Supplementary Material

A Smoothed Three Part Redescending M-Estimator

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1 Introduction

Closed form equations for asymptotic variance are not always possible. Here we derive equalities and equations to derive the asymptotic values and variance in terms of the normal cdf and pdf. These equalities are shown for the three-part, smoothed three-part, and Bachmaier ψ-functions, however could be applied to any ψ-function that is defined by polynomials. The proceeding asymptotic variance and value equalities have been translated to R code, which is computed more efficiently than numerical integration.

2 Contamination model

Let G be the contamination model of normal distributions defined by

$$G(x) = (1-\epsilon)\Phi(x) + \epsilon\Phi\left(\frac{x-\mu_2}{\alpha}\right)$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ and $\Phi(x) = \int_{-\infty}^{x}\phi(t)\ dt$. It follows then that the differential

$$dG(x) = (1-\epsilon)\phi(x)dx + \frac{\epsilon}{\alpha}\phi\left(\frac{x-\mu_2}{\alpha}\right)dx. \tag{1}$$

3 Asymptotic Variance

The asymptotic variance for simultaneous location and scale estimation is given by

$$\Sigma(\boldsymbol{\tau}, G) = M(\boldsymbol{\tau}, G)^{-1}\left\{\int_{-\infty}^{\infty}\Psi\left(x, \boldsymbol{\tau}\right)\Psi\left(x, \boldsymbol{\tau}\right)^T\ dG(x)\right\}\left\{M(\boldsymbol{\tau}, G)^{-1}\right\}^T = \begin{bmatrix}\nu_{11} & \nu_{12}\\ \nu_{21} & \nu_{22}\end{bmatrix}$$

Where, for $\boldsymbol{\tau} = (\mu, \sigma)^T$ and $\Psi\left(x, \boldsymbol{\tau}\right) = \Psi\left(\frac{x-\mu}{\sigma}\right)$, the matrices may be further defined as

follows:

$$M(\boldsymbol{\tau}, G) = \int \left\{ \frac{\partial}{\partial \boldsymbol{\tau}} \Psi(x, \boldsymbol{\tau}) \right\} \, dG(x)$$

$$= \begin{bmatrix} \int \frac{\partial}{\partial \mu} \psi_1 \left(\frac{x-\mu}{\sigma} \right) \, dG(x) & \int \frac{\partial}{\partial \sigma} \psi_1 \left(\frac{x-\mu}{\sigma} \right) \, dG(x) \\ \int \frac{\partial}{\partial \mu} \psi_2 \left(\frac{x-\mu}{\sigma} \right) \, dG(x) & \int \frac{\partial}{\partial \sigma} \psi_2 \left(\frac{x-\mu}{\sigma} \right) \, dG(x) \end{bmatrix}$$

$$= -\frac{1}{\sigma} \begin{bmatrix} \int \psi_1' \left(\frac{x-\mu}{\sigma} \right) \, dG(x) & \int \frac{x-\mu}{\sigma} \psi_1' \left(\frac{x-\mu}{\sigma} \right) \, dG(x) \\ \int \psi_2' \left(\frac{x-\mu}{\sigma} \right) \, dG(x) & \int \frac{x-\mu}{\sigma} \psi_2' \left(\frac{x-\mu}{\sigma} \right) \, dG(x) \end{bmatrix}$$

$$= \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

$$\int \Psi(x, \boldsymbol{\tau}) \, \Psi(x, \boldsymbol{\tau})^T \, dG(x) = \begin{bmatrix} \int \left[\psi_1 \left(\frac{x-\mu}{\sigma} \right) \right]^2 dG(x) & \int \psi_1 \left(\frac{x-\mu}{\sigma} \right) \psi_2 \left(\frac{x-\mu}{\sigma} \right) dG(x) \\ \int \psi_1 \left(\frac{x-\mu}{\sigma} \right) \psi_2 \left(\frac{x-\mu}{\sigma} \right) dG(x) & \int \left[\psi_2 \left(\frac{x-\mu}{\sigma} \right) \right]^2 dG(x) \end{bmatrix}$$

$$= \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

4 Useful Integral Equalities

Let $g_0(a, b) = \int_a^b \phi(x) dx = \Phi(b) - \Phi(a)$, and $g_1(a, b) = \int_a^b x\phi(x) dx = \phi(a) - \phi(b)$. The former is true by definition, and the latter with integration by parts.

