The Relationship between L–Moments and LH–Moments

Relations for LH–moments [1,6,7]:

$$L_1^\eta = E\left[X_{(\eta+1):(\eta+1)} \right]$$

$$L_2^\eta = \frac{1}{2} \cdot E\left[X_{(\eta+2):(\eta+2)} - X_{(\eta+1):(\eta+2)} \right]$$

$$L_3^\eta = \frac{1}{3} \cdot E\left[X_{(\eta+3):(\eta+3)} - 2 \cdot X_{(\eta+2):(\eta+3)} + X_{(\eta+1):(\eta+3)} \right]$$

$$L_4^\eta = \frac{1}{4} \cdot E\left[X_{(\eta+4):(\eta+4)} - 3 \cdot X_{(\eta+3):(\eta+4)} + 3 \cdot X_{(\eta+2):(\eta+4)} - X_{(\eta+1):(\eta+4)} \right]$$

unde $\eta = 0,1,2,3,...$ For $\eta = 0$, the method is identical to the L–moments method.

Based on these LH–moments, the reports $\tau_{H2}, \tau_{H3}, \tau_{H4}$ are defined as follows:

$$\tau_{H2} = \frac{L_2^\eta}{L_1^\eta} - \text{represents LH–CV (coefficient of LH–variation)}$$

$$\tau_{H3} = \frac{L_3^\eta}{L_2^\eta} - \text{represents LH–Cs (LH–skewness)}$$

$$\tau_{H4} = \frac{L_4^\eta}{L_2^\eta} - \text{represents LH–Ck (LH–kurtosis)}$$

The relations between the high–order linear moments and the normalized weighted moments are the following [19,24]:

$$L_1^\eta = B_\eta$$

$$L_2^\eta = \frac{(\eta+2)}{2} \cdot \left(B_{\eta+1} - B_\eta \right)$$

$$L_3^\eta = \frac{(\eta+3)}{6} \cdot \left((\eta+4) \cdot B_{\eta+2} - 2 \cdot (\eta+3) \cdot B_{\eta+1} + (\eta+2) \cdot B_\eta \right)$$

$$L_4^\eta = \frac{(\eta+4)}{24} \cdot \begin{pmatrix} (\eta+6) \cdot (\eta+5) \cdot B_{\eta+3} - 3 \cdot (\eta+5) \cdot (\eta+4) \cdot B_{\eta+2} + \\ 3 \cdot (\eta+4) \cdot (\eta+3) \cdot B_{\eta+1} - (\eta+3) \cdot (\eta+2) \cdot B_\eta \end{pmatrix}$$

where $B_\eta = (\eta+1) \cdot b_\eta$ and b_η represent the natural estimators expressed by the L–moments method. For $\eta = 0$, the method become the L–moments method.

The frequency factors for the analyzed distributions.

Table S1 presents the frequency factors $K_p(p)$ for the LH−moments.

Table S1. Frequency factors.

