_3(\alpha, \beta)/\partial\alpha = -n/\alpha^2 < 0$. Therefore, there is a solution for $f_3(\alpha, \beta) = 0$, and the root $\hat{\alpha}$ is unique. □

Proposition 4. *From Equation (13), let $f_4(\alpha,\beta) = \partial\ell(\alpha,\beta)/\partial\beta$. Then, there is a solution to $f_4(\alpha,\beta) = 0$ for $\beta \in (0,\infty)$, and the unique solution arises when $x > 1/e$ and $S_2 < 0$, where*

$$S_2 = -\frac{n}{\beta^2} - (\alpha+1)\frac{\sum_{i=1}^{n} \log(-\log(x_i))^2(-\log(x_i))^\beta}{\left(1+(-\log(x_i))^\beta\right)^2}.$$

Proof. We have

$$f_4(\alpha,\beta) = \frac{n}{\beta} + \sum_{i=1}^{n} \log(-\log(x_i)) - (\alpha+1)\sum_{i=1}^{n} \frac{\log(-\log(x_i))(-\log(x_i))^\beta}{1+(-\log(x_i))^\beta}.$$

The limiting values of $f_4(\alpha,\beta)$ as $\beta \to 0$ and $\beta \to \infty$ are obtained as follows: $\lim_{\beta\to 0} f_4(\alpha,\beta) = \infty$ and $\lim_{\beta\to\infty} f_4(\alpha,\beta) = \log(-\log(x)) < 0$ for $x > 1/e$. Thus, there exists at least one root, say $\hat{\beta} \in (0,\infty)$. To illustrate the uniqueness, we need to show that $\partial f_3(\alpha,\beta)/\partial\beta < 0$; that is,

$$-\frac{n}{\beta^2} - (\alpha+1)\frac{\sum_{i=1}^{n} \overbrace{\log(-\log(x_i))^2}^{>0}\overbrace{(-\log(x_i))^\beta}^{>0}}{\underbrace{\left(1+(-\log(x_i))^\beta\right)^2}_{>0}} < 0,$$

hence, $S_2 < 0$. Therefore, there exists a solution for $f_4(\alpha,\beta) = 0$, and the root $\hat{\beta}$ is unique. □

3.2. The Graphical Method

In this subsection, the graphical method is applied to the UB distribution. Let n independent and identically random variables $X_1, X_2, \ldots, X_n$ from the UB be represented by the observations $x_1, x_2, \ldots, x_n$. However, since there is uncertainty about the uniqueness of this solution, the estimation of the α parameter ($\hat{\alpha}$) with the help of Equation (12) is given below depending on the β parameter:

$$\widehat{\alpha} = \frac{n}{\sum_{i=1}^{n} \log\left(1+(-\log(x_i))^\beta\right)}, \tag{14}$$

and if Equation (14) is substituted in Equation (15), the following equation is obtained:

$$\frac{n}{\beta} = \left(\frac{n}{\sum_{i=1}^{n} \log\left(1+(-\log(x_i))^\beta\right)} + 1\right)\sum_{i=1}^{n} \frac{\log(-\log(x_i))(-\log(x_i))^\beta}{1+(-\log(x_i))^\beta} - \sum_{i=1}^{n} \log(-\log(x_i)),$$

where the expression above can be rearranged as follows:

$$\frac{1}{\beta} = \left(\frac{n}{\sum_{i=1}^{n} \log\left(1+(-\log(x_i))^\beta\right)} + 1\right)\sum_{i=1}^{n} \frac{\log(-\log(x_i))(-\log(x_i))^\beta}{1+(-\log(x_i))^\beta}\left(\frac{1}{n}\right) - \sum_{i=1}^{n} \log(-\log(x_i))\left(\frac{1}{n}\right). \tag{15}$$

Thus, a plot of the $\frac{1}{\beta}$ and the Right-Hand Side (RHS) of Equation (15) provides a straightforward graphical approach to figure out the ML estimator of the shape parameter β. As we have shown in the UB distribution that the RHS of the Equation (15) function increases depending on β, it can be proved using the Cauchy–Schwarz inequality.

4. Simulation Study

For the simulation study, we generated 5000 samples from the UB and UW distributions with sample sizes of $n = 50, 100, 200, 300, 500$ and selected values of $\alpha = 0.5, 0.4, 3$ and $\beta = 0.5, 3, 2$. The estimation performance was assessed using averages, incorporating metrics such as Average Absolute Biases (ABBs), Mean Squared Errors (MSEs), and Mean Relative Errors (MREs) for all methods. These criteria are computed as follows:

$$\widehat{ABB}_{\delta} = \frac{1}{5000}\sum_{i=1}^{5000}\left|\widehat{\delta}_1 - \delta\right|,$$

$$\widehat{MSE}_{\delta} = \frac{1}{5000}\sum_{i=1}^{5000}\left(\widehat{\delta}_1 - \delta\right)^2,$$

$$\widehat{MRE}_{\delta} = \frac{1}{5000}\sum_{i=1}^{5000}\frac{\left|\widehat{\delta}_1 - \delta\right|}{\delta}.$$

where $\delta = (\alpha, \beta)$ and $\widehat{\delta} = \left(\widehat{\alpha}, \widehat{\beta}\right)$.

