Technical Appendix

Proof of Lemma 1

The problem is solved backwards, in the sense that the decisions of F are determined first, assuming that the E strategies have been observed. The F's Hamilton-Jacobi-Bellman (HJB) equation for the problem of Model A is:

$$\rho V_f^{NG} = \max_{e_f}\left\{\pi_f(\eta e+\gamma s)-(1-\varepsilon_1)e_f^2+\frac{\partial V_f^{NG}}{\partial s}(\alpha e_r+\beta e_a-\sigma s)+\frac{\partial V_f^{NG}}{\partial e}(\lambda e_f+\mu e_a-\delta e)\right\} \quad \text{(S1)}$$

Taking the first derivative of (S1) with respect to e_f, and the F's optimal response production effort level can be obtained as follows:

$$e_f^{NG} = \frac{\partial V_f^{NG}}{\partial e}\frac{\lambda}{2(1-\varepsilon_1)} \quad \text{(S2)}$$

The HJB equation for the retailer is

$$\rho V_e^{NG} = \max_{e_a,e_r}\left\{\pi_e(\eta e+\gamma s)-(1-\varepsilon_2)e_a^2-e_r^2+\frac{\partial V_e^{NG}}{\partial s}(\alpha e_r+\beta e_a-\sigma s)+\frac{\partial V_e^{NG}}{\partial e}(\lambda e_f+\mu e_a-\delta e)\right\} \quad \text{(S3)}$$

Substituting (S2) into (S3) yields, and taking the first derivative of (S3) with respect to e_a and e_r, the level of optimal CSR and brand construction can be obtained as follows:

$$\begin{aligned} e_r^{NG} &= \frac{\partial V_e^{NG}}{\partial s}\frac{\alpha}{2}, \\ e_a^{NG} &= \frac{(\frac{\partial V_e^{NG}}{\partial s}\beta+\frac{\partial V_e^{NG}}{\partial e}\mu)}{2(1-\varepsilon_2)} \end{aligned} \quad \text{(S4)}$$

Substituting (S2) and (S4) into (S1) and (S3) yields

$$\begin{aligned} \rho V_f^{NG} &= (\pi_f\gamma-\frac{\partial V_f^{NG}}{\partial s}\sigma)s+\frac{\partial V_f^{NG}}{\partial s}(\frac{\beta}{2(1-\varepsilon_2)}(\frac{\partial V_e^{NG}}{\partial e}\mu+\frac{\partial V_e^{NG}}{\partial s}\beta)+\frac{\partial V_e^{NG}}{\partial s}\frac{\alpha^2}{2}) \\ &+\frac{\partial V_f^{NG}}{\partial e}(\frac{\mu}{2(1-\varepsilon_2)}(\frac{\partial V_e^{NG}}{\partial e}\mu+\frac{\partial V_e^{NG}}{\partial s}\beta)+\frac{\partial V_f^{NG}}{\partial e}\frac{\lambda^2}{2(1-\varepsilon_1)})-\frac{(1-\varepsilon_1)\lambda^2}{4(1-\varepsilon_1)^2}(\frac{\partial V_f^{NG}}{\partial e})^2+(\pi_f\eta-\frac{\partial V_f^{NG}}{\partial e}\delta)e \end{aligned} \quad \text{(S5)}$$

$$\begin{aligned} \rho V_e^{NG} &= (\pi_e\gamma-\frac{\partial V_e^{NG}}{\partial s}\sigma)s+(\pi_e\eta-\frac{\partial V_e^{NG}}{\partial e}\delta)e-\frac{\alpha^2}{4}(\frac{\partial V_e^{NG}}{\partial s})^2-\frac{(1-\varepsilon_2)}{4(1-\varepsilon_2)^2}(\frac{\partial V_e^{NG}}{\partial e}\mu+\frac{\partial V_e^{NG}}{\partial s}\beta)^2 \\ &+\frac{\partial V_e^{NG}}{\partial s}(\frac{\beta}{2(1-\varepsilon_2)}(\frac{\partial V_e^{NG}}{\partial e}\mu+\frac{\partial V_e^{NG}}{\partial s}\beta)+\frac{\partial V_e^{NG}}{\partial s}\frac{\alpha^2}{2})+\frac{\partial V_e^{NG}}{\partial e}(\frac{\mu}{2(1-\varepsilon_2)}(\frac{\partial V_e^{NG}}{\partial e}\mu+\frac{\partial V_e^{NG}}{\partial s}\beta)+\frac{\pi_f\eta\lambda^2}{2(1-\varepsilon_1)(\delta+\rho)}) \end{aligned} \quad \text{(S6)}$$

Conjecture the following functional forms for the value functions of F and E:

$$V_f^{NG}(s,e) = m_1^{NG}s+m_2^{NG}e+m_3^{NG} \quad \text{(S7)}$$

$$V_e^{NG}(s,e) = n_1^{NG} s + n_2^{NG} e + n_2^{NG} \tag{S8}$$

where m_1^{NG}, m_2^{NG}, m_3^{NG}, n_1^{NG}, n_2^{NG}, and n_3^{NG} are constants to be determined. Substituting (S7) and (S8) into (S5) and (S6), and sort it out can be obtained:

$$\begin{aligned}
m_1^{NG} &= \frac{\pi_f \gamma}{\rho+\sigma}, \\
m_2^{NG} &= \frac{\pi_f \eta}{\rho+\delta}, \\
n_1^{NG} &= \frac{\pi_e \gamma}{\rho+\sigma}, \\
n_2^{NG} &= \frac{\pi_e \eta}{\rho+\delta}, \\
m_3^{NG} &= \frac{2(1-\varepsilon_1)((\beta^2-\alpha^2(1-\varepsilon_2))\frac{\pi_f\gamma}{\rho+\sigma}+\frac{\pi_f\eta}{\rho+\delta}\mu\beta)\frac{\pi_e\gamma}{\rho+\sigma}+2\mu\beta\frac{\pi_f\gamma}{\rho+\sigma}\frac{\pi_e\eta}{\rho+\delta}(1-\varepsilon_1)}{4(1-\varepsilon_1)(1-\varepsilon_2)\rho+\delta} \\
&\quad - \frac{\frac{\pi_f\eta}{\rho+\delta}(2\varepsilon_1\mu^2\frac{\pi_e\eta}{\rho+\delta}-\lambda^2\frac{\pi_f\eta}{\rho+\delta}(1-\varepsilon_2)-2\mu^2\frac{\pi_e\eta}{\rho+\delta})}{4(1-\varepsilon_1)(1-\varepsilon_2)\rho+\delta}, \\
n_3^{NG} &= \frac{(1-\varepsilon_1)(\beta^2+\alpha^2(1-\varepsilon_2))(\delta+\rho)(\frac{\pi_e\gamma}{\rho+\sigma})^2-2\mu\beta\frac{\pi_e\gamma}{\rho+\sigma}\frac{\pi_e\eta}{\rho+\delta}(\delta+\rho)(1-\varepsilon_1)}{4\rho(1-\varepsilon_1)(1-\varepsilon_2)(\delta+\rho)} \\
&\quad + \frac{\frac{\pi_e\eta}{\rho+\delta}(\mu^2\frac{\pi_e\eta}{\rho+\delta}(1-\varepsilon_1)\rho+\mu^2\frac{\pi_e\eta}{\rho+\delta}\delta(1-\varepsilon_1)+2\eta\lambda^2\pi_f(1-\varepsilon_2))}{4\rho(1-\varepsilon_1)(1-\varepsilon_2)(\delta+\rho)}.
\end{aligned} \tag{S9}$$

Then, the equilibrium production effort level, CSR level and brand construction level are:

$$e_r^{NG*} = \frac{\pi_e\gamma\alpha}{2(\rho+\sigma)}, e_f^{NG*} = \frac{\pi_f\eta\lambda}{2(\delta+\rho)(1-\varepsilon_1)}, e_a^{NG*} = \frac{\pi_e(\beta\delta\gamma+\beta\gamma\rho+\eta\mu\rho+\eta\mu\sigma)}{2(1-\varepsilon_2)(\rho+\sigma)(\delta+\rho)} \tag{S10}$$

Substituting (S10) into the formula for goodwill and consumer-perceived value:

$$\begin{cases} \dot{e}(t) = \lambda e_f(t) + \mu e_a(t) - \delta e(t), & e(0) = e_0 \\ \dot{s}(t) = \alpha e_r(t) + \beta e_a(t) - \sigma s(t), & s(0) = s_0 \end{cases} \tag{S11}$$

Then, we have

$$\begin{cases} \dot{e} = \lambda\frac{\pi_f\eta\lambda}{2(\delta+\rho)(1-\varepsilon_1)} + \frac{\pi_e(\beta\delta\gamma+\beta\gamma\rho+\eta\mu\rho+\eta\mu\sigma)}{2(1-\varepsilon_2)(\rho+\sigma)(\delta+\rho)}, & e(0) = e_0 \\ \dot{s}(t) = \alpha\frac{\pi_e\gamma\alpha}{2(\rho+\sigma)} + \beta\frac{\pi_e(\beta\delta\gamma+\beta\gamma\rho+\eta\mu\rho+\eta\mu\sigma)}{2(1-\varepsilon_2)(\rho+\sigma)(\delta+\rho)} - \sigma s(t), & s(0) = s_0 \end{cases} \tag{S12}$$

Calculating the above differential equation, we obtain

$$
\begin{aligned}
s^{NG*}(t) &= (s_0 - s_\infty^N)e^{-\sigma t} + s_\infty^{NG}, \\
e^{NG*}(t) &= (e_0 - e_\infty^N)e^{-\delta t} + e_\infty^{NG} \\
s_\infty^{NG} &= \frac{\pi_e((\delta+\rho)((1-\varepsilon_2)\alpha^2+\beta^2)\gamma + \mu\beta\eta(\rho+\sigma))}{2\sigma(1-\varepsilon_2)(\rho+\sigma)(\delta+\rho)} \\
e_\infty^{NG} &= \frac{(1-\varepsilon_1)\mu\pi_e(\mu\eta(\rho+\sigma)+\beta\gamma(\delta+\rho)) + \pi_f\eta\lambda^2(\rho+\sigma)(1-\varepsilon_2)}{2\delta(1-\varepsilon_1)(1-\varepsilon_2)(\rho+\sigma)(\delta+\rho)}
\end{aligned}
\tag{S13}
$$

By further calculation, we obtain that the E and F present values of their monetary profits are:

$$
\begin{aligned}
V_f^{NG*}(s,e) &= \frac{\pi_f\gamma}{\rho+\sigma}s + \frac{\pi_f\eta}{\delta+\rho}e + m_3^{NG}, V_e^{NG}(s,e) = \frac{\pi_e\gamma}{\rho+\sigma}s + \frac{\pi_e\eta}{\delta+\rho}e + n_3^{NG} \\
V_{sc}^{NG*}(s,e) &= \frac{(\pi_e+\pi_f)\gamma}{\rho+\sigma}s + \frac{(\pi_e+\pi_f)\eta}{\delta+\rho}e + n_3^{NG} + m_3^{NG}
\end{aligned}
\tag{S14}
$$

Similarly, setting $\varepsilon_1 = 0, \varepsilon_2 = 0$, we will obtain the *Proof of Lemma 1*.