Distr.	LH−moments (first order)	LH−moments (second order)
DG	$$\frac{\left((1-p)^{-\frac{1}{\gamma}}-1\right)^{-\frac{1}{\alpha}}-\dfrac{\Gamma\left(2\cdot\gamma+\frac{1}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma(2\cdot\gamma)}}{\frac{3}{2}\cdot\Gamma\left(1-\frac{1}{\alpha}\right)\cdot\left(\dfrac{\Gamma\left(3\cdot\gamma+\frac{1}{\alpha}\right)}{\Gamma(3\cdot\gamma)}-\dfrac{\Gamma\left(2\cdot\gamma+\frac{1}{\alpha}\right)}{\Gamma(2\cdot\gamma)}\right)}$$	$$\frac{\left((1-p)^{-\frac{1}{\gamma}}-1\right)^{-\frac{1}{\alpha}}-\dfrac{\Gamma\left(3\cdot\gamma+\frac{1}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma(3\cdot\gamma)}}{2\cdot\Gamma\left(1-\frac{1}{\alpha}\right)\cdot\left(\dfrac{\Gamma\left(4\cdot\gamma+\frac{1}{\alpha}\right)}{\Gamma(4\cdot\gamma)}-\dfrac{\Gamma\left(3\cdot\gamma+\frac{1}{\alpha}\right)}{\Gamma(3\cdot\gamma)}\right)}$$
PR	$$\frac{\left(p^{-\frac{1}{\alpha}}-1\right)^{\frac{1}{\alpha}}-\frac{1}{\alpha}\cdot\Gamma\left(\frac{1}{\alpha}\right)\cdot\left(\dfrac{2\cdot\Gamma\left(\alpha-\frac{1}{\alpha}\right)}{\Gamma(\alpha)}-\dfrac{\Gamma\left(2\cdot\alpha-\frac{1}{\alpha}\right)}{\Gamma(2\cdot\alpha)}\right)}{\frac{3}{2}\cdot\Gamma\left(\frac{1}{\alpha}\right)\cdot\left(\dfrac{\Gamma\left(3\cdot\alpha-\frac{1}{\alpha}\right)}{2\cdot\Gamma(3\cdot\alpha)}-\dfrac{\Gamma\left(2\cdot\alpha-\frac{1}{\alpha}\right)}{\Gamma(2\cdot\alpha)}+\dfrac{\Gamma\left(\alpha-\frac{1}{\alpha}\right)}{\Gamma(\alpha)}\right)}$$	$$\frac{\left(p^{-\frac{1}{\alpha}}-1\right)^{\frac{1}{\alpha}}-\frac{1}{\alpha}\cdot\Gamma\left(\frac{1}{\alpha}\right)\cdot\left(\dfrac{\Gamma\left(3\cdot\alpha-\frac{1}{\alpha}\right)}{\Gamma(3\cdot\alpha)}-\dfrac{3\cdot\Gamma\left(2\cdot\alpha-\frac{1}{\alpha}\right)}{\Gamma(2\cdot\alpha)}+\dfrac{3\cdot\Gamma\left(\alpha-\frac{1}{\alpha}\right)}{\Gamma(\alpha)}\right)}{2\cdot\Gamma\left(\frac{1}{\alpha}\right)\cdot\left(\dfrac{3\cdot\Gamma\left(3\cdot\alpha-\frac{1}{\alpha}\right)}{\Gamma(3\cdot\alpha)}-\dfrac{\Gamma\left(4\cdot\alpha-\frac{1}{\alpha}\right)}{\Gamma(4\cdot\alpha)}-\dfrac{3\cdot\Gamma\left(2\cdot\alpha-\frac{1}{\alpha}\right)}{\Gamma(2\cdot\alpha)}+\dfrac{\Gamma\left(\alpha-\frac{1}{\alpha}\right)}{\Gamma(\alpha)}\right)}$$
IPR	$$\frac{\left(\dfrac{(1-p)^{\frac{1}{\alpha}}}{1-(1-p)^{\frac{1}{\alpha}}}\right)^{\frac{1}{\alpha}}-\dfrac{\Gamma\left(2\cdot\alpha+\frac{1}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma(2\cdot\alpha)}}{\frac{3}{2}\cdot\Gamma\left(1-\frac{1}{\alpha}\right)\cdot\left(\dfrac{\Gamma\left(3\cdot\alpha+\frac{1}{\alpha}\right)}{\Gamma(3\cdot\alpha)}-\dfrac{\Gamma\left(2\cdot\alpha+\frac{1}{\alpha}\right)}{\Gamma(2\cdot\alpha)}\right)}$$	$$\frac{\left(\dfrac{(1-p)^{\frac{1}{\alpha}}}{1-(1-p)^{\frac{1}{\alpha}}}\right)^{\frac{1}{\alpha}}-\dfrac{\Gamma\left(3\cdot\alpha+\frac{1}{\alpha}\right)\cdot\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma(3\cdot\alpha)}}{2\cdot\Gamma\left(1-\frac{1}{\alpha}\right)\cdot\left(\dfrac{\Gamma\left(4\cdot\alpha+\frac{1}{\alpha}\right)}{\Gamma(4\cdot\alpha)}-\dfrac{\Gamma\left(3\cdot\alpha+\frac{1}{\alpha}\right)}{\Gamma(3\cdot\alpha)}\right)}$$
BR4	$$\frac{\left((1-p)^{-\frac{1}{\alpha}}-1\right)^{-\frac{1}{\beta}}-\dfrac{\Gamma\left(2\cdot\alpha+\frac{1}{\beta}\right)\cdot\Gamma\left(1-\frac{1}{\beta}\right)}{\Gamma(2\cdot\alpha)}}{\frac{3}{2}\cdot\Gamma\left(1-\frac{1}{\beta}\right)\cdot\left(\dfrac{\Gamma\left(3\cdot\alpha+\frac{1}{\alpha}\right)}{\Gamma(3\cdot\alpha)}-\dfrac{\Gamma\left(2\cdot\alpha+\frac{1}{\alpha}\right)}{\Gamma(2\cdot\alpha)}\right)}$$	$$\frac{\left((1-p)^{-\frac{1}{\alpha}}-1\right)^{-\frac{1}{\beta}}-\dfrac{\Gamma\left(3\cdot\alpha+\frac{1}{\beta}\right)\cdot\Gamma\left(1-\frac{1}{\beta}\right)}{\Gamma(3\cdot\alpha)}}{2\cdot\Gamma\left(1-\frac{1}{\beta}\right)\cdot\left(\dfrac{\Gamma\left(4\cdot\alpha+\frac{1}{\alpha}\right)}{\Gamma(4\cdot\alpha)}-\dfrac{\Gamma\left(3\cdot\alpha+\frac{1}{\alpha}\right)}{\Gamma(3\cdot\alpha)}\right)}$$