Tables 1–7 provide the parameter estimates and iteration times for the graphical and ML methods for the simulation results. According to these results, as the sample size (n) increases, in both approaches, values become closer to the nominal value. However, it is observed that the graphical method has lower iteration times. Note that the previous sections have established the uniqueness of the estimators produced by the graphical technique.

Table 1. Average, MSE, ABB, and MRE values of the UW distribution for $\alpha = 0.5; \beta = 0.5$.

		Graphical Method			Maximum Likelihood Method		
		$\widehat{\beta}$	$\widehat{\alpha}$	Iteration Times	$\widehat{\beta}$	$\widehat{\alpha}$	Iteration Times
50	Average	0.5135	0.5020	3.2112	0.5009	0.5302	107.0762
	MSE	0.0035	0.0093		0.0027	0.0090	
	ABB	0.0460	0.0766		0.0412	0.0727	
	MRE	0.0920	0.1531		0.0825	0.1454	
100	Average	0.5077	0.5008	5.2085	0.5010	0.5144	126.5824
	MSE	0.0017	0.0046		0.0015	0.0043	
	ABB	0.0323	0.0541		0.0306	0.0517	
	MRE	0.0646	0.1082		0.0611	0.1035	
200	Average	0.5030	0.5002	10.0940	0.4939	0.5191	348.8838
	MSE	0.0008	0.0022		0.0006	0.0022	
	ABB	0.0220	0.0376		0.0203	0.0364	
	MRE	0.0440	0.0752		0.0407	0.0727	
300	Average	0.5024	0.5002	14.6583	0.4999	0.5080	390.4301
	MSE	0.0005	0.0015		0.0005	0.0015	
	ABB	0.0181	0.0305		0.0182	0.0302	
	MRE	0.0361	0.0609		0.0363	0.0604	
500	Average	0.5020	0.4998	24.1379	0.4997	0.5036	525.2444
	MSE	0.0003	0.0009		0.0003	0.0008	
	ABB	0.0140	0.0241		0.0131	0.0230	
	MRE	0.0279	0.0482		0.0263	0.0460	

Table 2. Average, MSE, ABB, and MRE values of the UW distribution for $\alpha = 0.5; \beta = 3$.

		Graphical Method			Maximum Likelihood Method		
		$\widehat{\beta}$	$\widehat{\alpha}$	Iteration Times	$\widehat{\beta}$	$\widehat{\alpha}$	Iteration Times
50	Average	3.0870	0.5018	3.7160	3.1452	0.4867	182.0035
	MSE	0.1298	0.0093		0.1347	0.8023	
	ABB	0.2793	0.0765		0.2799	0.0900	
	MRE	0.0931	0.1531		0.0933	0.1800	
100	Average	3.0445	0.4995	6.3259	3.0936	0.4937	155.8538
	MSE	0.0616	0.0045		0.0626	0.0044	
	ABB	0.1937	0.0530		0.1919	0.0521	
	MRE	0.0646	0.1060		0.0640	0.1043	
200	Average	3.0277	0.4988	11.7208	3.0576	0.4986	369.6908
	MSE	0.0295	0.0022		0.0276	0.0022	
	ABB	0.1353	0.0375		0.1304	0.0379	
	MRE	0.0451	0.0750		0.0435	0.0758	
300	Average	3.0117	0.5005	17.6820	3.0634	0.4924	552.2836
	MSE	0.0187	0.0015		0.0177	0.0013	
	ABB	0.1087	0.0306		0.1016	0.0286	
	MRE	0.0362	0.0612		0.0338	0.0572	
500	Average	3.0073	0.5001	29.4930	3.0377	0.4970	624.4005
	MSE	0.0111	0.0009		0.0109	0.0008	
	ABB	0.0840	0.0237		0.0812	0.0233	
	MRE	0.0280	0.0474		0.0271	0.0466	

Table 3. Average, MSE, ABB, and MRE values of the UW distribution for $\alpha = 0.4; \beta = 2$.