Proof of proposition 1

$$
\begin{aligned}
e_r^{NG*} - e_r^{N*} &= 0, \\
e_f^{NG*} - e_f^{N*} &= \frac{\pi_f\eta\lambda}{2(\delta+\rho)(1-\varepsilon_1)} - \frac{\pi_f\eta\lambda}{2(\delta+\rho)} > 0, \\
e_a^{NG*} - e_a^{N*} &= \frac{\pi_e(\beta\delta\gamma+\beta\gamma\rho+\eta\mu\rho+\eta\mu\sigma)}{2(1-\varepsilon_2)(\rho+\sigma)(\delta+\rho)} - \frac{\pi_e(\beta\delta\gamma+\beta\gamma\rho+\eta\mu\rho+\eta\mu\sigma)}{2(\rho+\sigma)(\delta+\rho)} > 0
\end{aligned}
\tag{S15}
$$

$$
\begin{aligned}
s^{NG*}(t) - s^{N*}(t) &= \frac{\pi_e(\beta\delta\gamma+\beta\gamma\rho+\eta\mu\rho+\eta\mu\sigma)\beta}{2\sigma(1-\varepsilon_2)(\rho+\sigma)(\delta+\rho)} - \frac{\pi_e(\beta\delta\gamma+\beta\gamma\rho+\eta\mu\rho+\eta\mu\sigma)\beta}{2\sigma(\rho+\sigma)(\delta+\rho)} > 0, \\
e^{NG*}(t) - e^{N*}(t) &= \left(\frac{\varepsilon_2}{1-\varepsilon_2}\right)\left(\frac{\pi_e(\beta\delta\gamma+\beta\gamma\rho+\eta\mu\rho+\eta\mu\sigma)\mu}{2\delta(\rho+\sigma)(\delta+\rho)}\right) + \left(\frac{\varepsilon_1}{1-\varepsilon_1}\right)\left(\frac{\pi_f\eta\lambda^2}{2\delta(\delta+\rho)}\right) > 0,
\end{aligned}
\tag{S16}
$$

$$
\begin{aligned}
&V_f^{NG*}(s,e) - V_f^{N*}(s,e) \\
&= \frac{1}{4\rho}\left[\left(\frac{2\varepsilon_2}{1-\varepsilon_2}\right)\left(\frac{\beta^2\gamma^2\pi_f\pi_e}{(\rho+\sigma)^2} + \frac{\eta^2\mu^2\pi_f\pi_e}{(\rho+\delta)^2} + \frac{2\eta\mu\rho\gamma\pi_f\pi_e}{(\rho+\sigma)(\delta+\rho)}\right) + \left(\frac{\varepsilon_1}{1-\varepsilon_1}\right)\frac{\eta^2\lambda^2\pi_f^2}{(\rho+\delta)^2}\right] > 0 \\
&V_e^{NG*}(s,e) - V_e^{N*}(s,e) = \frac{\left(\left[A_1\rho^2 + A_2\rho + A_3\right]\pi_e + A_4\right)}{4(\rho+\sigma)^2(\delta+\rho)^2\rho} > 0 \\
&V_{SC}^{NG*}(s,e) > V_{SC}^{N*}(s,e)
\end{aligned}
\tag{S17}
$$

where

$$
\begin{aligned}
A_1 &= \frac{\varepsilon_2}{1-\varepsilon_2}(\beta^2\gamma^2 + 2\mu\eta\beta\gamma + \mu^2\eta^2),\\
A_2 &= \frac{\varepsilon_2}{1-\varepsilon_2}\left[2\delta\gamma^2\beta^2 + 2\mu\eta\beta\gamma(\delta+\rho) + 2\sigma\mu^2\eta^2\right],\\
A_3 &= \frac{\varepsilon_2}{1-\varepsilon_2}(2\mu\eta\sigma\beta\gamma\delta + \mu^2\eta^2\sigma^2 + \beta^2\delta^2\gamma^2),\\
A_4 &= \frac{2\varepsilon_1}{1-\varepsilon_1}\lambda^2\eta^2(\rho+\sigma)^2\pi_f
\end{aligned}
\tag{S18}
$$

Proof of lemma 3

In this scenario, except that the F will make decisions based on utility maximization. The HJB equations for both sides are:

$$
\begin{aligned}
\rho V_{Uf}^{FG} &= \max_{e_f}\left\{\frac{1+N}{1+NM}\Pi_F - \frac{MN(1+N)}{1+MN}\Pi_E + \frac{\partial V_{Uf}^{FG}}{\partial s}(\alpha e_r + \beta e_a - \sigma s) + \frac{\partial V_{Uf}^{FG}}{\partial e}(\lambda e_f + \mu e_a - \delta e)\right\}\\
\rho V_{Ue}^{FG} &= \max_{e_r, e_a}\left\{\Pi_E + \frac{\partial V_{Uf}^{FG}}{\partial s}(\alpha e_r + \beta e_a - \sigma s) + \frac{\partial V_{Uf}^{FG}}{\partial e}(\lambda e_f + \mu e_a - \delta e)\right\}
\end{aligned}
\tag{S19}
$$

The game model is solved in the same order as the proof of *Lemma 2*, the equilibrium results of production effort level, CSR level, and Brand construction level are:

$$
e_r^{FG*} = \frac{\pi_e\gamma\alpha}{2(\rho+\sigma)}, e_f^{FG*} = \frac{(\pi_f - \pi_e MN)\eta\lambda}{2(1-\varepsilon_1)(\delta+\rho)}, e_a^{FG*} = \frac{\pi_e(\beta\delta\gamma + \beta\gamma\rho + \eta\mu\rho + \eta\mu\sigma)}{2(\rho+\sigma)(\delta+\rho)(1-\varepsilon_2)}
\tag{S20}
$$