The PR frequency factor.

For the L−moments method, the frequency factor can be estimated by:

$$K_p(p) = a + b \cdot \tau_3 + c \cdot \tau_3^2 + d \cdot \tau_3^3 + e \cdot \tau_3^4 + f \cdot \tau_3^5 + g \cdot \tau_3^6 + h \cdot \tau_3^7 + i \cdot \tau_3^8$$

Table S2. The frequency factor of the PR distribution, for the L−moments method.

P [%]	a	b	C	d	e	f	g	h	i
0.01	5.40984	93.06172	−721.3554	5707.80044	−22196.09176	56584.03944	−84914.76584	73986.01395	−28547.80261
0.1	6.35512	4.3471	183.49546	−779.47739	2855.20823	−5627.45694	6812.43663	−4455.90433	999.73507
0.5	4.7418082	9.7917006	29.4658093	−47.9076904	140.6917135	−137.92078	–	–	–
1	3.9973248	11.6282905	−8.6795797	48.4005748	−42.2810608	−14.153624	–	–	–
2	3.5411782	8.0411906	−8.5389187	35.8048583	−52.1522156	12.2677171	–	–	–
3	3.2709699	5.7449394	−5.2761016	19.8176078	−36.5282421	11.9531622	–	–	–
5	2.8841511	3.3115647	−2.2754563	4.3196483	−16.4631117	7.2179531	–	–	–
10	2.2787657	0.5901421	0.5374674	−7.8957795	3.4838008	–	–	–	–
20	1.4675792	−0.4788507	−2.5441964	−0.4120804	0.9697476	–	–	–	–
40	0.4477403	−1.6055923	−1.184721	1.8904259	−0.5477169	–	–	–	–
50	0.0099648	−1.7918176	0.0401133	1.2706535	−0.5296266	–	–	–	–
80	−1.4899255	−0.5705551	2.2793932	−1.6433576	0.4248825	–	–	–	–
90	−2.319694	1.5167835	0.1218483	−0.3248695	–	–	–	–	–

For the first level LH−moments method, the frequency factor can be estimated with:

$$K_p(p) = a + b \cdot \tau_{H3} + c \cdot \tau_{H3}^2 + d \cdot \tau_{H3}^3 + e \cdot \tau_{H3}^4 + f \cdot \tau_{H3}^5 + g \cdot \tau_{H3}^6 + h \cdot \tau_{H3}^7$$

Table S3. The frequency factor of the PR distribution, for the first level LH−moments.

P [%]	a	b	C	d	e	f	g	h
0.01	328.89326	−5942.86337	45868.19161	−190568.58841	466725.05262	−673630.33506	536894.23988	−182067.75228
0.1	0.46983	82.57344	−451.78112	1790.32986	−3367.88915	3372.13361	−678.47745	−934.05164
0.5	3.9068	4.95138	48.26228	−128.59639	291.58468	−251.44745	–	–
1	2.61715	16.79599	−39.93594	109.51929	−95.6647	−8.21658	–	–
2	2.5208	11.0132	−29.29168	82.33333	−106.79278	33.04472	–	–
3	3.13098	−0.89391	17.74291	−26.05593	0.90863	–	–	–
5	2.60007	−1.06978	12.17813	−25.86213	8.87753	–	–	–
10	1.72362	−0.42895	1.0052	−10.11637	5.82681	–	–	–
20	0.75096	−0.72032	−4.30071	2.78366	–	–	–	–
40	−0.4558519	−2.5056937	1.6923683	0.0268363	–	–	–	–
50	−1.1139134	−2.0740464	2.7934691	−0.8275351	–	–	–	–
80	−3.7549821	3.8300751	−1.1023443	−0.1675685	–	–	–	–
90	−5.2954083	8.9500255	−7.0887011	2.2945484	–	–	–	–

The IPR frequency factor.

