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Common Fixed-Point Theorem and Projection Method on a Hadamard Space

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Abstract: In this paper, we obtain an equivalent condition to the existence of a common fixed point of a given family of nonexpansive mappings defined on a Hadamard space. Moreover, if the space is bounded, we show that the generating process of the approximate sequence by a specific projection method will stop in finite steps if there is no common fixed point. It is a significant advantage to reveal the nonexistence of a common fixed point in a finite time.

Keywords: Hadamard space; nonexpansive mapping; common fixed point

MSC: 47H10

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1. Introduction

The study of fixed points of mappings on complete metric spaces is a central topic in nonlinear analysis, and it has been considered from various perspectives. Many researchers have investigated the existence of fixed points of nonlinear mappings and their approximation techniques. They adopt a subset of Hilbert and Banach spaces as the domain of mappings. One of the most important results is Kirk's fixed-point theorem for a nonexpansive mapping defined on a nonempty bounded closed convex subset of a reflexive Banach space having the normal structure [1]. On the other hand, approximation schemes of fixed points have also been actively studied. A nonlinear ergodic theorem by Baillon [2] can be regarded as an approximation scheme of a fixed point of nonexpansive mapping. The convergence of Mann's type [3] iterative scheme to a fixed point of a nonexpansive mapping was proved by Reich [4]. Wittmann [5] proved a strong convergence theorem of a Halpern's type [6] of iterative sequence in Hilbert spaces, and it was generalized to Banach spaces by Shioji and Takahashi [7].

In 2004, Kirk [8] proved the following remarkable theorem, a milestone in the history of fixed-point theory on geodesic spaces.

Theorem 1 (Kirk [8])**.** *Let X be a Hadamard space and U a bounded open subset of X. Let $T\colon \operatorname{cl} U \to X$ be a nonexpansive mapping. Suppose that there exists $p \in U$ such that $x \notin [p, Tx] \setminus \{Tx\}$ for every boundary point x of U. Then T has a fixed point.*

This result is a fixed-point theorem for nonexpansive mapping on a complete geodesic space. After this work, many researchers have studied fixed-point theory in geodesic spaces. In particular, the techniques to approximate a fixed point of given nonexpansive or other types of mappings have been investigated, and they obtained many valuable results. Saejung [9] got the convergence theorem of the iterative sequence generated by the Halpern scheme to the fixed point closest to a given anchor point. He, Fang, López, and Li [10] showed a Δ-convergence theorem of the Mann type iterative sequence.

The shrinking projection method was first proposed by Takahashi, Takeuchi, and Kubota [11]. There are many variations in projection methods, and it is one of the most critical schemes among them. For the recent works, see [12,13], for instance. This method has also been studied in the setting of complete geodesic space, and several convergence theorems were proved. Moreover, the following result was recently proved: a modified version of the shrinking projection method in a Hadamard space.

Theorem 2 (Kimura [14])**.** *Let X be a Hadamard space and suppose that a subset $\{z \in X \mid d(u,z) \le d(v,z)\}$ is convex for any $u, v \in X$. Let $T\colon X \to X$ a nonexpansive mapping with $\operatorname{Fix} T \ne \emptyset$. Generate a sequence $\{x_n\} \subset X$ as follows: $x_1 \in X$ is given, $C_1 = X$, and*

$$C_{n+1} = \{z \in X \mid d(Tx_n, z) \le d(x_n, z)\} \cap C_n,$$
$$x_{n+1} = P_{C_{n+1}} x_n$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ is Δ-convergent to $x_0 \in \operatorname{Fix} T$.

We will focus on this method in this study.

In a setting such as Banach or Hilbert spaces, some of approximate sequences mentioned above can be used to characterize the existence of a fixed point of a given mapping. In particular, the boundedness of a generated sequence often guarantees the existence of a fixed point; see [15] and references therein.

However, in a practical calculation, it is challenging to show the boundedness of the sequence because we need to calculate infinitely many points to confirm it.