Then we obtain the formula for goodwill and consumer-perceived value:

$$
s^{FG*}(t) = (s_0 - s_\infty^{FG})e^{-\sigma} + s_\infty^{FG},\ e^{FG*}(t) = (e_0 - e_\infty^{FG})e^{-\delta t} + e_\infty^{FG}
\tag{S21}
$$

where

$$
\begin{aligned}
s_\infty^{FG} &= \frac{\pi_e((\delta+\rho)((1-\varepsilon_2)\alpha^2 + \beta^2)\gamma + \mu\beta\eta(\rho+\sigma))}{2\sigma(1-\varepsilon_2)(\rho+\sigma)(\delta+\rho)},\\
e_\infty^{FG} &= \frac{\mu\pi_e(\mu\eta(\rho+\sigma) + \beta\gamma(\delta+\rho))}{2\delta(1-\varepsilon_2)(\rho+\sigma)(\delta+\rho)} + \frac{(\pi_f - \pi_e MN)\eta\lambda^2}{2\delta(\delta+\rho)(1-\varepsilon_1)}
\end{aligned}
\tag{S22}
$$

By further calculation, we obtain that the E and F present values of their utilities are:

$$
\begin{aligned}
V_{Ue}^{FG*}(s_0, e_0) &= \frac{\pi_e\gamma}{(\rho+\sigma)}s_0 + \frac{\pi_e\eta}{(\delta+\rho)}e_0 + n_{U3}^{FG}\\
V_{Uf}^{FG*}(s_0, e_0) &= \left(\frac{1+N}{1+MN}\right)\frac{(\pi_f - \pi_e MN)\gamma}{\rho+\sigma}s_0 + \left(\frac{1+N}{1+MN}\right)\frac{(\pi_f - \pi_e MN)\eta}{\delta+\rho}e_0 + m_{U3}^{FG}\\
V_{USC}^{FG*}(s_0, e_0) &= \left[\frac{(\pi_f + \pi_e + \pi_f N - \pi_e MN^2)\gamma}{(1+MN)(\rho+\sigma)}\right]s_0 + \left[\frac{(\pi_f + \pi_e + \pi_f N - \pi_e MN^2)\eta}{(1+MN)(\delta+\rho)}\right]e_0 + m_{U3}^{FG} + n_{U3}^{FG}
\end{aligned}
\tag{S23}
$$

where

$$
\begin{aligned}
n_{U3}^{FG} &= \frac{A+B}{4(1-\varepsilon_1)(1-\varepsilon_2)(\delta+\rho)\rho} \\
m_{U3}^{FG} &= \frac{(1+N)\left[M^2\lambda^2\eta^2\pi_e^2(\rho+\sigma)(1-\varepsilon_2)N^2+(\pi_e MN-2\pi_f)C\right]}{-4(1+MN)(1-\varepsilon_1)(1-\varepsilon_2)(\sigma+\rho)^2(\delta+\rho)^2\rho}
\end{aligned} \tag{S24}
$$

$$
\begin{aligned}
A &= \frac{\gamma(1-\varepsilon_1)\pi_e^2\left[(\delta+\rho)(\beta^2+(1-\varepsilon_2)\alpha^2)\gamma+\mu\eta\beta(\rho+\sigma)\right]}{(\rho+\sigma)^2} \\
B &= \frac{\pi_e\eta^2\left[(2MN(-1+\varepsilon_2)\lambda^2+\mu^2(1-\varepsilon_1))\pi_e+\lambda^2\pi_f(1-\varepsilon_2)\right]}{\delta+\rho} \\
C &= (1-\varepsilon_1)\left[\left(\beta^2+(1-\varepsilon_2)\alpha^2\right)(\delta+\rho)^2\gamma^2+2\mu\eta\beta(\sigma+\rho)\gamma(\delta+\rho)\right]\pi_e \\
&\quad +(1-\varepsilon_1)\mu^2\eta^2(\sigma+\rho)^2\pi_e+\frac{(1-\varepsilon_2)\lambda^2\eta^2(\sigma+\rho)^2\pi_f}{2}
\end{aligned} \tag{S25}
$$

By further calculation, we obtain that the F and the E present values of their monetary profits are

$$
\begin{aligned}
V_e^{FG*}(s_0,e_0) &= \frac{\pi_e\gamma}{(\rho+\sigma)}s_0+\frac{\pi_e\eta}{(\delta+\rho)}e_0+n_3^{FG} \\
V_f^{FG*}(s_0,e_0) &= \frac{\pi_f\gamma}{\rho+\sigma}s_0+\frac{\pi_f\eta}{\delta+\rho}e_0+m_3^{FG} \\
V_{SC}^{FG*}(s_0,e_0) &= \frac{(\pi_f+\pi_e)\gamma}{(\rho+\sigma)}s_0+\frac{(\pi_f+\pi_e)\eta}{(\delta+\rho)}e_0+m_3^{FG}+n_3^{FG}
\end{aligned} \tag{S26}
$$

$$
\begin{aligned}
n_3^{FG} &= \frac{C+D}{4(1-\varepsilon_1)(1-\varepsilon_2)(\delta+\rho)\rho} \\
C &= \frac{\gamma(1-\varepsilon_1)\pi_e^2\left[(\delta+\rho)(\beta^2+(1-\varepsilon_2)\alpha^2)\gamma+\mu\eta\beta(\rho+\sigma)\right]}{(\rho+\sigma)^2} \\
D &= \frac{\pi_e\eta^2\left[(2MN(-1+\varepsilon_2)\lambda^2+\mu^2(1-\varepsilon_1))\pi_e+\lambda^2\pi_f(1-\varepsilon_2)\right]}{\delta+\rho}
\end{aligned} \tag{S27}
$$

