Section S1. Changes in real wages

It can be obtained from equation (10):

$$P_j = \prod_{s=1}^{S} (P_j^{sF}/\alpha_j^s)^{\alpha_j^s} \tag{S1}$$

According to the definition of changing, take the logarithm of the change in real wages and substitute (S1) into the equation,

$$\begin{aligned} ln\frac{\widehat{w}_j}{\widehat{P}_j} = ln\widehat{w}_j \ - ln\widehat{P}_j \ &= ln\widehat{w}_j \ - ln\prod_{s=1}^{S} \left(\widehat{P}_j^{sF}\right)^{\alpha_j^s} \\ &= ln\widehat{w}_j \ - \sum_{s=1}^{S} \alpha_j^s \, \widehat{P}_j^{sF} \\ &= \sum_{s=1}^{S} \alpha_j^s \, ln\widehat{w}_j \ - \sum_{s=1}^{S} \alpha_j^s \, ln\widehat{P}_n^j \\ &= \sum_{s=1}^{S} \alpha_j^s \, (ln\widehat{w}_j \ - ln\widehat{P}_j^{sF}) \end{aligned} \tag{S2}$$

The model assumes that free domestic labor moves freely, then we get $\widehat{w}_j = \widehat{w}_j^s$. According to equation (17), we get $\widehat{w}_j^s$,

$$ln\hat{c}_j^s = ln\left(\widehat{w}_j^{\gamma_j^s} \prod_{r=1}^{S} \widehat{P}_j^{rs\gamma_j^{rs}}\right) = \gamma_j^s ln\widehat{w}_j \ + \gamma_j^{rs} \sum_{r=1}^{S} ln\widehat{P}_j^{rs} \tag{S3}$$

It can be obtained by converting：

$$ln\widehat{w}_j \ = \frac{1}{\gamma_j^s} ln\hat{c}_j^s - \gamma_j^{rs}/\gamma_j^s \sum_{r=1}^{S} ln\widehat{P}_j^{rs} \tag{S4}$$

The expression for $\hat{\pi}_{jj}^{sF}$ can be obtained from Equation (21):

$$\hat{\pi}_{jj}^{sF} = \left[\frac{\hat{c}_j^s \hat{\kappa}_{jj}^{sF}}{\widehat{P}_j^{sF}}\right]^{-\theta^s} \tag{S5}$$

And then it takes the log of both sides,

$$ln\hat{\pi}_{jj}^{sF} = ln\left[\frac{\hat{c}_j^s \hat{\kappa}_{jj}^{sF}}{\widehat{P}_j^{sF}}\right]^{-\theta^s} = -\theta^s ln\hat{c}_j^s - \theta^s ln\hat{\kappa}_{jj}^{sF} + \theta^s \widehat{P}_j^{sF} \tag{S6}$$

so we get:

$$ln\hat{c}_j^s = -\frac{1}{\theta^s} ln\hat{\pi}_{jj}^{sF} - ln\hat{\kappa}_{jj}^{sF} + ln\widehat{P}_j^{sF} \tag{S7}$$

Then we put $ln\frac{\widehat{w}_n}{\widehat{P}_n}$ into it:

$$
\begin{aligned}
ln\frac{\widehat{w}_n}{\widehat{P}_n} &= \sum_{s=1}^{S}\alpha_j^s\,(ln\widehat{w}_j \;\; - ln\widehat{P}_j^{sF}) = \sum_{s=1}^{S}\alpha_j^s\left(\frac{1}{\gamma_j^s}ln\widehat{c}_j^s - \frac{\gamma_j^{rs}}{\gamma_j^s}\sum_{r=1}^{S}ln\widehat{P}_j^{rs} - ln\widehat{P}_j^{sF}\right)\\
&= \sum_{s=1}^{S}\alpha_j^s\left(\frac{1}{\gamma_j^s}\left(\frac{-1}{\theta^s}ln\widehat{\pi}_{jj}^{sF} - ln\widehat{\kappa}_{jj}^{sF} + ln\widehat{P}_j^{sF}\right) - \frac{\gamma_j^{rs}}{\gamma_j^s}\sum_{r=1}^{S}ln\widehat{P}_j^{rs}\right.\\
&\quad \left. - ln\widehat{P}_j^{sF}\right)\\
&= \sum_{s=1}^{S}(\frac{-\alpha_j^s}{\theta^s}\frac{1}{\gamma_j^s}ln\widehat{\pi}_{jj}^{sF} + \frac{\alpha_j^s(1-\gamma_j^s)}{\gamma_j^s}ln\widehat{P}_j^{sF} - \alpha_j^s\frac{\gamma_j^{rs}}{\gamma_j^s}\sum_{r=1}^{S}ln\widehat{P}_j^{rs})\\
&= -\sum_{s=1}^{S}\frac{\alpha_j^s}{\theta^s}\frac{1}{\gamma_j^s}ln\widehat{\pi}_{jj}^{sF} + \sum_{s=1}^{S}\alpha_j^s\frac{\gamma_j^{rs}}{\gamma_j^s}(ln\widehat{P}_j^{sF} - \sum_{r=1}^{S}ln\widehat{P}_j^{rs}) \qquad \text{(S8)}\\
&= -\sum_{s=1}^{S}(\frac{\alpha_j^s}{\theta^s}\frac{1}{\gamma_j^s}ln\widehat{\pi}_{jj}^{sF} + \frac{\alpha_j^s}{\theta^s}ln\widehat{\pi}_{jj}^{sF} - \frac{\alpha_j^s}{\theta^s}ln\widehat{\pi}_{jj}^{sF})\\
&\quad - \sum_{s=1}^{S}\alpha_j^s\frac{\gamma_j^{rs}}{\gamma_j^s}ln\sum_{r=1}^{S}\frac{\widehat{P}_j^{rs}}{\widehat{P}_j^{sF}}\\
&= -\sum_{s=1}^{S}\frac{\alpha_j^s}{\theta^s}ln\widehat{\pi}_{jj}^{sF}\\
&\quad - \sum_{s=1}^{S}\frac{\alpha_j^s}{\theta^s}\frac{1-\gamma_j^s}{\gamma_j^s}ln\widehat{\pi}_{jj}^{sF} - \sum_{s=1}^{S}\alpha_j^s\frac{\gamma_j^{rs}}{\gamma_j^s}ln\sum_{r=1}^{S}\frac{\widehat{P}_j^{rs}}{\widehat{P}_j^{sF}}
\end{aligned}
$$

Proof finished.

Section S2. The derivation of welfare

According to equation (15), we obtain country j's income:

$$I_j = w_jL_j + R_j + D_j \qquad \text{(S9)}$$

Log-linearize the welfare

$$ln\mathrm{W}_j = lnI_j - lnP_j \qquad \text{(S10)}$$

We differentiate it to get:

$$
\begin{aligned}
dln\mathrm{W}_j = dlnI_j - dlnP_j &= \frac{dI_j}{I_j} - dlnP_j = \frac{d(w_jL_j + R_j + D_j)}{I_j} - dlnP_j\\
&= \frac{d(w_jL_j)}{I_j} + \frac{dR_j}{I_j} + \frac{dD_j}{I_j} - dlnP_j\\
&= \frac{w_jdL_j + L_jdw_j}{I_j} + \frac{dR_j}{I_j} + \frac{dD_j}{I_j} - dlnP_j = \frac{L_jdw_j}{I_j} + \frac{dR_j}{I_j} - dlnP_j \qquad \text{(S11)}\\
&= \frac{w_jL_j}{I_j}dlnw_j + \frac{R_j}{I_j}dlnR_j - dlnP_j
\end{aligned}
$$

From equation $\frac{R_j}{I_j}dlnR_j = \frac{dR_j}{I_j}$ above, $R_j = \sum_{s=1}^{S}\sum_{i=1}^{J}\tau_{ij}^{sF}M_{ij}^{sF} + \sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}\tau_{ij}^{sr}M_{ij}^{sr}$, We further represent the import and export share of country j to sector s, where M_{ij}^{sF} represents final imports from sector s of country i and $M_{ij}^{sF} = \alpha_j^sI_j\frac{\pi_{ij}^{sF}}{1+\tau_{ij}^{sF}}$. M_{ij}^{sr} represents the import of intermediate products from country i, sector s for the production of sector r in country j and $M_{ij}^{s} = \sum_{r=1}^{S}M_{ij}^{sr} = \sum_{r=1}^{S}\gamma_j^{sr}Y_j^r\frac{\pi_{ij}^{sr}}{1+\tau_{ij}^{sr}}$. With export of final goods $E_{ji}^{sF} = \alpha_i^sI_i\frac{\pi_{ji}^{sF}}{1+\tau_{ji}^{sF}}$ from country j, sector$s$, export of intermediate $E_{ji}^{s} = \sum_{r=1}^{S}E_{ji}^{sr} = \sum_{r=1}^{S}\frac{\pi_{ji}^{sr}}{1+\tau_{ji}^{sr}}\gamma_i^{sr}Y_i^r$ from country j, sector s, we obtain