		Graphical Method			Maximum Likelihood Method		
		$\widehat{\beta}$	$\widehat{\alpha}$	Iteration Times	$\widehat{\beta}$	$\widehat{\alpha}$	Iteration Times
50	Average	2.0554	0.4008	4.6646	2.0896	0.4012	50.2035
	MSE	0.0605	0.0072		0.0618	0.0070	
	ABB	0.1904	0.0673		0.1905	0.0668	
	MRE	0.0952	0.1682		0.0952	0.1670	
100	Average	2.0308	0.3978	6.0211	2.0652	0.3968	115.0413
	MSE	0.0263	0.0034		0.0274	0.0032	
	ABB	0.1276	0.0465		0.1274	0.0450	
	MRE	0.0638	0.01163		0.0637	0.1125	
200	Average	2.0165	0.3993	12.5732	2.0473	0.3945	204.3718
	MSE	0.0129	0.0017		0.0133	0.0015	
	ABB	0.0904	0.0326		0.0895	0.0313	
	MRE	0.0452	0.0816		0.0448	0.0782	
300	Average	2.0078	0.4011	16.8672	2.0275	0.3968	229.3947
	MSE	0.0081	0.0011		0.0088	0.0010	
	ABB	0.0713	0.0270		0.0738	0.0250	
	MRE	0.0356	0.0675		0.0369	0.0625	
500	Average	2.0067	0.3997	28.3637	2.0247	0.3972	529.7605
	MSE	0.0050	0.0007		0.0048	0.0007	
	ABB	0.0563	0.0202		0.0539	0.0206	
	MRE	0.0281	0.0506		0.0269	0.0516	

Table 4. Average, MSE, ABB, and MRE values of the UW distribution for $\alpha = 3; \beta = 2$.

		Graphical Method			Maximum Likelihood Method		
		$\hat{\beta}$	$\hat{\alpha}$	Iteration Times	$\hat{\beta}$	$\hat{\alpha}$	Iteration Times
50	Average	2.0548	3.1486	3.5716	2.0547	3.1351	41.2599
	MSE	0.0586	0.3397		0.0556	0.2858	
	ABB	0.1877	0.4279		0.1835	0.4079	
	MRE	0.0938	0.1426		0.0917	0.1360	
100	Average	2.0262	3.0731	6.1976	2.0303	3.0847	63.1969
	MSE	0.0258	0.1365		0.0263	0.1390	
	ABB	0.1258	0.2815		0.1282	0.2877	
	MRE	0.0629	0.0938		0.0641	0.0959	
200	Average	2.0138	3.0361	11.2080	2.0135	3.0359	170.5127
	MSE	0.0124	0.0632		0.0127	0.0628	
	ABB	0.0885	0.1972		0.0893	0.1964	
	MRE	0.0442	0.0657		0.0446	0.0655	
300	Average	2.0097	3.0181	18.7684	2.0105	3.0278	205.8142
	MSE	0.0084	0.0420		0.0083	0.0406	
	ABB	0.0726	0.1615		0.0722	0.1589	
	MRE	0.0363	0.0538		0.0361	0.0530	
500	Average	2.0045	3.0143	26.8866	2.0068	3.0171	270.5770
	MSE	0.0049	0.0241		0.0049	0.0229	
	ABB	0.0558	0.1220		0.0554	0.1194	
	MRE	0.0279	0.0407		0.0277	0.0398	

Table 5. Average, MSE, ABB, and MRE values of the UB distribution for $\alpha = 0.5; \beta = 3$.

		Graphical Method			Maximum Likelihood Method		
		$\hat{\beta}$	$\hat{\alpha}$	Iteration Times	$\hat{\beta}$	$\hat{\alpha}$	Iteration Times
50	Average	3.1272	0.5038	4.1265	3.2115	0.5044	129.0187
	MSE	0.3105	0.0098		0.3002	0.0096	
	ABB	0.4296	0.0782		0.4154	0.0778	
	MRE	0.1432	0.1564		0.1385	0.1555	
100	Average	3.0753	0.5008	7.4095	3.1783	0.4960	209.9537
	MSE	0.1493	0.0048		0.1366	0.0042	
	ABB	0.2994	0.0556		0.2782	0.0518	
	MRE	0.0690	0.0791		0.0673	0.0800	
300	Average	3.0181	0.5014	20.2056	3.1192	0.4935	629.8633
	MSE	0.0439	0.0015		0.0428	0.0013	
	ABB	0.1646	0.0311		0.1555	0.0291	
	MRE	0.0549	0.0622		0.0518	0.0582	
500	Average	3.0150	0.5003	33.9075	3.1023	0.4922	1133.2000
	MSE	0.0258	0.0009		0.0263	0.0008	
	ABB	0.1271	0.0243		0.1226	0.0227	
	MRE	0.0424	0.0487		0.0409	0.0453	

Table 6. Average, MSE, ABB, and MRE values of the UB distribution for $\alpha = 0.4$, $\beta = 2$.