$$
\begin{aligned}
m_3^{FG} &= -\frac{(\pi_f-\pi_e MN)^2\eta^2\lambda^2}{4\rho(1-\varepsilon_1)(\delta+\rho)^2}+\frac{\pi_f\gamma}{\rho(\rho+\sigma)}\left(\frac{\pi_e\gamma\alpha^2}{2(\rho+\sigma)}+\frac{\pi_e\beta(\beta\delta\gamma+\beta\gamma\rho+\eta\mu\rho+\eta\mu\sigma)}{2(\rho+\sigma)(\delta+\rho)(1-\varepsilon_2)}\right) \\
&\quad +\frac{\pi_f\eta}{\rho(\rho+\delta)}\left(\frac{(\pi_f-\pi_e MN)\eta\lambda^2}{2(1-\varepsilon_1)(\delta+\rho)}+\frac{\pi_e\mu(\beta\delta\gamma+\beta\gamma\rho+\eta\mu\rho+\eta\mu\sigma)}{2(\rho+\sigma)(\delta+\rho)(1-\varepsilon_2)}\right)
\end{aligned}
$$

Proof of proposition 2

$$
\begin{aligned}
&\frac{\partial e_r^{FG*}}{\partial N}=0, \\
&\frac{\partial e_a^{FG*}}{\partial N}=0, \\
&\frac{\partial e_f^{FG*}}{\partial N}=\frac{-\eta\lambda M\pi_e}{2(1-\varepsilon_1)(\delta+\rho)}<0, \\
&\frac{\partial e_f^{FG*}}{\partial M}=\frac{-\eta\lambda N\pi_e}{2(1-\varepsilon_1)(\delta+\rho)}<0,
\end{aligned} \tag{S28}
$$

Proof of proposition 3

$$\frac{\partial s^{FG*}}{\partial N}=0,$$
$$\frac{\partial e^{FG*}}{\partial N}=\frac{-\eta M\pi_e\lambda^2}{2(1-\varepsilon_1)(\delta+\rho)}<0$$
$$\frac{\partial V_f^{FG*}(s_0,e_0)}{\partial N}=\frac{-\eta^2\lambda^2\pi_e^2M^2N}{2(1-\varepsilon_1)(\delta+\rho)^2\rho}<0,$$
$$\frac{\partial^2 V_f^{FG*}(s_0,e_0)}{\partial N\partial M}=\frac{-\eta^2\lambda^2\pi_e^2MN}{(1-\varepsilon_1)(\delta+\rho)^2\rho}<0,$$
$$\frac{\partial V_e^{FG*}(s_0,e_0)}{\partial N}=\frac{-\eta^2\lambda^2\pi_e^2M}{2(1-\varepsilon_1)(\delta+\rho)^2\rho}<0,$$
$$\frac{\partial^2 V_e^{FG*}(s_0,e_0)}{\partial N\partial M}=\frac{-\eta^2\lambda^2\pi_e^2}{2(1-\varepsilon_1)(\delta+\rho)^2\rho}<0,$$
$$\frac{\partial V_f^{FG*}(s_0,e_0)}{\partial M}=\frac{-\eta^2\lambda^2\pi_e^2N^2M}{2(1-\varepsilon_1)(\delta+\rho)^2\rho}<0,$$
$$\frac{\partial V_e^{FG*}(s_0,e_0)}{\partial M}=\frac{-\eta^2\lambda^2\pi_e^2N}{2(1-\varepsilon_1)(\delta+\rho)^2\rho}<0,$$
$$\frac{\partial^2 e_f^{FG*}}{\partial N\partial M}=\frac{-\pi_e\lambda\eta}{2(1-\varepsilon_1)(\delta+\rho)}<0 \tag{S29}$$

Proof of lemma 4

In centralized scenario, the members of both parties take the maximum profit as the goal of decision-making. Meanwhile, the HJB equation of the decision function is:

$$\rho V_{SC}^{CG}=\max_{e_f,e_r,e_a}\left\{\begin{array}{l}(\pi_f+\pi_e)(\eta e(t)+\gamma s(t))-(1-\varepsilon_1)e_f^2-e_r^2-(1-\varepsilon_2)e_a^2\\+\dfrac{\partial v_{sc}^{CG}}{\partial s}(\alpha e_r+\beta e_a-\sigma s)+\dfrac{\partial v_{sc}^{CG}}{\partial e}(\lambda e_f+\mu e_a-\delta e)\end{array}\right\} \tag{S30}$$

Consistent with the proof process of Lemma 2, therefore, we can get the optimal decisions in centralized model.

$$e_r^{CG*}(t)=\frac{(\pi_e+\pi_f)\gamma\alpha}{2(\rho+\sigma)},e_f^{CG*}(t)=\frac{(\pi_f+\pi_e)\eta\lambda}{2(\delta+\rho)(1-\varepsilon_1)},e_a^{CG*}(t)=\frac{(\pi_e+\pi_f)(\beta\delta\gamma+\beta\gamma\rho+\eta\mu\rho+\eta\mu\sigma)}{2(\rho+\sigma)(\delta+\rho)(1-\varepsilon_2)} \tag{S31}$$

Substituting (S31) into the formula for goodwill and consumer-perceived value:

$$\left\{\begin{array}{l}\dot{e}(t)=\lambda e_f(t)+\mu e_a(t)-\delta e(t),\quad e(0)=e_0\\\dot{s}(t)=\alpha e_r(t)+\beta e_a(t)-\sigma s(t),\quad s(0)=s_0\end{array}\right.$$