Proof. Let

$$\begin{array}{ll} u = 1 & du = 0 \\ v = -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} & dv = x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \end{array}$$

then

$$\int_a^b x\phi(x)dx = \int_a^b x\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = -\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\Big|_a^b - 0 = \phi(a) - \phi(b)$$

□

Theorem 1. *For $n = 2, 3, 4, \ldots$,*

$$\begin{aligned} g_n(a,b) &= \int_a^b x^n\phi(x)dx \\ &= a^{n-1}\phi(a) - b^{n-1}\phi(b) + (n-1)g_{n-2}(a,b) \end{aligned} \tag{2}$$

Proof. We prove this by induction. There are two base cases. For $n = 2$, using integration by parts

$$\int_a^b x^2\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = -x\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\Big|_a^b + \underbrace{\int_a^b \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx}_{g_0(a,b)} = a\phi(a) - b\phi(b) + g_0(a,b)$$

For $n = 3$, again with integration by parts

$$\int_a^b x^3\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = -x^2\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\Big|_a^b + 2\underbrace{\int_a^b x\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx}_{g_1(a,b)} = a^2\phi(a) - b^2\phi(b) + 2g_1(a,b)$$

Now assume that for some $k \in \mathbb{N} > 2$ that (2) is true. It follows that

$$(k+1)g_k + a^{k+1}\phi(a) - b^{k+1}\phi(b) = (k+1)\int_a^b x^k\phi(x)dx - x^{k+1}\phi(x)\Big|_a^b$$

Letting $u = x^{k+1}$, $du = (k+1)x^k dx$, $v = -\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, and $dv = x\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$, we have

$$\begin{aligned} (k+1)\int_a^b x^k\phi(x)dx - x^{k+1}\phi(x)\Big|_a^b &= -\int_a^b v\ du + uv\Big|_a^b = \int_a^b u\ dv \\ &= \int_a^b x^{k+2}\phi(x)dx = g_{k+2}(a,b) \end{aligned}$$

Since the base cases and inductive step are proven, (2) is true for all $n \in \mathbb{N} > 2$, thus establishing Theorem 1. □

Theorem 2. *For* $n = 0, 1, 2, \ldots$

$$\mathrm{S}_g^n(a, b, \mu, \sigma) = \int_a^b \left(\frac{x-\mu}{\sigma}\right)^n \phi(x)dx = \sum_{k=0}^{n} \binom{n}{k} \frac{(-\mu)^k}{\sigma^n} g_{n-k}(a, b)$$

Proof. The term $\left(\frac{x-\mu}{\sigma}\right)^n = \frac{1}{\sigma^n}(x-\mu)^n$ may be written as a binomial expansion,

$$\frac{1}{\sigma^n} \sum_{k=0}^{n} \binom{n}{k} (-\mu)^k x^{n-k}.$$

As such, and making use of Theorem 1, it follows that

$$\begin{aligned}
&\int_a^b \left(\frac{x-\mu}{\sigma}\right)^n \phi(x)dx \\
&\quad = \frac{1}{\sigma^n}\left(\int_a^b x^n\phi(x)dx - \binom{n}{1}\mu\int_a^b x^{n-1}\phi(x)dx + \cdots \right. \\
&\qquad\qquad \left. + \binom{n}{n-1}\mu^{n-1}\int_a^b x\phi(x)dx - \mu^n\int_a^b \phi(x)dx\right) \\
&\quad = \frac{1}{\sigma^n}\left(g_n(a,b) - \binom{n}{1}\mu g_{n-1}(a,b) + \cdots + \binom{n}{n-1}\mu^{n-1}g_1(a,b) - \mu^n g_0(a,b)\right) \\
&\quad = \sum_{k=0}^{n}\binom{n}{k}\frac{(-\mu)^k}{\sigma^n}g_{n-k}(a,b)
\end{aligned}$$