$$
\begin{aligned}
dR_j = d\left(\sum_{s=1}^{S}\sum_{i=1}^{J}\tau_{ij}^{sF}M_{ij}^{sF} + \sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}\tau_{ij}^{sr}M_{ij}^{sr}\right) \\
&= \sum_{s=1}^{S}\sum_{i=1}^{J} d\left(\tau_{ij}^{sF}M_{ij}^{sF}\right) + \sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S} d(\tau_{ij}^{sr}M_{ij}^{sr}) \\
&= \sum_{s=1}^{S}\sum_{i=1}^{J}\tau_{ij}^{sF}dM_{ij}^{sF} + \sum_{s=1}^{S}\sum_{i=1}^{J}M_{ij}^{sF}d\,\tau_{ij}^{sF} + \sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}\tau_{ij}^{sr}dM_{i,}^{s} \\
&+ \sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}M_{ij}^{sr}d\tau_{ij}^{sr} \\
&= \sum_{s=1}^{S}\sum_{i=1}^{J}\tau_{ij}^{sF}M_{ij}^{sF}dlnM_{ij}^{sF} + \sum_{s=1}^{S}\sum_{i=1}^{J}M_{ij}^{sF}d\,\tau_{ij}^{sF} \\
&+ \sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}\tau_{ij}^{sr}M_{ij}^{sr}dlnM_{ij}^{sr} + \sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}M_{ij}^{sr}d\tau_{ij}^{sr} \qquad \text{(S12)} \\
&= \sum_{s=1}^{S}\sum_{i=1}^{J}\tau_{ij}^{sF}M_{ij}^{sF}dlnM_{ij}^{sF} + \sum_{s=1}^{S}\sum_{i=1}^{J}\alpha_j^s I_j\frac{\pi_{ij}^{sF}}{1+\tau_{ij}^{sF}}d\tau_{ij}^{sF} \\
&+ \sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}\tau_{ij}^{sr}M_{ij}^{sr}dlnM_{ij}^{sr} + \sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}M_{ij}^{sr}\frac{1+\tau_{ij}^{sr}}{1+\tau_{ij}^{sr}}d\tau_{ij}^{sr} \\
&= \sum_{s=1}^{S}\sum_{i=1}^{J}\tau_{ij}^{sF}M_{ij}^{sF}dlnM_{ij}^{sF} + \sum_{s=1}^{S}\sum_{i=1}^{J}\alpha_j^s I_j\frac{\pi_{ij}^{sF}}{1+\tau_{ij}^{sF}}d(1+\tau_{ij}^{sF}) \\
&+ \sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}\tau_{ij}^{sr}M_{ij}^{sr}dlnM_{ij}^{sr} + \sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}M_{ij}^{sr}\frac{1+\tau_{ij}^{sr}}{1+\tau_{ij}^{sr}}d(1+ \\
&= \sum_{s=1}^{S}\sum_{i=1}^{J}\tau_{ij}^{sF}M_{ij}^{sF}dlnM_{ij}^{sF} + \sum_{s=1}^{S}\alpha_j^s I_j\sum_{i=1}^{J}\pi_{ij}^{sF}dln\tilde{\tau}_{ij}^{sF} \\
&+ \sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}\tau_{ij}^{sr}M_{ij}^{sr}dlnM_{ij}^{sr} + \sum_{s=1}^{S}\sum_{r=1}^{S}\sum_{i=1}^{J}M_{ij}^{sr}(1+\tau_{ij}^{sr})dln\hat{}
\end{aligned}
$$

With $P_j = \prod_{s=1}^{S}(P_j^{sF}/\alpha_j^s)^{\alpha_j^s}$ and $dlnP_j$,

$$
\begin{aligned}
dlnP_j &= d\left(ln\left(\prod_{s=1}^{S}(P_j^{sF}/\alpha_j^s)^{\alpha_j^s}\right)\right) = d\left(\sum_{s=1}^{S}\alpha_j^s lnP_j^{sF} - \sum_{s=1}^{S}\alpha_j^s ln\alpha_j^s\right) \\
&= \sum_{s=1}^{S}\alpha_j^s dlnP_j^{sF} - \sum_{s=1}^{S}\alpha_j^s dln\alpha_j^s = \sum_{s=1}^{S}\alpha_j^s dlnP_j^{sF}
\end{aligned} \qquad \text{(S13)}
$$