We obtain that

$$
\begin{aligned}
s^{CG*}(t) &= (s_0 - s_\infty^{CG})e^{-\sigma t} + s_\infty^{CG}, \\
e^{CG*}(t) &= (e_0 - e_\infty^{CG})e^{-\delta t} + e_\infty^{CG}
\end{aligned}
\tag{S32}
$$

where

$$
\begin{aligned}
s_\infty^{CG} &= \frac{(\pi_e + \pi_f)(\mu\eta\rho + \mu\eta\sigma + \beta\gamma\delta + \beta\gamma\rho)\beta}{2\sigma(1-\varepsilon_2)(\rho+\sigma)(\delta+\rho)} + \frac{(\pi_e + \pi_f)\alpha^2\gamma}{2\sigma(\rho+\sigma)} \\
e_\infty^{CG} &= \frac{(\pi_e + \pi_f)(\mu\eta\rho + \mu\eta\sigma + \beta\gamma\delta + \beta\gamma\rho)\mu}{2\delta(1-\varepsilon_2)(\rho+\sigma)(\delta+\rho)} + \frac{(\pi_e + \pi_f)\alpha^2\gamma}{2\delta(\rho+\delta)(1-\varepsilon_1)}
\end{aligned}
\tag{S33}
$$

By further calculation, we obtain that the E and F present values of the PASC monetary profit is:

$$
V_{SC}^{CG*}(s_0, e_0) = \frac{(\pi_f + \pi_e)\gamma}{\rho+\sigma} s_0 + \frac{(\pi_f + \pi_e)\eta}{\delta+\rho} e_0 + h_3^{CG}
\tag{S34}
$$

where

$$
h_3^{CG} = \frac{(\pi_f + \pi_e)^2}{4\rho(1-\varepsilon_2)} \left\{ \frac{(-\alpha^2\varepsilon_2 + \alpha^2 + \beta^2)\gamma^2}{(\sigma+\rho)^2} + \frac{2\mu\eta\beta\gamma}{(\sigma+\rho)(\delta+\rho)} + \frac{\eta^2\left[(1-\varepsilon_2)\lambda^2 + (1-\varepsilon_1)\mu^2\right]}{(1-\varepsilon_1)(\delta+\rho)^2} \right\}
\tag{S35}
$$

Proof of conclusion 1

$$
\begin{aligned}
& e_r^{NG*} - e_r^{FG*} = 0, e_r^{CG*}(t) - e_r^{FG*} = \frac{\pi_f\gamma\alpha}{2(\rho+\sigma)} > 0 \\
& e_f^{CG*} - e_f^{NG*} = \frac{\pi_e\eta\lambda}{2(\delta+\rho)(1-\varepsilon_1)} > 0, e_f^{NG*} - e_f^{FG*} = \frac{\pi_e MN\eta\lambda}{2(1-\varepsilon_1)(\delta+\rho)} > 0 \\
& e_a^{NG*} - e_a^{FG*} = 0, e_a^{CG*}(t) - e_a^{FG*} = \frac{\pi_f(\beta\delta\gamma + \beta\gamma\rho + \eta\mu\rho + \eta\mu\sigma)}{2(\rho+\sigma)(\delta+\rho)(1-\varepsilon_2)} > 0 \\
& V_{SC}^{CG*}(s_0, e_0) - V_{SC}^{NG*}(s_0, e_0) = \frac{(1-\varepsilon_1)\pi_f^2 f(\varepsilon_2) + \lambda^2\eta^2\pi_e^2(\rho+\sigma)^2(1-\varepsilon_2)}{4\rho(1-\varepsilon_1)(1-\varepsilon_2)(\delta+\rho)^2(\rho+\sigma)^2} > 0
\end{aligned}
\tag{S36}
$$

where $f(\varepsilon_2) = \left[\beta^2 + (1-\varepsilon_2)\alpha^2\right]\gamma^2(\delta+\rho)^2 + 2\mu\eta\beta\gamma(\delta+\rho)(\sigma+\rho) + \mu^2\eta^2(\sigma+\rho)^2$. Noticed that

$$
\frac{\partial V_f^{FG*}(s_0, e_0)}{\partial N} = \frac{-\eta^2\lambda^2\pi_e^2 M^2 N}{2(1-\varepsilon_1)(\delta+\rho)^2\rho} < 0,
$$

$$
\frac{\partial V_e^{FG*}(s_0, e_0)}{\partial N} = \frac{-\eta^2\lambda^2\pi_e^2 M}{2(1-\varepsilon_1)(\delta+\rho)^2\rho} < 0,
$$

$$
\frac{\partial V_{SC}^{FG*}(s_0, e_0)}{\partial N} = \frac{\partial V_f^{FG*}(s_0, e_0)}{\partial N} + \frac{\partial V_e^{FG*}(s_0, e_0)}{\partial N} < 0.
$$

Therefore, $V_{SC}^{FG*}(s_0, e_0)$, $V_f^{FG*}(s_0, e_0)$, and $V_e^{FG*}(s_0, e_0)$ are decreasing function of N, so that $V_f^{FG*}(s_0, e_0) \le V_f^{NG*}(s_0, e_0)$, $V_e^{FG*}(s_0, e_0) \le V_e^{NG*}(s_0, e_0)$, $V_{SC}^{FG*}(s_0, e_0) \le V_{SC}^{NG*}(s_0, e_0) \le V_{SC}^{CG*}(s_0, e_0)$.