In this paper, we obtain an equivalent condition to the existence of a common fixed point of a family of nonexpansive mappings defined on a Hadamard space by generating an approximate sequence with an iterative process. Moreover, under the assumption that the space is bounded, we show that the generating process of the sequence will stop in finite steps if there is no common fixed point. We emphasize that judging the nonexistence of fixed points in a finite time is a significant advantage. As an application of our results, we also consider a convex minimization problem for a family of convex functions. The results characterize the existence of a common minimizer. We also consider the minimization problem on a given convex subset of the domain of the function.

2. Preliminaries

Let (X,d) be a metric space. We say $\gamma_{xy}\colon [0,1] \to X$ is a geodesic between $x, y \in X$ if $\gamma_{xy}(0) = x$, $\gamma_{xy}(1) = y$, and $d(\gamma_{xy}(s), \gamma_{xy}(t)) = |s-t| d(x,y)$ for any $s, t \in [0,1]$. If a geodesic γ_{xy} exists for any $x, y \in X$, then X is called a geodesic space. In particular, X is said to be uniquely geodesic if for any $x, y \in X$, a geodesic between them exists uniquely. In this case, the image of the geodesic γ_{xy} is denoted by $[x,y]$. In a uniquely geodesic space X, the convex combination between two points is naturally defined; for $x, y \in X$ and $t \in [0,1]$, we define

$$tx \oplus (1-t)y = \gamma_{xy}(1-t).$$

Using this notion, we can define the convexity of a subset of X; we say $C \subset X$ is convex if $tx \oplus (1-t)y \in C$ for any $x, y \in C$ and $t \in [0,1]$.

We usually define a $\mathrm{CAT}(0)$ space by using notions of geodesic triangles and comparison triangles on a model space. In this paper, we use the following definition which is equivalent to the original one. A uniquely geodesic space X is called $\mathrm{CAT}(0)$ space if for any $x, y, z \in X$ and $t \in [0,1]$, the inequality

$$d(tx \oplus (1-t)y, z)^2 \le t d(x,z)^2 + (1-t) d(y,z)^2 - t(1-t) d(x,y)^2$$

holds. For the formal definition, see [16,17] for instance.

A Hadamard space is defined as a complete CAT(0) space. This space includes some essential classes of sets for studying nonlinear mappings and their fixed points, such as closed convex subsets of a Hilbert space, real Hilbert balls, $\mathbb{R}$-trees, and others. Notice that a closed convex subset of a Banach space is not necessarily a Hadamard space.

Let X be a metric space. We say $x \in X$ is a fixed point of a mapping $T\colon X \to X$ if it satisfies $x = Tx$. The set of all fixed points of T is denoted by $\operatorname{Fix} T$, that is,

$$\operatorname{Fix} T = \{x \in X \mid x = Tx\}.$$

A mapping $T\colon X \to X$ is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in X$. It is easy to see that $\operatorname{Fix} T$ is always closed and convex if X is a CAT(0) space.

Let X be a Hadamard space and $C \subset X$ be a nonempty closed convex subset of X. It is known that, for $x \in X$, there exists a unique $y_x \in C$ which is closest to x in C, that is,

$$d(x, y_x) = \inf_{y \in C} d(x, y).$$

Using this point, we define the metric projection $P_C\colon X \to C$ by $P_C x = y_x$. We also know that P_C is a nonexpansive mapping with $\operatorname{Fix} P_C = C$.

For a bounded sequence $\{x_n\} \subset X$, we call z an asymptotic center of $\{x_n\}$ if

$$\limsup_{n\to\infty} d(x_n, z) = \inf_{y \in X} \limsup_{n\to\infty} d(x_n, y).$$

It is known that the asymptotic center of every bounded sequence in a Hadamard space is unique and it belongs to the closed convex hull of $\{x_n\}$.

A bounded sequence $\{x_n\}$ is said to be Δ-convergent to $x_0 \in X$ if every subsequence of $\{x_n\}$ has an identical asymptotic center x_0. In a Hadamard space, we know that every bounded sequence has a Δ-convergent subsequence [18].

For more details of Hadamard spaces and related notions, see [17].

Let X be a Hadamard space and $f : X \to \,]-\infty, \infty]$. We say f is proper if $f(x_0) < \infty$ for some $x_0 \in X$. f is said to be lower semicontinuous if

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n)$$

whenever $\{x_n\} \subset X$ converges to $x_0 \in X$. f is said to be convex if

$$f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y)$$

for any $x, y \in X$ and $t \in \,]0,1[$.

