Appendix A. Closed-Form Forward and Inverse Kinematics

This appendix derives the closed-form orientation kinematics of the spherical remote-centre-of-motion (RCM) mechanism referred to as Equations (8) and (9) in the main text. The mechanism is described by the two actuated base coordinates q_1 (base-axis rotation) and q_2 (the active joint of the extension–retraction spherical loop), the roll coordinate q_3, and the fixed structural angle α_1 (here $\alpha_1 = 15°$). The orientation of the instrument axis about the RCM is expressed through the azimuth ψ, the elevation θ and the roll φ. Throughout, the abbreviation

$$D(q_1) = \sqrt{1 - \sin^2 2\alpha_1 \sin^2 q_1}$$

is used.

A.1. Forward kinematics

Substituting the rotation operators of Equation (1) into the orientation parametrisation of Equations (6)–(7) and eliminating the passive coordinate u_1 through the spherical loop-closure constraint yields the orientation of the instrument axis as a function of the actuated coordinates. The roll is transmitted directly through the outer chain,

$$\varphi = q_3.$$

The elevation angle depends only on the base-rotation coordinate q_1. After trigonometric simplification it reduces to the compact form

$$\theta = 2\arcsin(\sin 2\alpha_1 \sin q_1),$$

which is equivalent to the expanded expression used in Equation (8),

$$\theta = \operatorname{atan2}(2D \sin 2\alpha_1 \sin q_1,\ \ \cos^2 2\alpha_1 + \sin^2 2\alpha_1 \cos 2q_1).$$

The azimuth couples both active coordinates,

$$\psi = -\arcsin\left(\frac{A \sin q_2 + B \cos q_2}{D}\right),$$

with the q_1-dependent coefficients

$$A(q_1) = 2\cos^2\alpha_1 \sin^2 q_1 - 1, \qquad B(q_1) = -\cos^2\alpha_1 \sin 2q_1.$$

A useful identity is that the coefficient amplitude equals the elevation factor, $\sqrt{A^2 + B^2} = D$, so that Equation (A2) may be written compactly as $\psi = -\arcsin[\sin(q_2 + \delta)]$ with the auxiliary angle $\delta(q_1) = \operatorname{atan2}(B, A)$. Equations (A1)–(A2) together with $\varphi = q_3$ constitute the forward map of Equation (8).

A.2. Inverse kinematics

Given a desired instrument orientation (ψ, θ, φ), the actuated coordinates are recovered in closed form. The roll follows immediately,

$$q_3 = \varphi.$$

Because θ depends only on q_1 and is strictly monotonic over the operating range ($\partial\theta/\partial q_1 > 0$ for $|q_1| < 90°$), inverting Equation (A1) gives q_1 uniquely,

$$q_1 = \arcsin\left(\frac{\sin(\theta/2)}{\sin 2\alpha_1}\right).$$

With q_1 known, the coefficients A, B and the auxiliary angle $\delta = \operatorname{atan2}(B, A)$ are evaluated. Solving Equation (A2) for q_2 gives $\sin(q_2 + \delta) = -\sin\psi$, whose two solutions are $q_2 = -\psi - \delta$ and $q_2 = \pi + \psi - \delta$. The operating workspace, which contains the reference configuration $q_1 = q_2 = 0$, corresponds to the branch

$$q_2 = \pi + \psi - \operatorname{atan2}(B, A) \pmod{2\pi},$$

reduced to the principal interval $(-\pi, \pi]$; the alternative branch lies beyond the extension–retraction singular locus and is not used. Equations (A3)–(A4) with $q_3 = \varphi$ constitute the inverse map of Equation (9).

A.3. Differential properties and singularities

Differentiating Equation (A1) gives the elevation sensitivity that appears in the orientation Jacobian of Equation (11),

$$\frac{\partial\theta}{\partial q_1} = \frac{2\sin 2\alpha_1 \cos q_1}{\sqrt{1 - \sin^2 2\alpha_1 \sin^2 q_1}},$$

which vanishes only at $q_1 = \pm 90°$. Since the reduced Jacobian is block-triangular, its determinant factorises as $\det J_3 = -(\partial\psi/\partial q_2)(\partial\theta/\partial q_1)$, so the mechanism is singular when either factor vanishes. The first factor gives the boundary singularity $q_1 = \pm 90°$; the second, $\partial\psi/\partial q_2 = 0$, is equivalent to

$$\cos q_2 = 2\sin q_1 \sin(q_1 + q_2)\cos^2\alpha_1,$$

which defines the extension–retraction singular locus of Equation (12). For $\alpha_1 = 15°$ both loci lie at the periphery of the reachable orientation region, so the central operating workspace is free of singularities.

A.4. Numerical verification

The closed-form maps were verified against the numerical kinematic model. Representative forward evaluations are $(q_1, q_2) = (20°, 10°) \mapsto (\psi, \theta) = (47.50°, 19.69°)$, $(30°, -15°) \mapsto (41.57°, 28.96°)$ and $(-25°, 20°) \mapsto (-26.99°, -24.40°)$. Applying the inverse relations (A3)–(A4) to each of these orientations recovers the original joint angles to numerical precision, confirming the mutual consistency of Equations (8) and (9).