Proof of conclusion 2

(1) Under decentralized channels without fairness concerns, the HJB equations of both parties under the coordination mechanism are:

$$\begin{aligned}\rho V_f^{XNG} &= \max_{e_f}\left\{\begin{array}{l}\bar{\pi}_f(\eta e(t)+\gamma s(t))-(1-\varphi_e)(1-\varepsilon_1)e_f^2-\varphi_f((1-\varepsilon_2)e_a^2+e_r^2)\\ +\dfrac{\partial V_f^{XNG}}{\partial s}(\alpha e_r+\beta e_a-\sigma s)+\dfrac{\partial V_f^{XNG}}{\partial e}(\lambda e_f+\mu e_a-\delta e)\end{array}\right\}\\ \rho V_e^{XNG} &= \max_{e_a,e_r}\left\{\begin{array}{l}\bar{\pi}_e(\eta e(t)+\gamma s(t))-(1-\varphi_f)((1-\varepsilon_2)e_a^2+e_r^2)-\varphi_e(1-\varepsilon_1)e_f^2\\ +\dfrac{\partial V_e^{XNG}}{\partial s}(\alpha e_r+\beta e_a-\sigma s)+\dfrac{\partial V_e^{XNG}}{\partial e}(\lambda e_f+\mu e_a-\delta e)\end{array}\right\}\end{aligned} \tag{S37}$$

Consistent with the proof process of Lemma 2, therefore, we can get the optimal decisions. The equilibrium results of production effort level, CSR level, and Brand construction level are:

$$e_r^{NG*}=\frac{\bar{\pi}_e\gamma\alpha}{2(\rho+\sigma)(1-\varphi_f)},e_f^{NG*}=\frac{\bar{\pi}_f\eta\lambda}{2(\delta+\rho)(1-\varepsilon_1)(1-\varphi_e)},e_a^{NG*}=\frac{\bar{\pi}_e(\beta\delta\gamma+\beta\gamma\rho+\eta\mu\rho+\eta\mu\sigma)}{2(1-\varepsilon_2)(\rho+\sigma)(\delta+\rho)(1-\varphi_f)} \tag{S38}$$

Letting $e_r^{XNG*}=e_r^{CG*},\ e_a^{XNG*}=e_a^{CG*},\ e_f^{XNG*}=e_f^{CG*}$, we obtain

$$\varphi_f=\frac{\bar{\pi}_f}{\pi_f+\pi_e},\varphi_e=\frac{\bar{\pi}_e}{\pi_f+\pi_e},(\varphi_f+\varphi_e=1) \tag{S39}$$

(2) Under decentralized channels with fairness concerns, the HJB equations of both parties under the coordination mechanism are:

$$\begin{aligned}\rho V_f^{XFG} &= \max_{e_f}\left\{\begin{array}{l}\dfrac{1+N}{1+NM}(\bar{\pi}_f(\eta e(t)+\gamma s(t))-(1-\varphi_e)(1-\varepsilon_1)e_f^2-\varphi_f((1-\varepsilon_2)e_a^2+e_r^2))\\ -\dfrac{MN(1+N)}{1+MN}(\bar{\pi}_e(\eta e(t)+\gamma s(t))-(1-\varphi_f)((1-\varepsilon_2)e_a^2+e_r^2)-\varphi_e(1-\varepsilon_1)e_f^2)\\ +\dfrac{\partial V_f^{XFG}}{\partial s}(\alpha e_r+\beta e_a-\sigma s)+\dfrac{\partial V_f^{XFG}}{\partial e}(\lambda e_f+\mu e_a-\delta e)\end{array}\right\}\\ \rho V_e^{XFG} &= \max_{e_a,e_r}\left\{\begin{array}{l}\bar{\pi}_e(\eta e(t)+\gamma s(t))-(1-\varphi_f)((1-\varepsilon_2)e_a^2+e_r^2)-\varphi_e(1-\varepsilon_1)e_f^2\\ +\dfrac{\partial V_e^{XFG}}{\partial s}(\alpha e_r+\beta e_a-\sigma s)+\dfrac{\partial V_e^{XFG}}{\partial e}(\lambda e_f+\mu e_a-\delta e)\end{array}\right\}\end{aligned} \tag{S40}$$

Similiarly, by solveing the HJB equation of both parties, we can get the optimal decisions of production effort level, CSR level, and Brand construction level are:

$$e_r^{XFG*}=\frac{\bar{\pi}_e\gamma\alpha}{2(\rho+\sigma)(1-\varphi_f)},e_f^{XFG*}=\frac{(\pi_f-\pi_eMN)\eta\lambda}{2(1-\varepsilon_1)(\delta+\rho)},e_a^{XFG*}=\frac{\bar{\pi}_e(\beta\delta\gamma+\beta\gamma\rho+\eta\mu\rho+\eta\mu\sigma)}{2(\rho+\sigma)(\delta+\rho)(1-\varepsilon_2)(1-\varphi_f)} \tag{S41}$$

Letting $e_r^{XNG*}=e_r^{CG*},\ e_a^{XNG*}=e_a^{CG*},\ e_f^{XNG*}=e_f^{CG*}$, we obtain

$$\varphi_f = \frac{\bar{\pi}_f}{\pi_f + \pi_e}, \varphi_e = \frac{\bar{\pi}_e}{\pi_f + \pi_e}, (\varphi_f + \varphi_e = 1, \bar{\pi}_e + \bar{\pi}_f = \pi_f + \pi_e) \tag{S42}$$

In summary, the decentralized model (fairness-neutral or fairness-concerned) can be coordinated through the "mixed cost-sharing and revenue sharing" contract if the contract parameters met:

$$\varphi_f = \frac{\bar{\pi}_f}{\pi_f + \pi_e}, \varphi_e = \frac{\bar{\pi}_e}{\pi_f + \pi_e}, (\varphi_f + \varphi_e = 1, \bar{\pi}_e + \bar{\pi}_f = \pi_f + \pi_e) \tag{S43}$$