A point $x_0 \in X$ is a minimizer of f if x_0 satisfies

$$f(x_0) = \inf_{x \in X} f(x).$$

The set of all minimizers of f on a subset $D \subset X$ is denoted by $\operatorname{argmin}_D f$.

3. Common Fixed Point Theorem

We consider the conditions equivalent to the existence of a common fixed point of a family of nonexpansive mappings in a Hadamard space. We begin with the following simple lemma.

Lemma 1. *Let X be a Hadamard space and let $\{C_n\}$ be a sequence of nonempty closed convex subsets of X which is decreasing with respect to inclusion, that is, $C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$. Let $\{y_n\} \subset X$ be a sequence such that $y_n \in C_n$ for every $n \in \mathbb{N}$. If $\{y_n\}$ is bounded, then its asymptotic center belongs to $\bigcap_{n=1}^{\infty} C_n$.*

Proof. Suppose that $\{y_n\}$ is bounded, and let y_0 be a unique asymptotic center of $\{y_n\}$. Fix $k \in \mathbb{N}$ arbitrarily. Letting $w_n = y_{n+k}$ for $n \in \mathbb{N}$, we have a sequence $\{w_n\}$ has the same asymptotic center y_0 as $\{y_n\}$. From the property of $\{C_n\}$, it follows that $\{w_n\} \subset C_k$. Since C_k is closed and convex, we have $y_0 \in C_k$. Since $k \in \mathbb{N}$ is arbitrary, we obtain $y_0 \in \bigcap_{k=1}^{\infty} C_k$, the desired result. □

The following main result shows that we can characterize the existence of a common fixed point of given nonexpansive mappings by using the generating procedure of its approximate sequence.

Theorem 3. *Let X be a Hadamard space and suppose that a subset $\{z \in X \mid d(u,z) \leq d(v,z)\}$ of X is convex for any $u, v \in X$. Let $\{T_i \colon X \to X \mid i = 1, 2, \ldots, m\}$ be a family of nonexpansive mappings. Generate a sequence $\{x_n\}$ in X with a sequence $\{C_n\}$ of subsets of X by the following steps:*

Step 0. $x_1 \in X$, $C_1 = X$, *and* $n = 1$;

Step 1. $C_{n+1} = \bigcap_{i=1}^{m} \{z \in X \mid d(T_i x_n, z) \leq d(x_n, z)\} \cap C_n$;

Step 2. (1) *if* $C_{n+1} \neq \varnothing$, *then let* $x_{n+1} = P_{C_{n+1}} x_n$, *increment* n *to* 1, *and go to Step 1;*

(2) *if* $C_{n+1} = \varnothing$, *then* $C_k = \varnothing$ *and leave* x_k *to be undefined for all* $k \geq n+1$, *and terminate the generating process.*

Then, the following conditions are equivalent:

(a) $\bigcap_{i=1}^{m} \operatorname{Fix} T_i \neq \varnothing$;

(b) $\bigcap_{k=1}^{\infty} C_k \neq \varnothing$.

Further, in this case, $\{x_n\}$ is well defined and Δ-convergent to some $x_0 \in \bigcap_{i=1}^{m} \operatorname{Fix} T_i$.

Proof. First we suppose $\bigcap_{k=1}^{\infty} C_k \neq \varnothing$ and show $\bigcap_{i=1}^{m} \operatorname{Fix} T_i \neq \varnothing$. Since $C_n \neq \varnothing$ for all $n \in \mathbb{N}$, the sequence $\{x_n\}$ is well defined. Let $p \in \bigcap_{k=1}^{\infty} C_k$. Then, since a metric projection is nonexpansive and $p \in C_{n+1} = \operatorname{Fix} P_{C_{n+1}}$, we have

$$d(x_{n+1}, p) = d(P_{C_{n+1}} x_n, p) \leq d(x_n, p)$$

for all $n \in \mathbb{N}$. It follows that a real sequence $\{d(x_n, p)\}$ is convergent to some non-negative number $c_p \in \mathbb{R}$, and that $\{x_n\}$ is bounded. Let $t \in \,]0, 1[$. Since $t x_{n+1} \oplus (1-t)p \in C_{n+1}$, we have

