

Supplementary File: Proofs

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In this supplementary document, we provide the complete proofs of Propositions 1 and 2 in the main article.

Proof of Proposition 1. Let $X \sim \text{Lindley}(\theta)$ and $x \in \mathbb{R}$. Then;

$$\begin{aligned} E|x - X| &= \int_0^\infty |x - y|f_X(y)dy \\ &= \int_0^x (x - y)f_X(y)dy + \int_x^\infty (y - x)f_X(y)dy \\ &= x \int_0^x f_X(y)dy - \int_0^x yf_X(y)dy + \int_x^\infty yf_X(y)dy - x \int_x^\infty f_X(y)dy \\ &= xF_X(x) - \int_0^x yf_X(y)dy + \left(\int_0^\infty yf_X(y)dy - \int_0^x yf_X(y)dy \right) - x(1 - F_X(x)) \\ &= 2xF(x) - x + E(X) - 2 \int_0^x yf_X(y)dy \end{aligned}$$

For the integral part in the above expectation, we have

$$\begin{aligned} \int_0^x yf_X(y)dy &= \int_0^x y\left(\frac{\theta^2}{1+\theta}\right)(1+y)e^{-\theta y}dy \\ &= \frac{\theta^2}{1+\theta} \int_0^x (ye^{-\theta y} + y^2e^{-\theta y})dy \\ &= \frac{\theta}{1+\theta} \left[(-xe^{-\theta x} - x^2e^{-\theta x}) + \left(\frac{-e^{-\theta x} + 1 - 2xe^{-\theta x}}{\theta} \right) - \frac{2xe^{-\theta x} - 2}{\theta^2} \right] \end{aligned}$$

Since the CDF and mean of the Lindley distribution are explicitly given in Eq. (2) and

Eq. (3), respectively, the distance expectation $E|x - X|$ is thus given as

$$\begin{aligned}
E|x - X| &= 2xF(x) - x + E(X) - 2 \int_0^x y f_X(y) dy \\
&= 2x \left(1 - \left(\frac{1 - \theta - \theta x}{1 + \theta}\right)\right) e^{-\theta x} - x + \frac{\theta + 2}{\theta(1 + \theta)} \\
&\quad - \frac{2\theta}{1 + \theta} \left[(-xe^{-\theta x} - x^2 e^{-\theta x}) + \left(\frac{-e^{-\theta x} + 1 - 2xe^{-\theta x}}{\theta}\right) - \frac{2xe^{-\theta x} - 2}{\theta^2} \right] \\
&= x + \frac{2xe^{-\theta x}}{1 + \theta} + \frac{2e^{-\theta x}}{1 + \theta} + \frac{4e^{-\theta x}}{\theta(1 + \theta)} + \frac{\theta - 2}{\theta(1 + \theta)} - \frac{2}{1 + \theta}.
\end{aligned}$$

□

Proof of Proposition 2. Let $X, X' \stackrel{iid}{\sim} \text{Lindley}(\theta)$.

$$\begin{aligned}
E|X - X'| &= \int_0^\infty \int_0^\infty |x - y| f_X(x) f_{X'}(y) dy dx \\
&= 2 \int_0^\infty \int_0^x (x - y) f_X(x) f_{X'}(y) dy dx \\
&= 2 \int_0^\infty \int_0^x (x - y) \frac{\theta^2}{1 + \theta} (1 + x) e^{-\theta x} \frac{\theta^2}{1 + \theta} (1 + y) e^{-\theta y} dy dx \\
&= 2m^2 \int_0^\infty \int_0^x (x + x^2 + x^2 y - y - y^2 - xy^2) e^{-\theta x} e^{-\theta y} dy dx, \quad m = \frac{\theta^2}{1 + \theta}.
\end{aligned}$$

We calculate each term in the above double integral separately using Integration-by-Parts techniques.