		Graphical Method			**Maximum Likelihood Method**		
		$\hat{\beta}$	$\hat{\alpha}$	**Iteration Times**	$\hat{\beta}$	$\hat{\alpha}$	**Iteration Times**
50	Average	2.1342	0.4071	4.4509	2.2082	0.4074	123.8491
	MSE	0.2193	0.0078		0.1906	0.0070	
	ABB	0.3377	0.0699		0.3246	0.0674	
	MRE	0.1689	0.1746		0.1623	0.1684	
100	Average	2.0742	0.4081	8.4490	2.1618	0.3992	214.0975
	MSE	0.0887	0.0038		0.0889	0.0028	
	ABB	0.2201	0.0484		0.2237	0.0429	
	MRE	0.1101	0.1212		0.1118	0.1072	
200	Average	2.0424	0.4079	18.7761	2.1357	0.3971	690.8692
	MSE	0.0374	0.0018		0.0434	0.0013	
	ABB	0.1499	0.0338		0.1544	0.0288	
	MRE	0.0750	0.0845		0.0772	0.0720	
300	Average	2.0293	0.4086	34.5835	2.1195	0.3952	480.1750
	MSE	0.0245	0.0013		0.0286	0.0008	
	ABB	0.1223	0.0284		0.1276	0.0223	
	MRE	0.0611	0.0710		0.0638	0.0557	
500	Average	2.0255	0.4090	102.1389	2.0749	0.4021	815.1217
	MSE	0.0149	0.0008		0.0152	0.0005	
	ABB	0.0958	0.0229		0.0933	0.0183	
	MRE	0.0479	0.0571		0.0467	0.0457	

Table 7. Average, MSE, ABB, and MRE values of the UB distribution for $\alpha = 3; \beta = 2$.

		Graphical Method			**Maximum Likelihood Method**		
		$\hat{\beta}$	$\hat{\alpha}$	**Iteration times**	$\hat{\beta}$	$\hat{\alpha}$	**Iteration Times**
50	Average	2.0499	3.1036	4.8906	2.0570	3.1236	59.6729
	MSE	0.0515	0.2519		0.0504	0.2507	
	ABB	0.1769	0.3807		0.1732	0.3782	
	MRE	0.0885	0.1269		0.0866	0.1264	
100	Average	2.0255	3.0562	6.7617	2.0330	3.0705	40.9404
	MSE	0.0241	0.1084		0.0238	0.1038	
	ABB	0.1225	0.2536		0.1209	0.2459	
	MRE	0.0613	0.0845		0.0604	0.0820	
200	Average	2.0121	3.0232	12.8648	2.0210	3.0535	133.0177
	MSE	0.0113	0.0516		0.0109	0.0472	
	ABB	0.0848	0.1776		0.0815	0.1680	
	MRE	0.0424	0.0592		0.0407	0.0560	
300	Average	2.0094	3.0184	19.0574	2.0184	3.0418	246.6066
	MSE	0.0074	0.0330		0.0076	0.0319	
	ABB	0.0688	0.1428		0.0681	0.1377	
	MRE	0.0344	0.0476		0.0341	0.0459	
500	Average	2.0027	3.0118	30.6004	2.0111	3.0357	234.0810
	MSE	0.0045	0.0202		0.0043	0.0181	
	ABB	0.0535	0.1131		0.0507	0.1042	
	MRE	0.0267	0.0377		0.0254	0.0347	

5. Data Analysis

In this section, we examine three actual data examples to demonstrate how the ML and graphical methods are put into practice. All computations made in this section were performed using the R programming. The R codes for the UW distribution are provided in Appendix A, which are used for the simulation study and data applications. To ensure that the log-likelihood function behaves appropriately and achieves a clear optimum, we produce plots showing the profiles of the log-likelihood (lk) function for both the

UW and UB distributions using the maximum likelihood (ML) estimation method and the considered graphical method (see Figures 1–6). The considered data examples are mentioned in the following.

Example 1. *In the first dataset, we consider the infection times of kidney dialysis patients as described by Klein and Moeschberger [20]. We then normalize these data by dividing them by 30 to scale them between 0 and 1, which are listed in Table 8.*

Table 8. Times of infection.

0.083	0.083	0.117	0.117	0.117	0.150	0.183	0.217	0.217	0.25
0.250	0.250	0.250	0.283	0.317	0.350	0.383	0.417	0.417	0.45
0.483	0.483	0.717	0.717	0.750	0.750	0.850	0.917		

The p-values for the UW and UB distributions are 0.7824 and 0.6971, respectively, which show that both distributions have good fit values for this dataset. Moreover, the ML estimates of $(\widehat{\alpha}, \widehat{\beta})$ for the UW and UB distributions are (0.6125, 1.6990) and (1.1077, 2.0975), respectively.