$$\begin{aligned} d(x_{n+1}, x_n)^2 &= d(P_{C_{n+1}} x_n, x_n)^2 \\ &\leq d(t x_{n+1} \oplus (1-t)p, x_n)^2 \\ &\leq t d(x_{n+1}, x_n)^2 + (1-t) d(p, x_n)^2 - t(1-t) d(x_{n+1}, p)^2, \end{aligned}$$

which implies that

$$d(x_{n+1}, x_n)^2 \leq d(x_n, p)^2 - t d(x_{n+1}, p)^2.$$

Letting $t \to 1$, we have $d(x_{n+1}, x_n)^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2$. Further, we obtain

$$0 \leq d(x_{n+1}, x_n)^2 \leq d(x_n, p)^2 - d(x_{n+1}, p)^2 \to c_p^2 - c_p^2 = 0.$$

as $n \to \infty$. Thus we have $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Since $x_{n+1} \in C_{n+1}$, from the definition of C_n, we have

$$d(T_i x_n, x_{n+1}) \leq d(x_n, x_{n+1})$$

for $i = 1, 2, \ldots, m$. It implies

$$0 \leq d(x_n, T_i x_n) \leq d(x_n, x_{n+1}) + d(T_i x_n, x_{n+1}) \leq 2d(x_n, x_{n+1}) \to 0$$

and thus $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for every $i = 1, 2, \ldots, m$.

On the other hand, since $\{x_n\}$ is bounded, its asymptotic center is a unique point $x_0 \in X$. For each $i = 1, 2, \ldots, m$, we have

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, T_i x_0) &\le \limsup_{n\to\infty}(d(x_n, T_i x_n) + d(T_i x_n, T_i x_0)) \\ &\le \limsup_{n\to\infty}(d(x_n, T_i x_n) + d(x_n, x_0)) \\ &\le \lim_{n\to\infty} d(x_n, T_i x_n) + \limsup_{n\to\infty} d(x_n, x_0) \\ &\le \limsup_{n\to\infty} d(x_n, x_0).\end{aligned}$$

By the uniqueness of the asymptotic center of $\{x_n\}$, we have $T_i x_0 = x_0$ for every $i = 1, 2, \ldots, m$, and hence $x_0 \in \bigcap_{i=1}^m \operatorname{Fix} T_i \neq \varnothing$.

Next, we suppose that $\bigcap_{i=1}^m \operatorname{Fix} T_i \neq \varnothing$ and prove $\bigcap_{k=1}^\infty C_k \neq \varnothing$. It is sufficient to show that $\bigcap_{i=1}^m \operatorname{Fix} T_i \subset C_k$ for every $k \in \mathbb{N}$. We prove this inclusion by induction. It is obvious for the case $k = 1$. Suppose $\bigcap_{i=1}^m \operatorname{Fix} T_i \subset C_k$ and we consider the case $k+1$. Notice that, in this case, x_k is defined. Let $z \in \bigcap_{i=1}^m \operatorname{Fix} T_i$. Then, since each T_i is nonexpansive, we have

$$d(T_i x_k, z) \le d(x_k, z)$$

for each $i = 1, 2, \ldots, m$. This fact and the assumption of induction imply $z \in C_{k+1}$. Consequently, we obtain

$$\bigcap_{k=1}^\infty C_k \supset \bigcap_{i=1}^m \operatorname{Fix} T_i \neq \varnothing,$$

and this is the desired result.

We now prove the latter part of the theorem. From the argument above, we have obtained the following:

- $\{d(x_n, p)\}$ is convergent to $c_p \in [0, \infty[$ for each $p \in \bigcap_{k=1}^\infty C_k$;
- $\{x_n\}$ is bounded;
- the asymptotic center x_0 of $\{x_n\}$ belongs to $\bigcap_{i=1}^m \operatorname{Fix} T_i$.