□

Theorem 3. *For* $n = 0, 1, j...,$

$$h_n(a, b, \mu_2, \alpha) = \int_a^b (x-\mu_2)^n \phi\left(\frac{x-\mu_2}{\alpha}\right) dx = \alpha^{n+1} g_n\left(\frac{a-\mu_2}{\alpha}, \frac{b-\mu_2}{\alpha}\right)$$

Proof. Let $y = \frac{x-\mu_2}{\alpha}$, which implies $dx = \alpha dy$, then

$$\begin{aligned}
\int_a^b (x-\mu_2)^n \phi\left(\frac{x-\mu_2}{\alpha}\right) dx &= \int_a^b (x-\mu_2)^n \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu_2)^2}{2\alpha^2}} dx \\
&= \alpha^n \int_a^b \left(\frac{x-\mu_2}{\alpha}\right)^n \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu_2)^2}{2\alpha^2}} dx \\
&= \alpha^{n+1} \int_{\frac{a-\mu_2}{\alpha}}^{\frac{b-\mu_2}{\alpha}} y^n \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \qquad \text{(substitution)} \\
&= \alpha^{n+1} g_n\left(\frac{a-\mu_2}{\alpha}, \frac{b-\mu_2}{\alpha}\right) \qquad \text{(Theorem 2)}
\end{aligned}$$

□

Theorem 4. *For* $n = 0, 1, \ldots,$

$$\mathrm{S}_h^n(a, b, \mu_2, \alpha) = \int_a^b x^n \phi\left(\frac{x - \mu_2}{\alpha}\right) dx = \sum_{k=0}^{n} \binom{n}{k} \mu_2^k h_{n-k}(a, b, \mu_2, \alpha)$$

Proof.

$$\begin{aligned}
\int_a^b x^n \phi\left(\frac{x - \mu_2}{\alpha}\right) dx &= \int_a^b \left((x - \mu_2) + \mu_2\right)^n \phi\left(\frac{x - \mu_2}{\alpha}\right) dx \\
&= \int_a^b \sum_{k=0}^{n} \binom{n}{k} \mu_2^k (x - \mu_2)^{n-k} \phi\left(\frac{x - \mu_2}{\alpha}\right) dx \qquad \text{(Binomial Theorem)} \\
&= \sum_{k=0}^{n} \binom{n}{k} \mu_2^k \left[\int_a^b (x - \mu_2)^{n-k} \phi\left(\frac{x - \mu_2}{\alpha}\right) dx\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} \mu_2^k h_{n-k}(a, b, \mu_2, \alpha) \qquad \text{(Theorem 3)}
\end{aligned}$$

□

5 Smoothed Three-Part Redescender

5.1 Location

Inserting polynomial smoothing functions in the 2δ regions around a, b, and c yields the following ψ function for location:

$$\psi_1 = \begin{cases} x & |x| \leq (a-\delta) \\ (\alpha_{1,4}x^4 + \alpha_{1,2}x^2 + \alpha_{1,0})\,\mathrm{sign}\,(x) + \alpha_{1,3}x^3 + \alpha_{1,1}x & (a-\delta) < |x| \leq (a+\delta) \\ a\,\mathrm{sign}\,(x) & (a+\delta) \leq |x| \leq (b-\delta) \\ (\beta_{1,4}x^4 + \beta_{1,2}x^2 + \beta_{1,0})\,\mathrm{sign}\,(x) + \beta_{1,3}x^3 + \beta_{1,1}x & (b-\delta) < |x| \leq (b+\delta) \\ a\,(c\,\mathrm{sign}\,(x) - x)\,/(c-b) & (b+\delta) \leq |x| \leq (c-\delta) \\ (\gamma_{1,4}x^4 + \gamma_{1,2}x^2 + \gamma_{1,0})\,\mathrm{sign}\,(x) + \gamma_{1,3}x^3 + \gamma_{1,1}x & (c-\delta) < |x| \leq (c+\delta) \\ 0 & \text{otherwise} \end{cases}$$

$$\psi_1' = \begin{cases} 1 & |x| \leq (a-\delta) \\ (4\alpha_{1,4}x^3 + 2\alpha_{1,2}x)\,\mathrm{sign}\,(x) + 3\alpha_{1,3}x^2 + \alpha_{1,1} & (a-\delta) < |x| \leq (a+\delta) \\ (4\beta_{1,4}x^3 + 2\beta_{1,2}x)\,\mathrm{sign}\,(x) + 3\beta_{1,3}x^2 + \beta_{1,1} & (b-\delta) < |x| \leq (b+\delta) \\ -a/(c-b) & (b+\delta) \leq |x| \leq (c-\delta) \\ (4\gamma_{1,4}x^3 + 2\gamma_{1,2}x)\,\mathrm{sign}\,(x) + 3\gamma_{1,3}x^2 + \gamma_{1,1} & (c-\delta) < |x| \leq (c+\delta) \\ 0 & \text{otherwise} \end{cases}$$

with

$$\begin{aligned} \alpha_{1,4} &= \zeta \\ \alpha_{1,3} &= -4a\zeta \\ \alpha_{1,2} &= 6\left(a^2 - \delta^2\right)\zeta \\ \alpha_{1,1} &= -4\left(a^3 - 3a\delta^2 - 2\delta^3\right)\zeta \\ \alpha_{1,0} &= \left(a^4 - 6a^2\delta^2 + 8a\delta^3 - 3\delta^4\right)\zeta \end{aligned}$$

$$\beta_{1,4} = -a\xi$$

$$\beta_{1,3} = 4ab\xi$$

$$\beta_{1,2} = -6a(b^2 - \delta^2)\xi$$

$$\beta_{1,1} = 4a\left(b^3 - 3b\delta^2 + 2\delta^3\right)\xi$$

$$\beta_{1,0} = -a(b^4 - 6b^2\delta^2 - 8b\delta^3 - 3\delta^4 + 16c\delta^3)\xi$$

$$\gamma_{1,4} = a\xi$$

$$\gamma_{1,3} = -4ac\xi$$

$$\gamma_{1,2} = 6a\left(c^2 - \delta^2\right)\xi$$

$$\gamma_{1,1} = -4a(c^3 - 3c\delta^2 - 2\delta^3)\xi$$

$$\gamma_{1,0} = a\left(c^4 - 6c^2\delta^2 - 8c\delta^3 - 3\delta^4\right)\xi$$

where $\zeta = (16\delta^3)^{-1}$, and $\xi = (16\delta^3(b-c))^{-1}$.

5.2 Scale

The smoothing procedure for the scale ψ-function yields the following definition:

$$\psi_2 = \begin{cases} x^2 - 1 - P & |x| \le (a-\delta) \\ (\alpha_{2,5}x^5 + \alpha_{2,3}x^3 + \alpha_{2,1}x)\,\mathrm{sign}\,(x) + \alpha_{2,4}x^4 + \alpha_{2,2}x^2 + \alpha_{2,0} & (a-\delta) \le |x| \le (a+\delta) \\ \eta & (a+\delta) \le |x| \le (b-\delta) \\ \beta_{2,4}x^4 + \beta_{2,2}x^2 + (\beta_{2,3}x^3 + \beta_{2,1}x)\,\mathrm{sign}\,(x) + \beta_{2,0} & (b-\delta) < |x| \le (b+\delta) \\ \eta\,(c - x\,\mathrm{sign}\,(x))\,/(c-b) & (b+\delta) \le |x| \le (c-\delta) \\ \gamma_{2,4}x^4 + \gamma_{2,2}x^2 + (\gamma_{2,3}x^3 + \gamma_{2,1}x)\,\mathrm{sign}\,(x) + \gamma_{2,0} & (c-\delta) \le |x| \le (c+\delta) \\ 0 & \text{otherwise} \end{cases}$$

$$\psi_2' = \begin{cases} 2x & |x| \le (a-\delta) \\ (5\alpha_{2,5}x^4 + 3\alpha_{2,3}x^2 + \alpha_{2,1})\,\mathrm{sign}\,(x) + 4\alpha_{2,4}x^3 + 2\alpha_{2,2}x & (a-\delta) \le |x| \le (a+\delta) \\ 4\beta_{2,4}x^3 + 2\beta_{2,2}x + (3\beta_{2,3}x^2 + \beta_{2,1})\,\mathrm{sign}\,(x) & (b-\delta) < |x| \le (b+\delta) \\ -\eta\,\mathrm{sign}\,(x)\,/(c-b) & (b+\delta) \le |x| \le (c-\delta) \\ 4\gamma_{2,4}x^3 + 2\gamma_{2,2}x + (3\gamma_{2,3}x^2 + \gamma_{2,1})\,\mathrm{sign}\,(x) & (c-\delta) \le |x| \le (c+\delta) \\ 0 & \text{otherwise} \end{cases}$$

where

$$\alpha_{2,5} = \zeta$$

$$\alpha_{2,4} = -3a\zeta$$

$$\alpha_{2,3} = 2(a^2 - 3\delta^2)\zeta$$

$$\alpha_{2,2} = 2(a^3 + 3a\delta^2 + 4\delta^3)\zeta$$

$$\alpha_{2,1} = -3(a^4 - 2a^2\delta^2 + \delta^4)\zeta$$

$$\alpha_{2,0} = -\left(16P\delta^3 + 3a\delta^4 - a^5 + 16\delta^3 - 8a^2\delta^3 + 6a^3\delta^2\right)\zeta$$

$$\beta_{2,4} = -\eta\xi$$

$$\beta_{2,3} = 4b\eta\xi$$

$$\beta_{2,2} = -6(b^2 - \delta^2)\eta\xi$$

$$\beta_{2,1} = 4(b^3 + 2\delta^3 - 3b\delta^2)\eta\xi$$

$$\beta_{2,0} = \left(6b^2\delta^2 + 8b\delta^3 - 16c\delta^3 + 3\delta^4 - b^4\right)\eta\xi$$

$$\gamma_{2,4} = \eta\xi$$

$$\gamma_{2,3} = -4c\eta\xi$$

$$\gamma_{2,2} = 6(c^2 - \delta^2)\eta\xi$$

$$\gamma_{2,1} = 4(2\delta^3 - c^3 + 3c\delta^2)\eta\xi$$

$$\gamma_{2,0} = -(3\delta^4 - c^4 + 6c^2\delta^2 + 8c\delta^3)\eta\xi$$

and, as before, $\zeta = (16\delta^3)^{-1}$ and $\xi = (16\delta^3(b-c))^{-1}$, and additionally $\eta = a^2 - 1 - P$,

5.3 Finding P

In the the same manner as for the three-part redescender, we may find an explicit form for P, albeit in a form that is a little more unwieldy. Note that

