Supplementary Materials

In the Appendix, we discuss two subjects, i.e., the role of η-Function in Fermi-Dirac statistics and symmetry breaking Beyond the Mermin–Wagner theorem

S1. The alternating Dirichlet series of the η-Function, Fermi-Dirac Statistics, and Role of λ

In this extended analysis, we further elaborate on the significance of the Dirichlet eta function, η(s), and its role in Fermi-Dirac (FD) statistics, particularly in contrast with the Riemann zeta function ζ(s) and Bose-Einstein (BE) statistics. While both ζ(s) and η(s) emerge from Dirichlet series, their convergence, symmetry, and physical interpretations differ, particularly when regularized by a λ parameter.

The η-function is defined by the alternating Dirichlet series:

$$\eta_\lambda(s) = \frac{1}{\Gamma(s)}\int_0^\infty dx \frac{x^{s-1}}{e^{x+\lambda}+1} = \sum_{n=1}^\infty (-1)^{n+1}\frac{e^{-n\lambda}}{n^s}, \quad \text{(S1)}$$

Unlike the zeta function, which diverges at s = 1, the alternating nature of the η-function ensures convergence for all Re(s) > 0, including at s = 1. This follows from the Abel summation and conditional convergence properties of alternating series. Thus, η(1) = ln(2) is finite and well-defined.

Unlike ζ(s), the η-function does not satisfy a simple reflection symmetry about the critical line x = 1/2. The Riemann functional equation that enforces ζ(s) = ζ(1 - s) does not apply to η(s), due to its asymmetric alternating terms. Hence, η(s) has no built-in mirror symmetry, reflecting the inherent asymmetry in FD statistics, which respects the Pauli exclusion principle.

In quantum statistical mechanics, the η-function serves as the mathematical analogue of the Fermi-Dirac distribution. The logarithmic partition function for an ideal Fermi gas involves sums of the form:

$$Z_{FD} = \sum_{n=1}^\infty 1/(\boldsymbol{exp}((\varepsilon_n - \mu)/k_BT) + 1) \approx \sum_{n=1}^\infty (-1)^{n+1}\frac{e^{-n\lambda}}{n^s}, \quad \text{(S2)}$$

where the alternating sign matches that of η(s), and the exponential damping with λ = βμ encodes the temperature and chemical potential. The use of η(s) reflects fermionic anti-symmetry and quantized occupation constraints (0 or 1).

For $\zeta(s)$, λ plays a dual role: Mathematically, it regularizes divergence at $s = 1$ and breaks symmetry; Physically, it encodes inverse temperature or chemical potential, governing BE condensation.

For $\eta(s)$, λ mainly controls convergence smoothness and thermodynamic behavior, but does not correct a divergence at $s = 1$ (since $\eta(1)$ is finite). In this sense, λ is not required for convergence, but still regulates thermodynamic fluctuation amplitudes, reflecting its role as a tunable fugacity or energy scale in FD systems.

Here, in Table S1, it shows the comparison between $\zeta(s)$ and $\eta(s)$

Table S1. Comparison between $\zeta(s)$ and $\eta(s)$

Function	Series Type	Convergence Domain	|Symmetry Property	Quantum System	Role of λ
$\zeta(s)$	Non-alternating	Re(s) > 1 (diverges at 1	$\xi(s) = \xi(1-s)$	Bose-Einstein (bosons)	Ensures convergence, breaks symmetry
$\eta(s)$	Alternating series	Re(s) > 0 (converges at 1)|	No reflection symmetry	Fermi-Dirac (fermions)	Regulates thermal response

In conclusion, the λ parameter serves as a unifying regularization tool but assumes distinct mathematical and physical roles across $\zeta(s)$ and $\eta(s)$. While $\zeta(s)$ uses λ to remove divergence and analyze symmetric structure (essential for RH), $\eta(s)$ naturally converges and inherently reflects fermionic anti-symmetry. Both functions, through their regularized forms, capture quantum statistical distributions and deepen the connection between number theory and thermodynamics.

S.2. **Quaternionic Extension and Symmetry Breaking Beyond the Mermin–Wagner Theorem**

The Mermin–Wagner theorem forbids spontaneous breaking of continuous symmetries in one- and two-dimensional systems with short-range interactions at finite temperature. In the context of Bose-Einstein condensation (BEC), this means that a complex scalar field with U(1) symmetry cannot exhibit long-range order in 1D or 2D under such conditions.

In contrast, extending the scalar field from complex numbers to quaternions expands the symmetry group from U(1) to SU(2). The quaternionic order parameter can be written as $q = x + a\,e_1 + b\,e_2 + c\,e_3$. This non-Abelian structure introduces three imaginary units (e_1, e_2, e_3) that do not commute, and the unit quaternions form a 3-sphere (S^3), a higher-dimensional manifold compared to the unit circle of complex phases. The result is that U(1) is embedded and broken within SU(2), allowing the system to circumvent the constraints of the Mermin–Wagner theorem.

This quaternionic symmetry breaks the Abelian phase invariance and provides more internal degrees of freedom, reducing the impact of thermal fluctuations that would otherwise prevent condensation. Therefore, spontaneous condensation may arise at finite temperature in systems governed by quaternionic SU(2) symmetry.

Table S2 shows the comparison between the symmetry and the physical properties represented by the complex and quaternionic frameworks.

Table S2. Comparison between the complex and quaternionic zeta functions

Framework	**Symmetry**	**Order Parameter**	**Condensation at T > 0**
Complex (Standard BEC)	U(1) (Abelian)	Magnitude times phase: psi = |psi| × exp(i theta)	Forbidden in 2D (M-W theorem)
Quaternionic Extension	SU(2) (Non-Abelian)	Quaternion: $q = x + a\,e_1 + b\,e_2 + c\,e_3$,	Allowed via extended symmetry

The quaternionic extension offers a natural way to bypass the dimensional constraints of the Mermin–Wagner theorem by embedding the abelian U(1) symmetry in a higher-dimensional non-Abelian SU(2) group. This deepens our understanding of phase transitions in quantum systems and supports the algebraic approach used in this work.