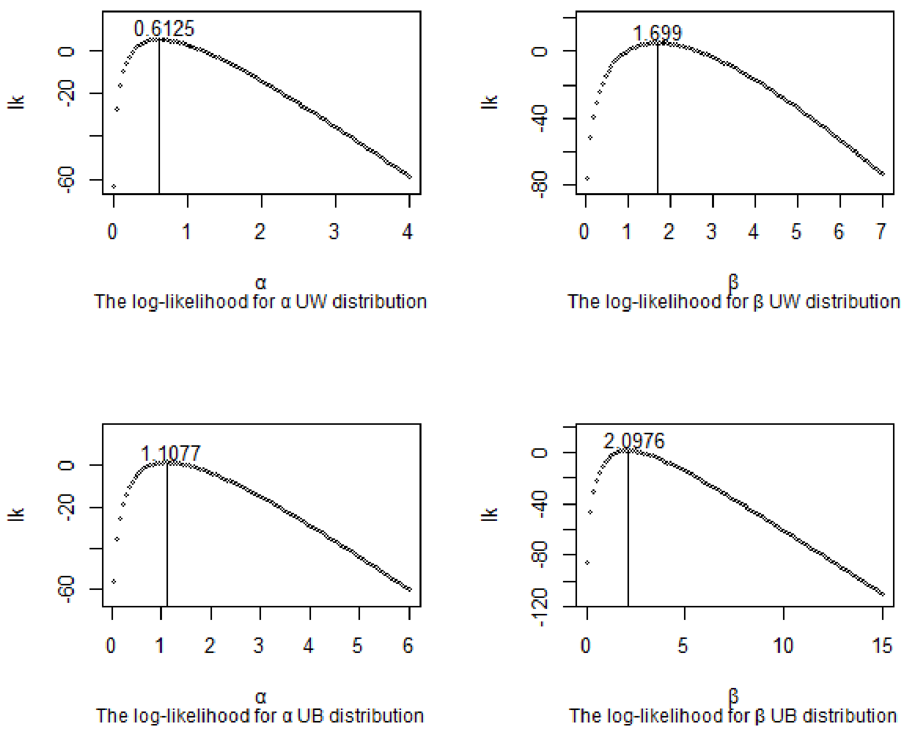

Figure 1. A fitted profile of the log-likelihood function obtained from the UW and UB distributions for the ML estimators based on Example 1.

Example 2. *The second dataset relates to the failure times of an airplane's air conditioning system as documented by Linhart and Zucchini [21]. To normalize these data, we divided the values by 265, resulting in data that range from 0 to 1. These normalized values are presented in Table 9.*

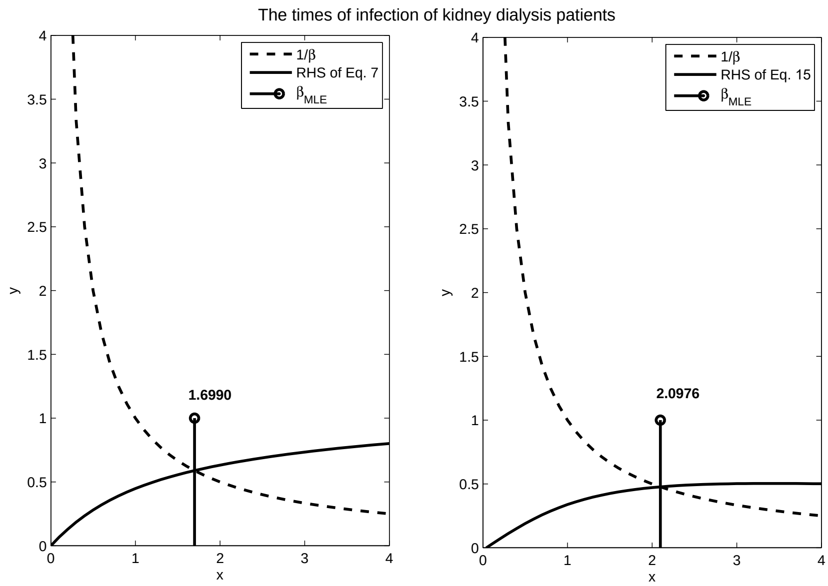

Figure 2. RHS of Equation (7) vs. $\frac{1}{\beta}$ plot for UW and RHS of Equation (15) vs. $\frac{1}{\beta}$ plot for UB, in Example 1.

Table 9. Times of infection.

0.0868	0.9849	0.3283	0.0264	0.4528	0.0528	0.2340	0.1774	0.8491
0.9283	0.0792	0.1585	0.0754	0.0188	0.0453	0.4528	0.0415	0.0113
0.2679	0.0415	0.0528	0.0415	0.0603	0.3396	0.0038	0.2679	0.0528
0.0604	0.1962	0.3585						

The p-values for the UW and UB distributions are 0.3229 and 0.0979, respectively. Since the p-values are greater than 0.05, the dataset fits both UW and UB distributions. Furthermore, the ML estimates of $(\widehat{\alpha}, \widehat{\beta})$ for the UW and UB distributions are (0.2786, 1.4563) and (0.6897, 1.6794), respectively.