Let $\{x_{n_j}\}$ be an arbitrary subsequence of $\{x_n\}$. Since $\{x_{n_j}\}$ is also bounded, there exists a unique asymptotic center $y_0 \in X$. We show that y_0 is identical to the asymptotic center x_0 of $\{x_n\}$. Since every C_n is a closed convex subset of X for $n \in \mathbb{N}$, and $\{C_n\}$ is a decreasing sequence with respect to inclusion, by Lemma 1, we have

$$y_0 \in \bigcap_{j=1}^\infty C_{n_j} = \bigcap_{n=1}^\infty C_n.$$

Therefore, a sequence $\{d(x_n, y_0)\}$ has a limit $c_{y_0} \in [0, \infty[$. It follows that

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, y_0) = c_{y_0} &= \lim_{n\to\infty} d(x_n, y_0) \\ &= \lim_{j\to\infty} d(x_{n_j}, y_0) \\ &\le \limsup_{j\to\infty} d(x_{n_j}, x_0) \le \limsup_{n\to\infty} d(x_n, x_0).\end{aligned}$$

This inequality shows that y_0 is an asymptotic center of $\{x_n\}$. From its uniqueness, we have $x_0 = y_0$. Hence $\{x_n\}$ is Δ-convergent to $x_0 \in \bigcap_{i=1}^m \operatorname{Fix} T_i$. $\square$

This result deals with a finite family of nonexpansive mappings, and we note that it can be generalized to the case of an arbitrary infinite family of mappings. We can change the proof for this case in a trivial way. However, in the view of practical calculations such as computer experiments, it is almost impossible to handle an infinite family of mappings.

4. The Case That the Underlying Space is Bounded

In this section, we consider the case where the underlying space X is bounded. Notice that we do not assume the boundedness of X in Theorem 3. Thus, in the procedure in the theorem, $\bigcap_{k=1}^{\infty} C_k$ might be empty even if every C_k is nonempty, as in the following example.

Example 1. *Consider the graph of the function $f\colon]0,\infty[\ \to \mathbb{R}$ defined by $f(x) = \log x$ for $x \in]0,\infty[$. Then, the tangent line of the curve at the point $p = (p_1, \log p_1)$ on the graph intersects with the y-axis at $u = (0, -1 + \log p_1)$, and the normal line at p intersects with the y-axis at $v = (0, p_1^2 + \log p_1)$. The midpoint of u and v is $w = (0, (p_1^2 - 1)/2 + \log p_1)$; see Figure 1.*

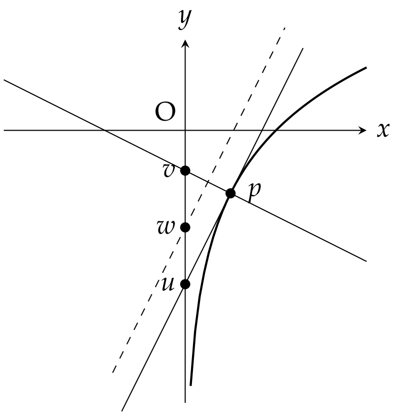

Figure 1. The graph of f.

Using this fact, we consider the following procedure. Let T_1 and T_2 be the metric projections onto

$$\begin{aligned} D_1 &= \{z = (z_1, z_2) \in \mathbb{R}^2 \mid z_1 > 0,\ z_2 \leq \log z_1\}, \\ D_2 &= \{z = (z_1, z_2) \in \mathbb{R}^2 \mid z_1 < 0,\ z_2 \leq \log(-z_1)\}, \end{aligned}$$

respectively. If the initial point x_1 lies on the y-axis and we generate the sequence $\{x_n\}$ by the scheme in Theorem 3, then, by symmetry, every x_n will be on y-axis if it is defined. Now, we assume that $x_1, x_2, \ldots, x_n$ are defined and lie on the y-axis with descent order; x_{k+1} lies below x_k for $k = 1, 2, \ldots, n-1$. Let

$$p = T_1 x_n = P_{D_1} x_n = (p_1, \log p_1).$$

Then, by symmetry, we have

$$T_2 x_n = P_{D_2} x_n = (p_1, \log(-p_1)).$$

Further, from the calculations above, x_n can be expressed by $x_n = (0, p_1^2 + \log x)$. Since the points $x_1, x_2, \ldots, x_{n-1}$ lie above x_n, by the simple calculation, we obtain