For the L−moments method, the frequency factor can be estimated using by:

$$K_p(p) = a + b \cdot \tau_3 + c \cdot \tau_3^2 + d \cdot \tau_3^3 + e \cdot \tau_3^4 + f \cdot \tau_3^5 + g \cdot \tau_3^6 + h \cdot \tau_3^7 + i \cdot \tau_3^8$$

Table S4. The frequency factor of the IPR distribution, for the L−moments method.

P [%]	a	b	C	d	e	f	g	h	i
0.01	−345.2689	6562.6442	−51661.2117	226645.6636	−598680.6533	986650.2494	−990573.1864	559435.1263	−138025.5546
0.1	−52.6697	1046.8422	−7817.9022	33304.0462	−84320.3313	132013.698	−124778.344	65284.0191	−14679.6286
0.5	2.9526714	21.225288	3.5005404	27.3987597	−31.0664423	27.9923751	−52.59999	−	−
1	3.5500577	9.3630888	36.0627588	−102.9663387	168.1820787	−151.1667814	36.19326	−	−
2	3.5123821	4.8775520	28.4044598	−95.8057252	151.0283175	−133.6328173	40.73529	−	−
3	2.6786283	12.1156060	−25.2840454	43.1213241	−46.7740367	13.2066034	−	−	−
5	2.6171426	7.2621147	−18.2335705	28.341378	−31.6462423	10.6978611	−	−	−
10	2.3013644	1.8122701	−8.5373716	11.3635705	−13.1343828	5.21305	−	−	−
20	1.6379728	−1.6030694	−1.3897333	1.1366457	−1.9847799	1.21019	−	−	−
40	0.5076777	−2.4125272	1.6401858	−2.0314792	1.7419422	−0.4447	−	−	−
50	−0.0275478	−1.7300404	0.5294398	0.2286324	−	−	−	−	−
80	−1.5538165	0.1722079	0.5190002	−0.1345256	−	−	−	−	−
90	−2.2804644	1.4965624	−0.0578446	−0.1571371	−	−	−	−	−

For the first level LH−moments method, the frequency factor can be estimated by:

$$K_p(p) = a + b \cdot \tau_{H3} + c \cdot \tau_{H3}^2 + d \cdot \tau_{H3}^3 + e \cdot \tau_{H3}^4 + f \cdot \tau_{H3}^5 + g \cdot \tau_{H3}^6 + h \cdot \tau_{H3}^7 + i \cdot \tau_{H3}^8$$

Table S5. The frequency factor of the IPR distribution, for the first level LH−moments.

P [%]	a	b	C	d	e	f	g	h	i
0.01	−767.6908	13654.387	−104036.3744	447822.297	−1189551.1958	2010036.758	−2116441.842	1282341.5755	−345882.5164
0.1	−14.4065	268.1497	−1603.7447	5965.8294	−12749.9538	17193.6413	−14129.8029	6956.0145	−2087.5865
0.5	4.36425	−15.78366	182.17918	−469.89640	707.87903	−496.25482	54.96872	−	−
1	3.22765	−1.78049	89.30195	−269.96465	455.38503	−404.02723	112.73221	−	−
2	1.18714	22.45955	−62.35309	122.82644	−127.67726	36.11891	−	−	−
3	3.02675	1.104	10.93435	−18.56752	−1.71188	−	−	−	−
5	2.83453	−1.46458	10.65611	−22.64894	7.28810	−	−	−	−
10	2.147555	−2.734561	5.042787	−12.925056	6.431907	−	−	−	−
20	1.095351	−3.268998	2.180755	−4.185255	2.702168	−	−	−	−
40	−0.555994	−2.045112	1.100031	0.247911	−	−	−	−	−
50	−1.262905	−1.269213	1.526737	−0.203663	−	−	−	−	−
80	−3.703033	3.69368	−1.141327	−	−	−	−	−	−
90	−4.825085	6.246594	−2.601531	−	−	−	−	−	−