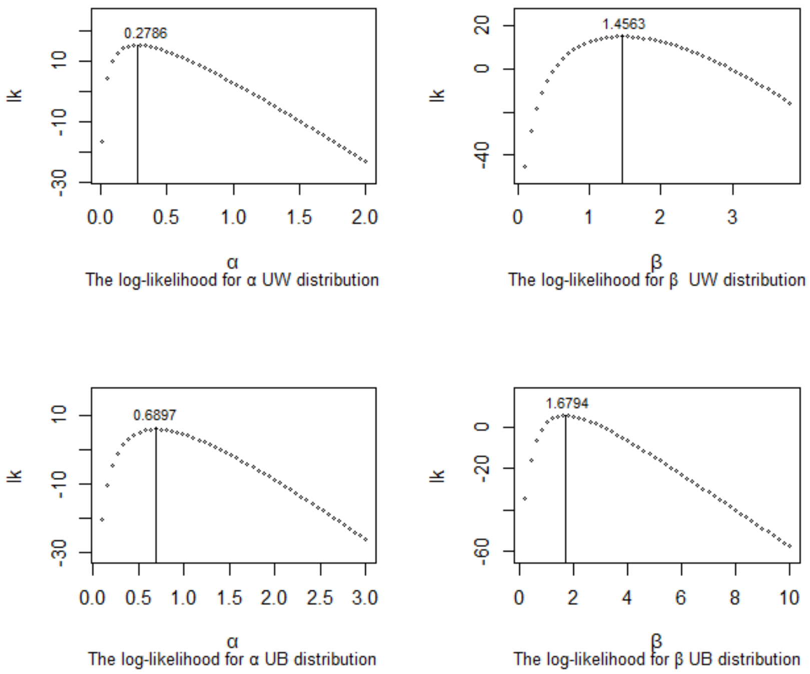

Figure 3. A fitted profile of the log-likelihood function obtained from the UW and UB distributions for the ML estimators based on Example 2.

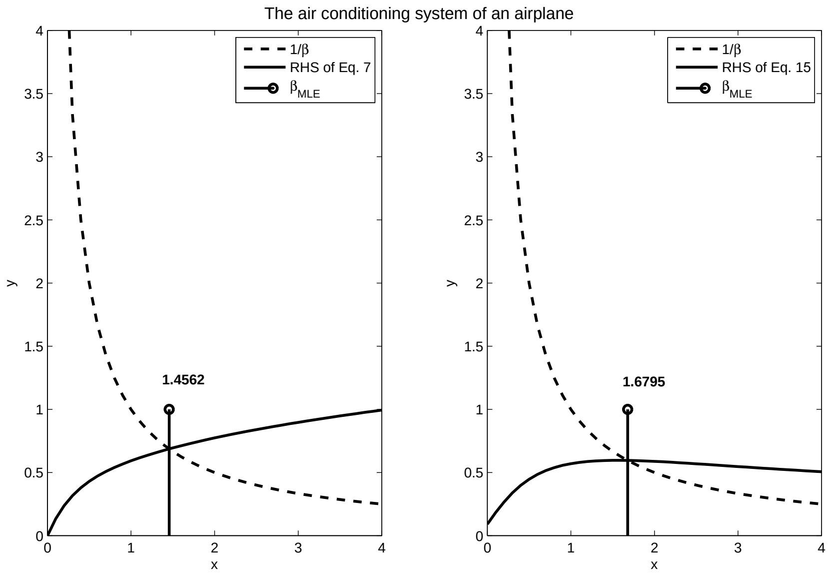

Figure 4. RHS of Equation (7) vs. $\frac{1}{\beta}$ plot for UW and RHS of Equation (15) vs. $\frac{1}{\beta}$ plot for UB, in Example 2.

Example 3. *The third dataset comes from the field of civil engineering and records the hailing times. This dataset has previously been discussed by Kotz and Van Dorp [22]. Once again, we normalized the data by dividing by 85, resulting in values that range between 0 and 1. These normalized values are detailed in Table 10.*

Table 10. The times that represent the hailing times.