$$\begin{aligned} C_{n+1} &= \{z \in \mathbb{R}^2 \mid \|P_{D_1} x_n - z\| \leq \|x_n - z\|\} \\ &\quad \cap \{z \in \mathbb{R}^2 \mid \|P_{D_2} x_n - z\| \leq \|x_n - z\|\} \\ &= \left\{ z = (z_1, z_2) \in \mathbb{R}^2 \;\middle|\; z_2 \leq \frac{1}{p_1} z_1 + \frac{p_1^2 - 1}{2} + \log p_1 \right\} \\ &\quad \cap \left\{ z = (z_1, z_2) \in \mathbb{R}^2 \;\middle|\; z_2 \leq -\frac{1}{p_1} z_1 + \frac{p_1^2 - 1}{2} + \log p_1 \right\} \\ &= \left\{ z = (z_1, z_2) \in \mathbb{R}^2 \;\middle|\; z_2 \leq -\frac{1}{p_1} |z_1| + \frac{p_1^2 - 1}{2} + \log p_1 \right\}. \end{aligned}$$

This set forms a cone with the apex at $(0, (p_1^2 - 1)/2 + \log p_1)$*, and therefore we have* $x_{n+1} = (0, (p_1^2 - 1)/2 + \log p_1)$*; see Figure 2. Thus* x_{n+1} *is on the y-axis again, and we also have*

$$\|x_n - x_{n+1}\| = \frac{p_1^2 + 1}{2} > \frac{1}{2}.$$

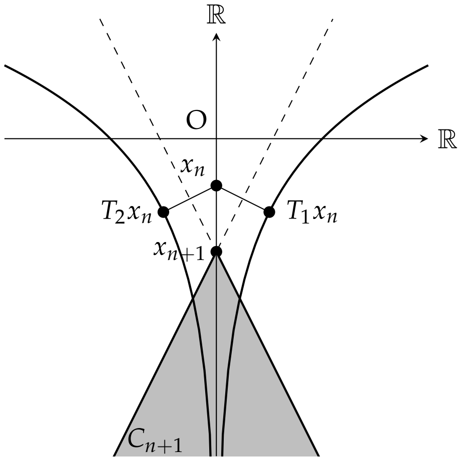

Figure 2. Generating x_{n+1} from x_n.

From these facts, the sequence $\{x_n\}$ *generated by this procedure with the initial point* $x_1 = (0, 0)$ *has the following properties:*

- $C_n \neq \varnothing$ *for every* $n \in \mathbb{N}$*;*
- $\|x_n - x_{n+1}\| > 1/2$ *for every* $n \in \mathbb{N}$*.*

Since every x_n *is defined and lies on the y-axis with descent order, the second property above implies that*

$$\|x_n\| = \sum_{k=1}^{n-1} \|x_{k+1} - x_k\| > \frac{n-1}{2} \to \infty$$

as $n \to \infty$*. Thus we have*

$$\bigcap_{k=1}^{\infty} C_k = \varnothing.$$

Suppose that the underlying space X is bounded. In this case, Kirk's fixed-point theorem guarantees that each nonexpansive mapping T_i has a fixed point. However, we do not know whether a finite family $\{T_i\}$ of mapping has a common fixed point or not.

The following result shows that we can obtain the non-existence of a common fixed point of $\{S_i\}$ within a finite repeating time.

Theorem 4. *Let X be a bounded Hadamard space, and suppose that a subset* $\{z \in X \mid d(u, z) \leq d(v, z)\}$ *of X is convex for any* $u, v \in X$*. Let* $\{T_i \colon X \to X \mid i = 1, 2, \dots, m\}$ *be a family of nonexpansive mappings, and let* $\{x_n\}$ *be a sequence generated by the process in Theorem 3. Then, the following hold:*

(i) *If* $\bigcap_{i=1}^m \operatorname{Fix} T_i \neq \varnothing$*, then* $\{x_n\}$ *is* Δ*-convergent to* $x_0 \in \bigcap_{i=1}^m \operatorname{Fix} T_i$*;*
(ii) *if* $\bigcap_{i=1}^m \operatorname{Fix} T_i = \varnothing$*, then there exists* $n_0 \in \mathbb{N}$ *such that* $C_{n_0} = \varnothing$*.*