Term 1: Consider the integral

$$\int_0^\infty \int_0^x x e^{-\theta x} e^{-\theta y} dy dx.$$

Then by use of integration-by-parts, let $u = x$, and $dv = e^{-\theta x} dx$, we have

$$\begin{aligned}
\int_0^\infty \int_0^x x e^{-\theta x} e^{-\theta y} dy dx &= -\frac{1}{\theta} \int_0^\infty x e^{-2\theta x} - x e^{-\theta x} dx \\
&= -\frac{1}{\theta} \left[\int_0^\infty x e^{-2\theta x} - \left(-\frac{x e^{-\theta}}{\theta} \Big|_0^\infty \right) - \int_0^\infty -\frac{1}{\theta} e^{-\theta x} dx \right] \\
&= -\frac{1}{\theta} \left[\int_0^\infty x e^{-2\theta x} dx - \frac{1}{\theta^2} \right]
\end{aligned}$$

let $u = x$, and $dv = e^{-2\theta x} dx$ and apply Integration-by-parts

$$\begin{aligned}
&= -\frac{1}{\theta} \left(\left(-\frac{x e^{-2\theta x}}{2\theta} \Big|_0^\infty \right) + \int_0^\infty \frac{1}{2\theta} e^{-2\theta x} dx - \frac{1}{\theta^2} \right) \\
&= -\frac{1}{\theta} \left[\frac{1}{4\theta^2} - \frac{1}{\theta^2} \right] \\
&= \frac{3}{4\theta^3}.
\end{aligned}$$

Term 2: Consider the integral and we use integration-by-parts as follows.

$$\begin{aligned}
\int_0^\infty \int_0^x x^2 e^{-\theta x} e^{-\theta y} dy dx &= -\frac{1}{\theta} \int_0^\infty x^2 e^{-2\theta x} - x^2 e^{-\theta x} dx \\
&\text{let } u = x^2, \text{ and } dv = e^{-2\theta x} dx \text{ and apply I.b.p} \\
&= -\frac{1}{\theta} \left(\left(\frac{x^2 e^{-2\theta x}}{-2\theta} \right) \Big|_0^\infty + \frac{1}{\theta} \int_0^\infty x e^{-2\theta x} dx - \int_0^\infty x^2 e^{-\theta x} dx \right) \\
&= -\frac{1}{\theta} \left(\frac{1}{\theta} \int_0^\infty x e^{-2\theta x} dx - \int_0^\infty x^2 e^{-\theta x} dx \right) \\
&\text{Apply Integration by Parts with } u = x, \text{ and } dv = e^{-2\theta x} dx \\
&= -\frac{1}{\theta} \left(\frac{1}{\theta} \left[\frac{x e^{-2\theta x}}{-2\theta} \right] \Big|_0^\infty + \int_0^\infty \frac{1}{2\theta} e^{-2\theta x} dx \right) - \int_0^\infty x^2 e^{-\theta x} dx \\
&= -\frac{1}{\theta} \left(\frac{1}{4\theta^3} - \int_0^\infty x^2 e^{-\theta x} dx \right) \\
&\text{Apply Integration by Parts with } u = x^2 \text{ and } dv = e^{-\theta x} dx \\
&= -\frac{1}{\theta} \left(\frac{1}{4\theta^3} - \left[-\frac{x^2 e^{-\theta x}}{\theta} \right] \Big|_0^\infty + \int_0^\infty \frac{2x}{\theta} e^{-\theta x} dx \right) \\
&= -\frac{1}{\theta} \left(\frac{1}{4\theta^3} + \frac{2}{\theta} \int_0^\infty x e^{-\theta x} dx \right) \\
&= -\frac{1}{\theta} \left[\frac{1}{4\theta^3} - \frac{2}{\theta} \left[-\frac{x e^{-\theta x}}{\theta} \right] \Big|_0^\infty + \int_0^\infty \frac{1}{\theta} e^{-\theta x} dx \right] \\
&= -\frac{1}{\theta} \left[\frac{1}{4\theta^3} - \frac{2}{\theta^3} \right] = \frac{7}{4\theta^4}.
\end{aligned}$$

Term 3: Consider the following integral,

$$\int_0^\infty \int_0^x x^2 y e^{-\theta x} e^{-\theta y} dy dx.$$

Let $u = y$, $dv = e^{-\theta y} dy$ and apply Integration-by-Parts. Then, we have

$$\begin{aligned}
\int_0^\infty \int_0^x x^2 y e^{-\theta x} e^{-\theta y} dy dx &= \int_0^\infty x^2 e^{-\theta x} \left[-\frac{y e^{-\theta y}}{\theta} \right] \Big|_0^x - \int_0^x -\frac{1}{\theta} e^{-\theta y} dy dx \\
&= \int_0^\infty x^2 e^{-\theta x} \left[-\frac{x e^{-\theta x}}{\theta} + \frac{1}{\theta} \int_0^x e^{-\theta y} dy \right] dx \\
&= \int_0^\infty x^2 e^{-\theta x} \left[\frac{x e^{-\theta x}}{\theta} - \frac{e^{-\theta x}}{\theta^2} + \frac{1}{\theta^2} \right] dx \\
&= -\frac{1}{\theta} \int_0^\infty x^3 e^{-2\theta x} dx - \frac{1}{\theta^2} \int_0^\infty x^2 e^{-2\theta x} dx + \frac{1}{\theta^2} \int_0^\infty x^2 e^{-\theta x} dx
\end{aligned}$$