By making some algebraic calculations, we can derive that

$$\begin{aligned} V_f^{XG*}(e_0, s_0) &= \varphi_f V_{SC}^{CG*}(e_0, s_0) = (1 - \varphi_e) V_{SC}^{CG*}(e_0, s_0) \\ V_e^{XG*}(e_0, s_0) &= (1 - \varphi_f) V_{SC}^{CG*}(e_0, s_0) = \varphi_e V_{SC}^{CG*}(e_0, s_0) \end{aligned} \tag{S44}$$

When the coordination is realized, the total profit values of both parties in PASC are

$$\begin{aligned} V_f^{XG*}(e_0, s_0) &= \varphi V_{SC}^{CG*}(e_0, s_0) \\ V_e^{XG*}(e_0, s_0) &= (1 - \varphi) V_{SC}^{CG*}(e_0, s_0) \end{aligned} \tag{S45}$$

So that

$$\varphi_f = \varphi \tag{S46}$$

Therefore, through this contract, the proportion of cost shared by F is consistent with the proportion of PASC profit.

Thus

$$V_{Uf}^{XG*}(e_0, s_0) = \frac{(1+N) V_{SC}^{CG*}(e_0, s_0)}{1 + MN} \left[(1 + MN)\varphi - MN \right] \tag{S47}$$

In the decentralized model without fairness concerns, both F and E accept the coordination mechanism only when their present values of the monetary profits satisfy

$$\begin{cases} V_f^{XNG*}(e_0, s_0) = \varphi V_{SC}^{CG*}(e_0, s_0) \geq V_f^{NG*}(e_0, s_0) \\ V_e^{XNG*}(e_0, s_0) = (1 - \varphi) V_{SC}^{CG*}(e_0, s_0) \geq V_e^{NG*}(e_0, s_0) \end{cases} \tag{S48}$$

Thus

$$\frac{V_f^{NG*}(e_0, s_0)}{V_{SC}^{CG*}(e_0, s_0)} \leq \varphi \leq 1 - \frac{V_e^{NG*}(e_0, s_0)}{V_{SC}^{CG*}(e_0, s_0)} \tag{S49}$$

In the decentralized model with fairness concerns, both F and E accept the coordination mechanism only when their present values of the monetary profits satisfy

$$\begin{cases} V_f^{XG*}(e_0,s_0)=\varphi V_{SC}^{CG*}(e_0,s_0)\geq V_f^{FG*}(e_0,s_0) \\ V_e^{XG*}(e_0,s_0)=(1-\varphi)V_{SC}^{CG*}(e_0,s_0)\geq V_e^{FG*}(e_0,s_0) \\ V_{Uf}^{XG*}(e_0,s_0)=\dfrac{(1+N)V_{SC}^{CG*}(e_0,s_0)}{1+MN}\left[(1-MN)\varphi-MN\right]\geq V_{Uf}^{FG*}(e_0,s_0) \end{cases} \tag{S50}$$

Thus

$$\frac{V_f^{FG*}(e_0,s_0)-MNV_e^{FG*}(e_0,s_0)}{(1+N)V_{SC}^{CG*}(e_0,s_0)}+\frac{MN}{1+MN}\leq\varphi\leq 1-\frac{V_e^{FG*}(e_0,s_0)}{V_{SC}^{CG*}(e_0,s_0)} \tag{S51}$$

Proof of proposition 4

Consistent with the proof process of Lemma 2, therefore, we can get the optimal decisions. The equilibrium results of production effort level, CSR level, and Brand construction level are:

$$\begin{aligned} &\varepsilon_1^{EG*}=\frac{\pi_e}{\pi_e+\pi_f},\varepsilon_2^{EG*}=\frac{\pi_f}{\pi_e+\pi_f},e_r^{EG*}=\frac{\gamma\alpha\pi_e}{k_3(\rho+\sigma)}, \\ &e_f^{EG*}=\frac{(\pi_f+\pi_e)\eta\lambda}{k_1(\delta+\rho)},e_a^{EG*}=\frac{(\pi_e+\pi_f)(\beta\delta\gamma+\beta\gamma\rho+\eta\mu\rho+\eta\mu\sigma)}{k_2(\rho+\sigma)(\delta+\rho)} \end{aligned} \tag{S52}$$

According to the structure of ε_1^{EG*} and ε_2^{EG*}, it is not hard to see that proposition 4 is true.

Proof of proposition 5

Consistent with the proof process of Lemma 2, therefore, we can get the optimal decisions. The equilibrium results of production effort level, CSR level, and Brand construction level are:

$$\begin{aligned} &p^{WG*}=\frac{a+bw}{2b},e_a^{WG*}=\frac{(a-bw)^2(\beta\delta\gamma+\beta\gamma\rho+\eta\mu\rho+\eta\mu\sigma)}{4bk_2(\rho+\sigma)(\delta+\rho)(1-\varepsilon_2)} \\ &e_r^{WG*}=\frac{(a-bw)^2\gamma\alpha}{4bk_3(\rho+\sigma)},e_f^{WG*}=\frac{(a-bw)\left[(Ma-(M+2)wb)N-2bw\right]\eta\lambda}{4bk_1(1-\varepsilon_1)(\delta+\rho)(1+N)(1-\varepsilon_1)} \end{aligned} \tag{S53}$$

Thus

$$\begin{aligned} &\frac{\partial p^{WG*}}{\partial N}=0,\frac{\partial e_r^{WG*}}{\partial N}=0,\frac{\partial e_a^{WG*}}{\partial N}=0, \\ &\frac{\partial e_f^{WG*}}{\partial N}=\frac{-(a-bw)^2M\eta\lambda}{4bk_1(1-\varepsilon_1)(\delta+\rho)(1+N)^2}<0 \end{aligned} \tag{S54}$$