0.320	0.340	0.350	0.350	0.360	0.360	0.390	0.415	0.430	0.440
0.440	0.440	0.454	0.460	0.470	0.470	0.473	0.475	0.479	0.480
0.480	0.482	0.490	0.495	0.510	0.510	0.515	0.520	0.520	0.530
0.540	0.540	0.540	0.540	0.540	0.560	0.560	0.570	0.570	0.570
0.580	0.580	0.580	0.580	0.590	0.590	0.590	0.590	0.590	0.590
0.590	0.590	0.590	0.590	0.600	0.600	0.600	0.600	0.600	0.600
0.600	0.600	0.620	0.630	0.640	0.650	0.650	0.650	0.660	0.670
0.680	0.680	0.690	0.700	0.700	0.700	0.710	0.730	0.740	0.750
0.790	0.800	0.820	0.850	0.860					

The p-values for the UW and UB distributions are 0.2292 and 0.1401, respectively. Both the UW and UB distributions are fitted by the dataset since the p-values are greater than 0.05. Also, the ML estimates of $(\widehat{\alpha}, \widehat{\beta})$ for the UW and UB distributions are (3.5129, 2.9745) and (4.8865, 3.3733), respectively.

Figures 2, 4, and 6 show that the curves of $\frac{1}{\beta}$ and RHS of Equations (7) and (15) are decreasing and increasing, respectively, which adapts the theoretical investigations of the graphical method of them. Furthermore, the intersection of both curves in the two plots of all the mentioned figures gives the ML estimators of the shape parameter β for the UW and UB, respectively, which is confirmed by the profiles of the log-likelihood in Figures 1, 3, and 5 and also by their numerical ML estimates values.

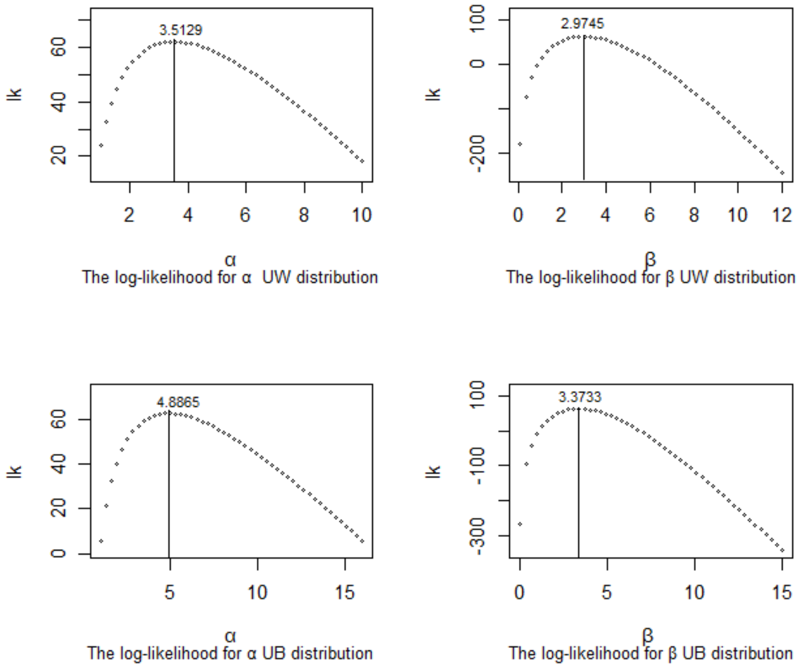

Figure 5. A fitted profile of the log-likelihood function obtained from the UW and UB distributions for the ML estimators based on Example 3.

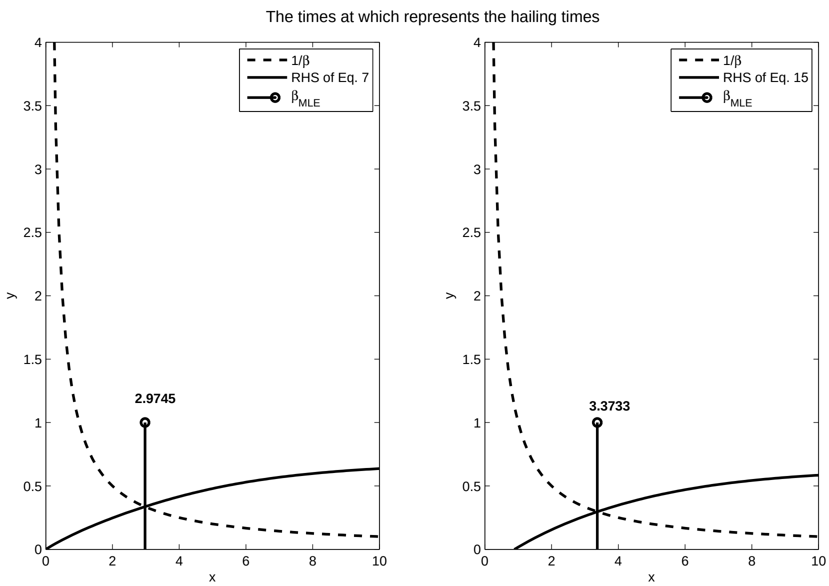

Figure 6. RHS of Equation (7) vs. $\frac{1}{\beta}$ plot for UW and RHS of Equation (15) vs. $\frac{1}{\beta}$ plot for UB, in Example 3.