Proof. (i) is a direct result of Theorem 3. For (ii), we show its contrapositive; we suppose that C_n is nonempty for all $n \in \mathbb{N}$ and obtain $\bigcap_{i=1}^m \operatorname{Fix} T_i \neq \varnothing$. Take a sequence $\{y_n\} \subset X$ such that $y_n \in C_n$ for all $n \in \mathbb{N}$. Since $\{y_n\}$ is bounded, it follows from Lemma 1 that its unique asymptotic center belongs to $\bigcap_{n=1}^\infty C_n$. Thus $\bigcap_{n=1}^\infty C_n$ is nonempty, and hence $\bigcap_{i=1}^m \operatorname{Fix} T_i$ is also nonempty by Theorem 3. This is the desired result. □

5. Applications to a Convex Minimization Problem

In this section, we attempt to apply the results discussed in the previous sections to the problem of finding a common minimizer of a family of convex functions.

Let $f\colon X \to]-\infty, \infty]$ be a proper lower semicontinuous convex function defined on a Hadamard space X. Then, for each $x \in X$, there exists unique $y_x \in X$ such that

$$f(y_x) + \frac{1}{2}d(x, y_x)^2 = \inf_{y \in X}\left(f(y) + \frac{1}{2}d(x, y)^2\right)$$

Using this point, we define the resolvent $R_f\colon X \to X$ of f by $R_f x = y_x$. Namely, $R_f x \in X$ is a unique minimizer of the function $g(y) = f(y) + (1/2)d(x, y)^2$. It is known [19] that R_f satisfies the following inequality:

$$2d(R_f x, R_f y)^2 + d(R_f x, x)^2 + d(R_f y, y)^2 \le d(R_f x, y)^2 + d(R_f y, x)^2$$

for $x, y \in X$. Moreover, since the inequality

$$(p, s)^2 + d(q, r)^2 - d(p, r)^2 - d(q, s)^2 \le 2d(p, q)d(r, s)$$

holds for all $p, q, r, s \in X$, we have

$$\begin{aligned} 2d(R_f x, R_f y)^2 &\le d(R_f x, y)^2 + d(R_y, x)^2 - d(R_f x, x)^2 - d(R_f y, y)^2 \\ &\le 2d(R_f x, R_f y)d(x, y) \end{aligned}$$

for $x, y \in X$, and thus R_f is nonexpansive. See also [16,20,21].

The resolvent operator has the following important property: the set of minimizers of f is identical to the set of fixed points of R_f. From this fact, we can apply our results to find a common minimizer of a given family of convex functions.

Theorem 5. *Let X be a Hadamard space and suppose that a subset $\{z \in X \mid d(u, z) \le d(v, z)\}$ of X is convex for any $u, v \in X$. Let $\{f_i\colon X \to]-\infty, \infty] \mid i = 1, 2, \dots, m\}$ be a family of proper, lower semicontinuous convex functions on X, and $R_{f_i}\colon X \to X$ be the resolvent operator of f_i for $i = 1, 2, \dots, m$. Generate a sequence $\{x_n\}$ in X with a sequence $\{C_n\}$ of subsets of X by the following steps:*

Step 0. $x_1 \in X$, $C_1 = X$, *and* $n = 1$;

Step 1. $C_{n+1} = \bigcap_{i=1}^{m}\{z \in X \mid d(R_{f_i} x_n, z) \le d(x_n, z)\} \cap C_n$;

Step 2. (1) *if* $C_{n+1} \ne \varnothing$, *then let* $x_{n+1} = P_{C_{n+1}} x_n$, *increment* n *to* 1, *and go to Step 1;*

(2) *if* $C_{n+1} = \varnothing$, *then* $C_k = \varnothing$ *and leave* x_k *to be undefined for all* $k \ge n + 1$, *and terminate the generating process.*

Then, the following conditions are equivalent:

(a) $\bigcap_{i=1}^{m} \operatorname{argmin}_X f_i \ne \varnothing$;

(b) $\bigcap_{k=1}^{\infty} C_k \ne \varnothing$.

Further, in this case, $\{x_n\}$ is well defined and Δ-convergent to some $x_0 \in \bigcap_{i=1}^{m} \operatorname{argmin}_X f_i$.