From $P_j^{rF} = A^r\left[\sum_{i=1}^{J}T_i^r\left(c_i^r\kappa_{ij}^{rF}\right)^{-\theta^r}\right]^{-\frac{1}{\theta^r}}$ and $\pi_{ij}^{sF} = \frac{T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s}}{\sum_{i=1}^{J}T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s}}$, we can get:

$$
\begin{aligned}
dlnP_j^{sF} &= dln\left(A^s\left[\sum\nolimits_{i=1}^{J} T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s}\right]^{-\frac{1}{\theta^s}}\right)\\
&= d\left(lnA^s + (-\frac{1}{\theta^s})ln\sum\nolimits_{i=1}^{J} T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s}\right)\\
&= dlnA^s + \left(-\frac{1}{\theta^s}\right)dln(\sum\nolimits_{i=1}^{J} T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s})\\
&= dA^s/A^s + \left(-\frac{1}{\theta^s}\right)d(\sum\nolimits_{i=1}^{J} T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s})/\sum\nolimits_{i=1}^{J} T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s}\\
&= \left(-\frac{1}{\theta^s}\right)d(\sum\nolimits_{i=1}^{J} T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s})/\sum\nolimits_{i=1}^{J} T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s}\\
&= \left(-\frac{1}{\theta^s}\right)(\sum\nolimits_{i=1}^{J} \left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s}dT_i^s\\
&+\sum\nolimits_{i=1}^{J} T_i^s d\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s})/\sum\nolimits_{i=1}^{J} T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s} \qquad (S14)\\
&= \left(-\frac{1}{\theta^s}\right)(\sum\nolimits_{i=1}^{J} T_i^s d\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s})/\sum\nolimits_{i=1}^{J} T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s}\\
&= \left(-\frac{1}{\theta^s}\right)(\sum\nolimits_{i=1}^{J} T_i^s(dln\left(\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s}\right)*\left(\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s}\right)\\
&/\sum\nolimits_{i=1}^{J} T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s}\\
&= \left(-\frac{1}{\theta^s}\right)(\sum\nolimits_{i=1}^{J}(-\theta^s)\,dln\left(c_i^s\kappa_{ij}^{sF}\right)*\left(T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s}\right))\\
&/\sum\nolimits_{i=1}^{J} T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s}\\
&= \sum\nolimits_{i=1}^{J} d\left(lnc_i^s + ln\kappa_{ij}^{sF}\right)*\left(T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s}\right)/\sum\nolimits_{i=1}^{J} T_i^s\left(c_i^s\kappa_{ij}^{sF}\right)^{-\theta^s}\\
&= \sum\nolimits_{i=1}^{J} d\left(lnc_i^s + ln\kappa_{ij}^{sF}\right)*\pi_{ij}^{sF} = \sum\nolimits_{i=1}^{J}\pi_{ij}^{sF}\left(dlnc_i^s + dln\tilde{\tau}_{ij}^{sF}\right)
\end{aligned}
$$

As $dlnP_j^{sF} = \sum_{i=1}^{J}\pi_{ij}^{sF}\left(dlnc_i^s + dln\tilde{\tau}_{ij}^{sF}\right)$,we obtain $dlnP_j = \sum_{s=1}^{S}\alpha_j^s dlnP_j^{sF} = \sum_{s=1}^{S}\alpha_j^s\sum_{i=1}^{J}\pi_{ij}^{sF}\left(dlnc_i^s + dln\tilde{\tau}_{ij}^{sF}\right)$, the same with $d\,ln\,P_j^{sr} = \sum_{i=1}^{J}\pi_{ij}^{sr}\left(dlnc_i^s + dln\tilde{\tau}_{ij}^{sr}\right)$ and $c_j^s = \gamma_j^s w_j^{\gamma_j^s}\prod_{r=1}^{S}P_j^{rs\gamma_j^{rs}}$. we get:

$$
\begin{aligned}
dlnc_j^s = dln\left(\gamma_j^s w_j^{\gamma_j^s}\prod\nolimits_{r=1}^{S}P_j^{rs\gamma_j^{rs}}\right) &= d\left(ln\gamma_j^s + lnw_j^{\gamma_j^s} + \sum_{r=1}^{S}\ln P_j^{rs\gamma_j^{rs}}\right)\\
&= dln\gamma_j^s + d\left(\gamma_j^s lnw_j + \sum_{s=1}^{S}\gamma_j^{rs}\,ln\,P_j^{rs}\right) \qquad (S15)\\
&= \gamma_j^s dlnw_j + \sum_{r=1}^{S}\gamma_j^{rs}d\,ln\,P_j^{rs}
\end{aligned}
$$

$$
\begin{aligned}
dln\mathrm{W}_j = \frac{w_jL_j}{I_j}dlnw_j + \frac{R_j}{I_j}dlnR_j - dlnP_j &= \frac{w_jL_j}{I_j}dlnw_j + \frac{dR_j}{I_j} - dlnP_j\\
&= \frac{w_jL_j}{I_j}dlnw_j + \frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\tau_{ij}^{sF}M_{ij}^{sF}dlnM_{ij}^{sF}\\
&+\frac{1}{I_j}\sum_{s=1}^{S}\alpha_j^s I_j\sum_{i=1}^{J}\pi_{ij}^{sF}dln\tilde{\tau}_{ij}^{sF} + \sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}\tau_{ij}^{sr}M_{ij}^{sr}dlnM_{ij}^{sr} \qquad (S16)\\
&+\sum_{s=1}^{S}\sum_{r=1}^{S}\sum_{i=1}^{J}M_{ij}^{sr}(1+\tau_{ij}^{sr})dln\tilde{\tau}_{ij}^{sr}\\
&-\sum_{s=1}^{S}\alpha_j^s\sum\nolimits_{i=1}^{J}\pi_{ij}^{sF}\left(dlnc_i^s + dln\tilde{\tau}_{ij}^{sF}\right)
\end{aligned}
$$

First, we substitute $\alpha_j^s I_j$,

$$
\begin{aligned}
ln\mathrm{W}_j = {} & \frac{w_j L_j}{I_j} dlnw_j + \frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J} \tau_{ij}^{sF} M_{ij}^{sF}\, dlnM_{ij}^{sF} \\
& + \frac{1}{I_j}\sum_{s=1}^{S} \alpha_j^s I_j \sum_{i=1}^{J} \pi_{ij}^{sF}\, dln\tilde{\tau}_{ij}^{sF} \\
& + \frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S} \tau_{ij}^{sr} M_{ij}^{sr}\, dlnM_{ij}^{sr} \\
& + \sum_{s=1}^{S}\sum_{r=1}^{S}\sum_{i=1}^{J} M_{ij}^{sr}(1+\tau_{ij}^{sr}) dln\tilde{\tau}_{ij}^{sr} \\
& - \frac{1}{I_j}\sum_{s=1}^{S} \alpha_j^s I_j \sum\nolimits_{i=1}^{J} \pi_{ij}^{sF} \left(dlnc_i^s + dln\tilde{\tau}_{ij}^{sF}\right) \qquad \text{(S17)} \\
& = \frac{w_j L_j}{I_j} dlnw_j + \frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J} \tau_{ij}^{sF} M_{ij}^{sF}\, dlnM_{ij}^{sF} \\
& - \frac{1}{I_j}\sum_{s=1}^{S}\sum\nolimits_{i=1}^{J} M_{ij}^{sF}\,(1+\tau_{ij}^{sF}) dlnc_i^s \\
& + \frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S} \tau_{ij}^{sr} M_{ij}^{sr}\, dlnM_{ij}^{sr} \\
& + \sum_{s=1}^{S}\sum_{r=1}^{S}\sum_{i=1}^{J} M_{ij}^{sr}(1+\tau_{ij}^{sr}) dln\tilde{\tau}_{ij}^{sr}
\end{aligned}
$$

