Supplementary Materials: Full equation set for AHP and TOPSIS methodology

AHP

Establish the AHP decision matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

Where: (S1)

(i) $a_{ij} = \frac{1}{a_{ji}}$ for $i, j = 1, \dots, n$ and $i \neq j$

(ii) $a_{ij} = 1$ for $i, j = 1, \dots, n$ and $i = j$

Normalizing the matrix by column

$$\hat{A} = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \cdots & \hat{a}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{a}_{n1} & \hat{a}_{n2} & \dots & \hat{a}_{nn} \end{bmatrix}$$

Where: (S2)

(i) $\hat{a}_{ij} = \frac{a_{ij}}{\sum_{i=1}^{n} a_{ij}}$

Define the priority weight vector by the geometric mean of the rows

$$\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix} = \begin{pmatrix} \sqrt[n]{\hat{a}_{11} \cdot \hat{a}_{12} \cdot \dots \cdot \hat{a}_{1n}} \\ \sqrt[n]{\hat{a}_{21} \cdot \hat{a}_{22} \cdot \dots \cdot \hat{a}_{2n}} \\ \dots \\ \sqrt[n]{\hat{a}_{n1} \cdot \hat{a}_{n2} \cdot \dots \cdot \hat{a}_{nn}} \end{pmatrix} \quad \text{(S3)}$$

Where:

(i) $w_i > 0$ and $i = 1,2, \dots n$

(ii) $\sum_{i=1}^{n} w_i = 1$

A note on consistency

In Saaty's consistency eigenvector method, $\boldsymbol{w}$ is the eigenvector of the matrix so that λ would be the principle eigenvalue of the vector in accordance with the Perron-Frobenius theorem and $\lambda = n.$ Equation S4 demonstrates a relationship with matrix A assuming perfect consistency. However, because of inconsistency, Saaty proposes supplementing λ with the λ_{max} where $\lambda_{max} > n$ and the difference between λ_{max} and n is representative of the inconsistency of the judgements.

$$A\boldsymbol{w} = \lambda \boldsymbol{w} \quad \text{(S4)}$$

Where

(i) λ is the principle eigenvalue of matrix A

Geometric consistency index for the geometric mean priority weight vector method

$$GCI = \frac{2}{(n-1)(n-2)} \sum_{\substack{i,j=1 \\ i<j}}^{n} ln^2 e_{ij}$$

Where (S5)

(i) $e_{ij} = a_{ij} w_j / w_i$ is the error obtained when the ratio $\boldsymbol{w}_i/\boldsymbol{w}_j$ is approximated by a_{ij}

Determine the consistency measure of each participant in group aggregation.

$$CM^k = 1 - CR^k$$

Where: (S6)

(i) $k = 1, 2, \ldots, r$ for the set of decision-makers

Determine the individual aggregation weight of each participant.

$$aiw^k = \frac{CM^k}{\sum_{k=1}^{r} CM^k}$$

Where: (S7)

(i) $\sum_{k=1}^{r} aiw^k = 1$

Determine the group priority weight vector

$$\overline{\boldsymbol{w}} = \prod\nolimits_{k=1}^{r} (\boldsymbol{w}^k)^{(aiw^k)}$$

So that:

$$\overline{\boldsymbol{w}} = \begin{pmatrix} w_1^1 \\ w_2^1 \\ \ldots \\ w_n^1 \end{pmatrix}^{aiw^1} \cdot \begin{pmatrix} w_1^2 \\ w_2^2 \\ \ldots \\ w_n^2 \end{pmatrix}^{aiw^2} \cdot \ldots \cdot \begin{pmatrix} w_1^K \\ w_2^K \\ \ldots \\ w_n^K \end{pmatrix}^{aiw^r} = \begin{pmatrix} \overline{w}_1 \\ \overline{w}_2 \\ \ldots \\ \overline{w}_n \end{pmatrix} \tag{S8}$$