Proof. From the properties of a resolvent operator, we have $\operatorname{argmin}_X f_i = \operatorname{Fix} R_{f_i}$ for every $i = 1, 2, \dots, m$. Therefore, the condition (a) is equivalent to

$$\bigcap_{i=1}^{m} \operatorname{Fix} R_{f_i} \ne \varnothing.$$

Applying Theorem 3 with this fact, we have the condition (a) if and only if (b), which is the desired result. The latter part of the theorem is also deduced directly. □

Next, we consider the problem of finding a minimizer of a single function f on a given closed convex subset D. If f minimizes at some point x_0 in D, then it is a solution to the common fixed-point problem of the resolvent operator R_f of f and the metric projection P_D onto D. On the other hand, if every minimizer of f does not belong to D, then a minimizing point of f on D must be another point that belongs to D. In this case, we have

$$\operatorname{Fix} R_f \cap \operatorname{Fix} P_D = \operatorname*{argmin}_X f \cap D = \varnothing.$$

From this observation, we obtain the following result related to the convex minimization problem on a given convex set.

Theorem 6. *Let X be a Hadamard space and suppose that a subset $\{z \in X \mid d(u,z) \leq d(v,z)\}$ of X is convex for any $u, v \in X$. Let $f\colon X \to]-\infty, \infty]$ be a proper, lower semicontinuous convex function on X, and $R_f\colon X \to X$ be the resolvent operator of f. Let D be a nonempty closed convex subsets of X. Generate a sequence $\{x_n\}$ in X with a sequence $\{C_n\}$ of subsets of X by the following steps:*

Step 0. $x_1 \in X$, $C_1 = X$, *and* $n = 1$;

Step 1. $C_{n+1} = \{z \in X \mid d(R_f x_n, z) \leq d(x_n, z)\} \cap \{z \in X \mid d(P_D x_n, z) \leq d(x_n, z)\} \cap C_n$;

Step 2. (1) *if $C_{n+1} \neq \varnothing$, then let $x_{n+1} = P_{C_{n+1}} x_n$, increment n to 1, and go to Step 1;*

(2) *if $C_{n+1} = \varnothing$, then $C_k = \varnothing$ and leave x_k to be undefined for all $k \geq n+1$, and terminate the generating process.*

Then, the following conditions are equivalent:

(a) *f minimizes at a point in D;*

(b) $\bigcap_{k=1}^{\infty} C_k \neq \varnothing$.

Further, in this case, the sequence $\{x_n\}$ is well defined and Δ-convergent to some $x_0 \in \operatorname{argmin}_X f \cap D$.

Furthermore, if we assume that X is bounded, then D is also bounded and thus $f|_D\colon D \to]-\infty, \infty]$ always has a minimizer, that is, $\operatorname{argmin}_D f \neq \varnothing$. We can use Theorem 4 to check whether this minimizer is also a global minimizer of f.

Theorem 7. *Let X be a bounded Hadamard space, and suppose that a subset $\{z \in X \mid d(u,z) \leq d(v,z)\}$ of X is convex for any $u, v \in X$. Let $f\colon X \to]-\infty, \infty]$ be a proper lower semicontinuous convex function on X, and let $\{x_n\}$ be a sequence generated by the process in Theorem 5. Then, the following hold:*

(i) *If f minimizes at some point in D, then $\{x_n\}$ is Δ-convergent to $x_0 \in \operatorname{argmin}_X f \cap D$;*

(ii) *If f does not minimize at any point in D, then there exists $n_0 \in \mathbb{N}$ such that $C_{n_0} = \varnothing$.*

6. Conclusions

In this paper, we study the existence of a common fixed point of a family of nonexpansive mapping defined on a Hadamard space. Using an iterative scheme by a projection method, we obtained an equivalent condition to the existence of a common fixed point.

The sequence $\{C_n\}$ of subsets in Theorem 3 may have an empty intersection even if each C_n is not empty. We showed this fact in Example 1. However, if the whole space is bounded, then $\bigcap_{n=1}^{\infty} C_n = \varnothing$ implies that some C_{n_0} is empty. This fact tells us that the generating process of the iterative sequence will stop in finite steps if there is no common fixed point. Therefore, the advantage of this method is that we can find the emptiness of a common fixed point of mappings can be revealed in a finite time.

These results can be applied to convex minimization problems. We discussed this problem and obtained several results deduced from our main results.

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