6. Conclusions

In this study, the uniqueness and existence of ML estimators for the UB and UW distributions were investigated by analytical and graphical approaches. By employing those approaches, reliance on the Newton–Raphson method for obtaining ML estimations is feasible. Balakrishnan and Kateri [13] demonstrated the uniqueness and existence of ML estimations for the Weibull distribution using the graphical method by utilizing the Cauchy–Schwartz inequality. However, this method has not been previously studied for the UW and UB distributions. Here, theoretical and applied studies have evidenced the uniqueness and existence of ML estimations for the UW and UB distributions, with parameter estimations obtained through graphical methods shown to be unique across three real-life datasets. Furthermore, to verify the suitability of the log-likelihood function and confirm a precise optimum, corresponding profiles of the log-likelihood function for ML estimation methods for the UW and UB distributions were depicted graphically. Lastly, a detailed simulation study was conducted for the proposed graphical method and traditional ML methods for the UB and UW distributions. In this study, the iteration times of the methods are provided. According to these results, the graphical method achieves similar outcomes in a shorter period compared to traditional methods. In a future study, we will investigate the efficiency of this graphical method to some recent bounded distributions, including Kies families [23–25].

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Data Availability Statement: The data that were used are listed with their citations in the study's application section.

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Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Unit Weibull: Graphical Method

```
library(boot)
library(rootSolve)
n=50
x=NULL
alpha=0.4
beta=2
fullb=fullm=fullabb=fullmre=NULL
betamle=alphamle=NULL
ds=5000
baslangic <- Sys.time()
for (j in 1:ds)
cat("\14",j)
dev=F
while(dev==F) {
for(i in 1:n) {
u=runif(1)
x[i]=1/exp(exp(log(-log(u)/alpha)/beta)) }
modelm<-function(par) c(F1=((1/par[1])-((1/(sum((-log(x))^par[1])))*
sum(((-(log(x)))^
par[1])*log(-log(x)))-(1/n)*sum(log(-log(x))))))
ssm=try(multiroot(f = modelm, start =c(0.05)))
if (!is.character(ssm)) if (ssm$root[1]>0)dev=T
} betamle[j]=ssm$root[1] alphamle[j]=n/(sum(((-log(x))^ssm$root[1]))) } bitis<- Sys.time()
fark <- difftime(bitis, baslangic, units = "secs")
print(fark)
b1=mean(betamle)
b2=mean(alphamle)
m1=c(mean((betamle-beta)^2))
m2=c(mean((alphamle-alpha)^2))
abb1=c(mean(abs(betamle-beta)))
```

```
abb2=c(mean(abs(alphamle-alpha)))
mre1=c(mean(abs(betamle-beta)/beta))
mre2=c(mean(abs(alphamle-alpha)/alpha))
fullb= rbind(fullb,n,b1,b2)
fullm= rbind(fullm,n,m1,m2)
fullabb= rbind(fullabb,n,abb1,abb2)
fullmre= rbind(fullmre,n,mre1,mre2)
print(fark)
fullb
fullm
fullabb
fullmre
```

"Maximum Likelihood Method"

```
alphamle=betamle=NULL
fullb=fullm=fullabb=fullmre=NULL
loglk=function(par){
t=0
beta=par[1]
alpha=par[2]
f=(1/x)*alpha*beta*(-log(x))^(beta-1)*exp(-alpha*(-
log(x))^beta)
t=t+sum(log(f))
return(-t)
}
for (j in 1:ds) { cat("\14",j) dev=F
while(dev==F) { for(i in 1:n) {
u=runif(1)
x[i]=1/exp(exp(log(-log(u)/alpha)/beta))
} aa=try(optim(c(beta,alpha),loglk, method = "CG", hessian = T),silent = T)
if (!is.character(aa))if (aa$par[1] >0& aa$par[2] >0)
dev=T
}
betamle[j]=aa$par[1]
alphamle[j]=aa$par[2]
} bitis <- Sys.time()
fark <- difftime(bitis, baslangic, units = "secs")
print(fark)
b1=mean(betamle)
b2=mean(alphamle)
m1=c(mean((betamle-beta)^2))
m2=c(mean((alphamle-alpha)^2))
abb1=c(mean(abs(betamle-beta)))
abb2=c(mean(abs(alphamle-alpha)))
mre1=c(mean(abs(betamle-beta)/beta))
mre2=c(mean(abs(alphamle-alpha)/alpha))
fullb= rbind(fullb,n,b1,b2)
fullm= rbind(fullm,n,m1,m2)
fullabb= rbind(fullabb,n,abb1,abb2)
fullmre= rbind(fullmre,n,mre1,mre2)
print(fark)
fullb
fullm)
fullabb
fullmre
```

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