The L−moments diagram.
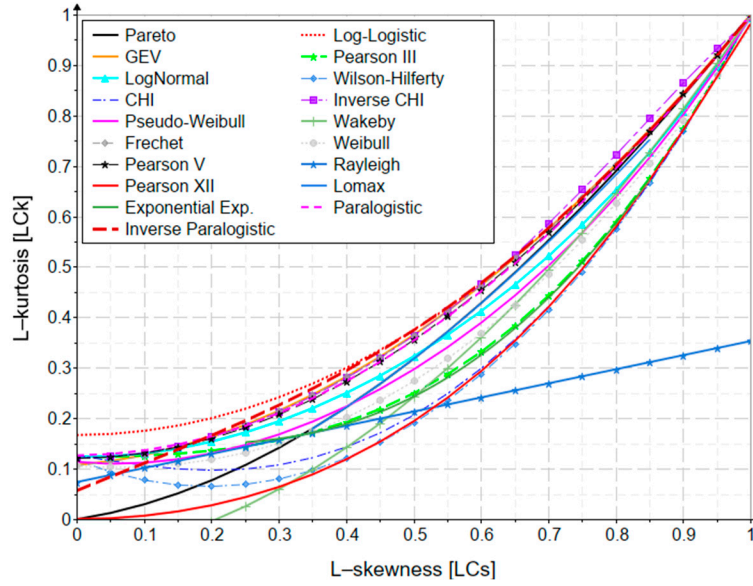
Figure S1 shows the L−moments diagram.

Figure S1. The variation diagram for L−skewness and L−kurtosis.

Paralogistic:

$$\tau_4 = 0.1262814 + 0.0078207 \cdot \tau_3 + 0.9179335 \cdot \tau_3^2 - 0.0328508 \cdot \tau_3^3 - 0.0190348 \cdot \tau_3^4$$

Inverse Paralogistic:

$$\tau_4 = 0.0577651 + 0.5568896 \cdot \tau_3 - 0.2198157 \cdot \tau_3^2 + 0.9069583 \cdot \tau_3^3 - 0.3025029 \cdot \tau_3^4$$

In the case of the Dagum and Burr distributions, unique curves regarding the variation of these 2 statistical indicators cannot be established, because they are characterized by two shape parameters.

The first level LH−moments diagram.

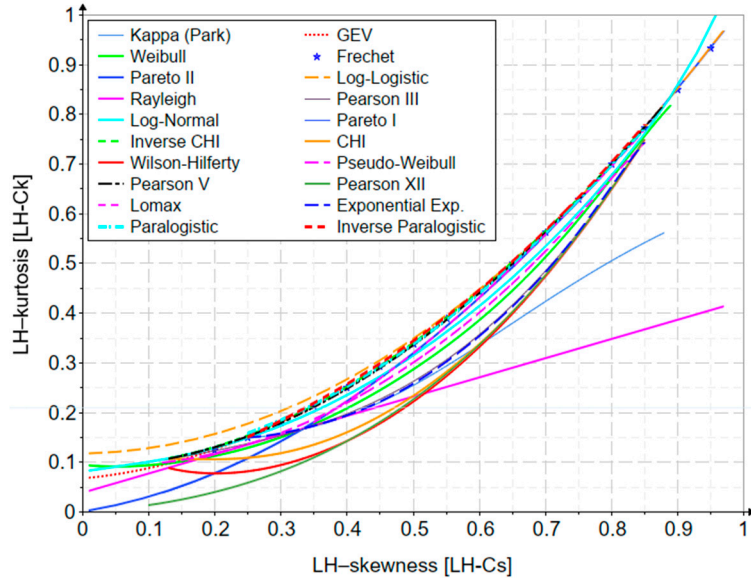Figure S2 shows the first level LH−moments diagram.

Figure S2. The variation diagram for LH−skewness and LH−kurtosis.

Paralogistic:

$$\tau_{H4} = 0.0949132 + 0.0207519 \cdot \tau_{H3} + 0.9413469 \cdot \tau_{H3}^2 - 0.0225815 \cdot \tau_{H3}^3 + 0.0002503 \cdot \tau_{H3}^4 + 0.0023262 \cdot \tau_{H3}^5 - 0.027725 \cdot \tau_{H3}^6 + 0.0084845 \cdot \tau_{H3}^7$$

Inverse Paralogistic:

$$\tau_{H4} = 0.0111007 + 0.4687435 \cdot \tau_{H3} + 0.5140639 \cdot \tau_{H3}^2 - 1.2020277 \cdot \tau_{H3}^3 + 2.866485 \cdot \tau_{H3}^4 - 2.0532499 \cdot \tau_{H3}^5 + 0.2781232 \cdot \tau_{H3}^6 + 0.1437387 \cdot \tau_{H3}^7$$

In the case of the Dagum and Burr distributions, unique curves regarding the variation of these 2 statistical indicators cannot be established, because they are characterized by two shape parameters.