Second, we obtain export and import:

$$
\begin{aligned}
dln\mathrm{W}_j =& \frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\tau_{ij}^{sF}M_{ij}^{sF}(dlnM_{ij}^{sF}-dlnc_i^s) \\
&+\frac{1}{I_j}\sum_{s=1}^{S}\sum\nolimits_{i=1}^{J}(E_{ji}^{sF}\,dlnc_j^s - M_{ij}^{sF}\,dlnc_i^s) \\
&-\frac{1}{I_j}\sum_{s=1}^{S}\sum\nolimits_{i=1}^{J}E_{ji}^{sF}\,dlnc_j^s + \frac{w_jL_j}{I_j}dlnw_j \\
&+\frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}\tau_{ij}^{sr}M_{ij}^{sr}dlnM_{ij}^{sr} \\
&+\sum_{s=1}^{S}\sum_{r=1}^{S}\sum_{i=1}^{J}M_{ij}^{sr}(1+\tau_{ij}^{sr})dln\tilde{\tau}_{ij}^{sr} \\
&-\frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}M_{ij}^{sr}\left(1+\tau_{ij}^{sr}\right)dlnc_i^s \\
&+\frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}M_{ij}^{sr}\left(1+\tau_{ij}^{sr}\right)dlnc_i^s \\
&+\frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}E_{ji}^{sr}dlnc_j^s - \frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}E_{ji}^{sr}dlnc_j^s \\
=& \frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\tau_{ij}^{sF}M_{ij}^{sF}(dlnM_{ij}^{sF}-dlnc_i^s) \\
&+\frac{1}{I_j}\sum_{s=1}^{S}\sum\nolimits_{i=1}^{J}(E_{ji}^{sF}\,dlnc_j^s - M_{ij}^{sF}\,dlnc_i^s) \\
&+\frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}\tau_{ij}^{sr}M_{ij}^{sr}(dlnM_{ij}^{sr}-dlnc_i^s) \\
&+\frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}(E_{ji}^{sr}dlnc_j^s - M_{ij}^{sr}\,dlnc_i^s) + \frac{w_jL_j}{I_j}dlnw_j \\
&+\frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}M_{ij}^{sr}\left(1+\tau_{ij}^{sr}\right)(dln\tilde{\tau}_{ij}^{sr}+dlnc_i^s) \\
&-\frac{1}{I_j}\sum_{s=1}^{S}\sum\nolimits_{i=1}^{J}E_{ji}^{sF}\,dlnc_j^s - \frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}E_{ji}^{sr}dlnc_j^s
\end{aligned}
\tag{S18}
$$

Then we eliminate the residual term:

$$
\begin{aligned}
\frac{w_jL_j}{I_j}dlnw_j &+ \frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}M_{ij}^{sr}\left(1+\tau_{ij}^{sr}\right)(dln\tilde{\tau}_{ij}^{sr}+dlnc_i^s) \\
&-\frac{1}{I_j}\sum_{s=1}^{S}\sum\nolimits_{i=1}^{J}E_{ji}^{sF}\,dlnc_j^s - \frac{1}{I_j}\sum_{s=1}^{S}\sum_{i=1}^{J}\sum_{r=1}^{S}E_{ji}^{sr}dlnc_j^s = 0
\end{aligned}
\tag{S19}
$$

$$
\begin{aligned}
0 &= \frac{w_j L_j}{I_j} dlnw_j + \frac{1}{I_j}\sum_{s=1}^{S}\sum_{r=1}^{S}\gamma_j^{sr} Y_j^r \sum_{i=1}^{J}\pi_{ij}^{sr}(dln\tilde{\tau}_{ij}^{sr} + dlnc_i^s) \\
&- \frac{1}{I_j}\sum_{s=1}^{S}\sum\nolimits_{i=1}^{J}\left(\sum_{r=1}^{S} E_{ji}^{sr} + E_{ji}^{sF}\right) dlnc_j^s \\
&= \frac{w_j L_j}{I_j} dlnw_j + \frac{1}{I_j}\sum_{s=1}^{S}\sum_{r=1}^{S}\gamma_j^{sr} Y_j^r \, d\,ln\,P_j^{sr} \\
&- \frac{1}{I_j}\sum_{s=1}^{S} Y_j^s dlnc_j^s \\
&= \frac{w_j L_j}{I_j} dlnw_j \\
&- \frac{1}{I_j}\sum_{s=1}^{S}\gamma_j^s Y_j^s\left(\frac{1}{\gamma_j^s} dlnc_j^s - \sum_{r=1}^{S}\frac{\gamma_j^{sr}}{\gamma_j^s} d\,ln\,P_j^{rs}\right) \\
&= \frac{w_j L_j}{I_j} dlnw_j - \frac{w_j L_j}{I_j} dlnw_j = 0
\end{aligned}
\tag{S20}
$$

Proof finished.