To evaluate the above integrals, we will use the property of the Gamma function $\Gamma(\alpha)$ for a nonnegative $\alpha \in \mathbb{R}$ that

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt,$$

and by the substitution technique, let $t = bx$ then lead to the scaled property so that

$$\int_0^\infty x^k e^{-bx} dx = \frac{\Gamma(k+1)}{b^{k+1}}, \quad \text{for } k > -1. \quad (1)$$

With appropriate values of b and k , we apply the property in Eq. (1) to solve each term in the expression as follows.

$$\begin{aligned} \int_0^\infty \int_0^x x^2 y e^{-\theta x} e^{-\theta y} dy dx &= -\frac{1}{\theta} \int_0^\infty x^3 e^{-2\theta x} dx - \frac{1}{\theta^2} \int_0^\infty x^2 e^{-2\theta x} dx + \frac{1}{\theta^2} \int_0^\infty x^2 e^{-\theta x} dx \\ &= -\frac{1}{\theta} \left[\frac{6}{16\theta^4} \right] - \frac{1}{\theta^2} \left[\frac{2}{8\theta^3} \right] + \frac{1}{\theta} \left[\frac{2}{\theta^3} \right] \\ &= -\frac{3}{8\theta^5} - \frac{1}{4\theta^5} + \frac{2}{\theta^5} \\ &= \frac{11}{8\theta^5}. \end{aligned}$$

Term 4: Consider the following integral and use integration-by-parts. Let $u = y$ and $dv = e^{-\theta y} dy$. Thus, we have

$$\begin{aligned} \int_0^\infty \int_0^x y e^{-\theta x} e^{-\theta y} dy dx &= \int_0^\infty e^{-\theta x} \left(-\frac{y e^{-\theta y}}{\theta} \Big|_0^x + \int_0^x \frac{1}{\theta} e^{-\theta y} dy \right) dx \\ &= \frac{1}{\theta} \int_0^\infty e^{-\theta x} \left(-x e^{-\theta x} - \frac{e^{-\theta x}}{\theta} + \frac{1}{\theta} \right) dx \\ &= -\frac{1}{\theta} \left(\int_0^\infty x e^{-2\theta x} dx + \int_0^\infty \frac{e^{-2\theta x}}{\theta} dx - \int_0^\infty \frac{e^{-\theta x}}{\theta} dx \right) \\ &\text{Let } u = x \text{ and } dv = e^{-2\theta x} dx \text{ and use Integration-by-parts} \\ &= -\frac{1}{\theta} \left(-\frac{x}{2\theta} e^{-2\theta x} \Big|_0^\infty + \int_0^\infty \frac{1}{2\theta} e^{-2\theta x} dx \right) - \frac{1}{\theta} \left(-\frac{1}{2\theta} e^{-2\theta x} \Big|_0^\infty \right) \\ &\quad + \frac{1}{\theta} \left(e^{-\theta x} \Big|_0^\infty \right) = -\frac{1}{\theta} \left(\frac{1}{4\theta^2} + \frac{1}{2\theta^2} - \frac{1}{\theta^2} \right) = \frac{1}{4\theta^3}. \end{aligned}$$

Term 5: Consider the following integral and us Integration-by-Parts. Let $u = y^2$ and

$dv = e^{-\theta y} dy$. We then have

$$\begin{aligned} \int_0^\infty \int_0^x y^2 e^{-\theta x} e^{-\theta y} dy dx &= \int_0^\infty e^{-\theta x} \left(-\frac{y^2 e^{-\theta y}}{\theta} \Big|_0^x + \frac{2}{\theta} \int_0^x y e^{-\theta y} dy \right) dx \\ &= \int_0^\infty e^{-\theta x} \left(-\frac{x^2 e^{-\theta x}}{\theta} + \frac{2}{\theta} \int_0^x y e^{-\theta y} dy \right) dx \\ &= \int_0^\infty e^{-\theta x} \left(-\frac{x^2 e^{-\theta x}}{\theta} + \frac{2}{\theta} \left[-\frac{y e^{-\theta y}}{\theta} \Big|_0^x + \frac{1}{\theta} \int_0^x e^{-\theta y} dy \right] \right) dx \end{aligned}$$