Normalize the group priority weight vector

$$\hat{\boldsymbol{w}} = \frac{\overline{w}_i}{\sum_{i=1}^{n} \overline{w}_i} = \begin{pmatrix} \widehat{w}_1 \\ \widehat{w}_2 \\ \dots \\ \widehat{w}_n \end{pmatrix} \tag{S9}$$

Determine the global weights on the criteria considering their parent criterion

$$\boldsymbol{w}_{global\ sub-criteria\ matrix} = \widehat{w}_i \widehat{\boldsymbol{w}}_{sub-criteria\ matrix} \tag{S10}$$

Where:

(i) w_i is the weight of the parent criterion in the decision hierarchy from the aggregated parent priority weight vector defined in equation S9

Definition of the final global priority weight vector for all criteria interacting with the alternatives.

$$\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{pmatrix} \tag{S11}$$

Where:

(i) w_j is the global weight of criterion j that interacts with the alternatives in the decision tree

(ii) $w_j > 0$ and $j = 1, \dots, n$

(iii) $\sum_{j=1}^{n} w_j = 1$

<u>TOPSIS</u>

Establish the TOPSIS decision matrix

$$B^k = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_n \end{array} \\ \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mn} \end{bmatrix} \end{array} \tag{S12}$$

Where:

(i) A_i represents the alternative i and C_j represents the criteria j, for $i = 1, \dots, m$ and $j = 1, \dots, n$

(ii) And f_{ij} represents the performance rating of A_i under C_j

(iii) For k=1,2,…,r for the number of decision-makers

Normalize the TOPSIS matrix

$$Z^k = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{bmatrix}$$

Where: (S13)

(i) $z_{ij} = \frac{f_{ij}}{\sqrt{\sum_{i=1}^{m} f_{ij}^2}}$

Determine the Positive Ideal Solution (PIS) for each criterion

$$Z^{k+} = \left\{ \left(Max\, z_{ij}^k | j \in J \right), \left(Min\, z_{ij}^k | j \in J' \right) | i = 1,2, \dots, m \right\} = \left\{ z_1^{k+}, \dots, z_j^{k+} | j = 1,2, \dots, n \right\}$$

Where: (S14)

(i) J is associated with positive criteria or benefits while J' is associated with negative criteria or costs

Determine the Negative Ideal Solution (NIS) for each criterion

$$Z^{k-} = \left\{ \left(Min\, z_{ij}^k | j \in J \right), \left(Max\, z_{ij}^k | j \in J' \right) | i = 1,2, \dots, m \right\} = \left\{ z_1^{k-}, \dots, z_j^{k-} | j = 1,2, \dots, n \right\}$$

Where: (S15)

(i) J is associated with positive criteria or benefits while J' is associated with negative criteria or costs

Establish the separation distance of each alternative from the PIS

$$S_i^{k+} = \sqrt{\sum_{j=1}^{n} w_j \left(z_{ij}^k - z_j^{k+} \right)^2}$$

Where:

(i) w_j is the weight of criterion j from the priority weight vector $\boldsymbol{w}$ defined in Equation S11

So that: (S16)

$$S_i^{k+} = \begin{pmatrix} \sqrt{w_1(z_{11}^k - z_1^{k+})^2 + w_2(z_{12}^k - z_2^{k+})^2 + \cdots + w_n(z_{1n}^k - z_n^{k+})^2} \\ \sqrt{w_1(z_{21}^k - z_1^{k+})^2 + w_2(z_{22}^k - z_2^{k+})^2 + \cdots + w_n(z_{2n}^k - z_n^{k+})^2} \\ \vdots \\ \sqrt{w_1(z_{m1}^k - z_1^{k+})^2 + w_2(z_{m2}^k - z_2^{k+})^2 + \cdots + w_n(z_{mn}^k - z_n^{k+})^2} \end{pmatrix}$$

Establish the separation distance of each alternative from the NIS

$$S_i^{k-} = \sqrt{\sum_{j=1}^{n} w_j \left(z_{ij}^k - z_j^{k-}\right)^2}$$

Where:

(i) w_j is the weight of criterion j from the priority weight vector $\boldsymbol{w}$ defined in Equation S11

so that: (S17)

$$S_i^{k+} = \begin{pmatrix} \sqrt{w_1(z_{11}^k - z_1^{k-})^2 + w_2(z_{12}^k - z_2^{k-})^2 + \cdots + w_j\left(z_{1j}^k - z_j^{k-}\right)^2} \\ \sqrt{w_1(z_{21}^k - z_1^{k-})^2 + w_2(z_{22}^k - z_2^{k-})^2 + \cdots + w_j\left(z_{2j}^k - z_j^{k-}\right)^2} \\ \vdots \\ \sqrt{w_1(z_{i1}^k - z_1^{k-})^2 + w_2(z_{i2}^k - z_2^{k-})^2 + \cdots + w_j\left(z_{ij}^k - z_j^{k-}\right)^2} \end{pmatrix}$$

Aggregate the separation distances from the PIS' for the group

$$\overline{S_i^+} = \left(\prod_{k=1}^{r} S_i^{k+}\right)^{1/r} = \begin{pmatrix} (S_1^{1+} \cdot S_1^{2+} \cdot \ldots \cdot S_1^{r+})^{1/r} \\ (S_2^{1+} \cdot S_2^{2+} \cdot \ldots \cdot S_2^{r+})^{1/r} \\ \vdots \\ (S_m^{1+} \cdot S_m^{2+} \cdot \ldots \cdot S_m^{r+})^{1/r} \end{pmatrix} \tag{S18}$$

Aggregate the separation distances from the NIS' for the group

$$\overline{S_i^-} = \left(\prod_{k=1}^{K} S_i^{k-}\right)^{1/r} = \begin{pmatrix} (S_1^{1-} \cdot S_1^{2-} \cdot \ldots \cdot S_1^{K-})^{1/r} \\ (S_2^{1-} \cdot S_2^{2-} \cdot \ldots \cdot S_2^{K-})^{1/r} \\ \vdots \\ (S_m^{1-} \cdot S_m^{2-} \cdot \ldots \cdot S_m^{K-})^{1/r} \end{pmatrix} \tag{S19}$$

Determine the closeness measure for each alternative

$$\overline{C_i^*} = \frac{\overline{S_i^-}}{\overline{S_i^+} + \overline{S_i^-}} \tag{S20}$$

Sensitivity

Initiate a disturbance on a weight and adjust the remaining weights in accordance.

$$\boldsymbol{w}' = \begin{pmatrix} w_1' = \dfrac{w_1}{w_1 + w_2 + \cdots w_q^* + \cdots w_n} = \dfrac{w_1}{1 + (\gamma_q - 1)w_q} \\ w_2' = \dfrac{w_2}{w_1 + w_2 + \cdots w_q^* + \cdots w_n} = \dfrac{w_2}{1 + (\gamma_q - 1)w_q} \\ \vdots \\ w_q' = \dfrac{w_q^*}{w_1 + w_2 + \cdots w_q^* + \cdots w_n} = \dfrac{\gamma_q w_q}{1 + (\gamma_q - 1)w_q} \\ \vdots \\ w_n' = \dfrac{w_n}{w_1 + w_2 + \cdots w_q^* + \cdots w_n} = \dfrac{w_n}{1 + (\gamma_q - 1)w_q} \end{pmatrix} \tag{S21}$$

Where:

(i) w_1', w_2', w_q', and w_n' are the new weights for criteria 1, 2, q, and n after the disturbance of w_q

(ii) $\sum_{j=1}^{n} w_j' = 1$ and $j = 1, \ldots, n$

Establish the unitary variation ratio of the disturbed weight

$$\beta_q = \frac{w_q'}{w_q} \tag{S22}$$

The initial variation ratio in terms of the unitary variation ratio

$$\gamma_q = \frac{\beta_q - \beta_q w_q}{1 - \beta_q w_q} \tag{S23}$$