Apply Integration-by-parts with $u = y$ and $dv = e^{-\theta y} dy$.

$$\begin{aligned} &= \int_0^\infty e^{-\theta x} \left(-\frac{x^2 e^{-\theta x}}{\theta} + \frac{2}{\theta} \left[-\frac{x e^{-\theta x}}{\theta} - \frac{e^{-\theta x}}{\theta^2} + \frac{1}{\theta^2} \right] \right) dx \\ &= -\frac{1}{\theta} \int_0^\infty x^2 e^{-2\theta x} dx - \frac{2}{\theta^2} \int_0^\infty x e^{-2\theta x} dx - \frac{2}{\theta^3} \int_0^\infty e^{-2\theta x} dx \\ &\quad + \frac{2}{\theta^3} \int_0^\infty e^{-\theta x} dx = \left[-\frac{2}{8\theta^4} - \frac{2}{4\theta^4} - \frac{1}{\theta^4} + \frac{2}{\theta^4} \right] = \frac{1}{4\theta^4}. \end{aligned}$$

Term 6: Consider the integral

$$\int_0^\infty \int_0^x -xy^2 e^{-\theta x} e^{-\theta y} dy dx.$$

From the previous derivation in Term 5, we see that the integration with respect to y will give us the following.

$$\begin{aligned} \int_0^\infty x \left(\int_0^x y^2 e^{-\theta x} e^{-\theta y} dy \right) dx &= \int_0^\infty x e^{-\theta x} \left(-\frac{x^2 e^{-\theta x}}{\theta} + \frac{2}{\theta} \left[-\frac{x e^{-\theta x}}{\theta} - \frac{e^{-\theta x}}{\theta^2} + \frac{1}{\theta^2} \right] \right) dx \\ &= -\frac{1}{\theta} \int_0^\infty x^3 e^{-2\theta x} dx - \frac{2}{\theta^2} \int_0^\infty x^2 e^{-2\theta x} dx - \frac{2}{\theta^3} \int_0^\infty x e^{-2\theta x} dx \\ &\quad + \frac{2}{\theta^3} \int_0^\infty x e^{-\theta x} dx \end{aligned}$$

Apply the same properties of Gamma function from Term 3

$$\begin{aligned} &= -\frac{1}{\theta} \left(\frac{3!}{(2\theta)^4} \right) - \frac{2}{\theta^2} \left(\frac{2!}{(2\theta)^3} \right) - \frac{2}{\theta^3} \left(\frac{1}{(2\theta)^2} \right) + \frac{2}{\theta^3} \left(\frac{1}{\theta^2} \right) \\ &= -\frac{6}{16\theta^5} - \frac{8}{16\theta^5} - \frac{8}{16\theta^5} + \frac{32}{16\theta^5} = \frac{5}{8\theta^5}. \end{aligned}$$

Thus, combining all these terms we obtain the distance expectation $E|X - Y|$ as

follows.

$$\begin{aligned} E|X - Y| &= 2m^2 \int_0^\infty \int_0^x (x + x^2 + x^2y - y - y^2 - xy^2)e^{-\theta x}e^{-\theta y}dydx \\ &= 2m^2 \left(\frac{3}{4\theta^3} + \frac{7}{4\theta^4} - \frac{3}{8\theta^5} - \frac{1}{4\theta^5} + \frac{2}{\theta^5} - \frac{1}{4\theta^3} - \frac{1}{4\theta^4} - \frac{5}{8\theta^5} \right) \end{aligned}$$

Using the fact that $m = \frac{\theta^2}{1 + \theta}$

$$= 2 \left(\frac{\theta^2}{1 + \theta} \right)^2 \left(\frac{2}{4\theta^3} + \frac{6}{4\theta^4} + \frac{6}{8\theta^5} \right) = \frac{2\theta^2 + 6\theta + 3}{2\theta(1 + \theta)^2}.$$

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