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \psi_2(x)\phi(x)dx \\
&\quad = \int_{-a+\delta}^{a-\delta} \left(x^2 - 1 - P\right)\phi(x)dx \\
&\quad + \alpha_{2,5}\int_{a-\delta}^{a+\delta} x^5\phi(x)dx + \alpha_{2,3}\int_{a-\delta}^{a+\delta} x^3\phi(x)dx + \alpha_{2,1}\int_{a-\delta}^{a+\delta} x\phi(x)dx \\
&\qquad\quad + \alpha_{2,4}\int_{a-\delta}^{a+\delta} x^4\phi(x)dx + \alpha_{2,2}\int_{a-\delta}^{a+\delta} x^2\phi(x)dx + \alpha_{2,0}\int_{a-\delta}^{a+\delta} \phi(x)dx \\
&\quad - \alpha_{2,5}\int_{-a-\delta}^{-a+\delta} x^5\phi(x)dx - \alpha_{2,3}\int_{-a-\delta}^{-a+\delta} x^3\phi(x)dx - \alpha_{2,1}\int_{-a-\delta}^{-a+\delta} x\phi(x)dx \\
&\qquad\quad + \alpha_{2,4}\int_{-a-\delta}^{-a+\delta} x^4\phi(x)dx + \alpha_{2,2}\int_{-a-\delta}^{-a+\delta} x^2\phi(x)dx + \alpha_{2,0}\int_{-a-\delta}^{-a+\delta} \phi(x)dx \\
&\quad + \eta\left(\int_{a+\delta}^{b-\delta}\phi(x)dx + \int_{-b+\delta}^{-a-\delta}\phi(x)dx\right) \\
&\quad + \beta_{2,4}\int_{b-\delta}^{b+\delta} x^4\phi(x)dx + \beta_{2,2}\int_{b-\delta}^{b+\delta} x^2\phi(x)dx + \beta_{2,0}\int_{b-\delta}^{b+\delta} \phi(x)dx \\
&\qquad\quad + \beta_{2,3}\int_{b-\delta}^{b+\delta} x^3\phi(x)dx + \beta_{2,1}\int_{b-\delta}^{b+\delta} x\phi(x)dx \\
&\quad + \beta_{2,4}\int_{-b-\delta}^{-b+\delta} x^4\phi(x)dx + \beta_{2,2}\int_{-b-\delta}^{-b+\delta} x^2\phi(x)dx + \beta_{2,0}\int_{-b-\delta}^{-b+\delta} \phi(x)dx \\
&\qquad\quad - \beta_{2,3}\int_{-b-\delta}^{-b+\delta} x^3\phi(x)dx - \beta_{2,1}\int_{-b-\delta}^{-b+\delta} x\phi(x)dx \\
&\quad + \frac{\eta}{c-b}\left(\int_{b+\delta}^{c-\delta}(c-x)\,\phi(x)dx + \int_{-c+\delta}^{-b-\delta}(c+x)\,\phi(x)dx\right) \\
&\quad + \gamma_{2,4}\int_{c-\delta}^{c+\delta} x^4\phi(x)dx + \gamma_{2,2}\int_{c-\delta}^{c+\delta} x^2\phi(x)dx + \gamma_{2,0}\int_{c-\delta}^{c+\delta} \phi(x)dx \\
&\qquad\quad + \gamma_{2,3}\int_{c-\delta}^{c+\delta} x^3\phi(x)dx + \gamma_{2,1}\int_{c-\delta}^{c+\delta} x\phi(x)dx \\
&\quad + \gamma_{2,4}\int_{-c-\delta}^{-c+\delta} x^4\phi(x)dx + \gamma_{2,2}\int_{-c-\delta}^{-c+\delta} x^2\phi(x)dx + \gamma_{2,0}\int_{-c-\delta}^{-c+\delta} \phi(x)dx \\
&\qquad\quad - \gamma_{2,3}\int_{-c-\delta}^{-c+\delta} x^3\phi(x)dx - \gamma_{2,1}\int_{-c-\delta}^{-c+\delta} x\phi(x)dx
\end{aligned}
$$

Therefore, letting $\widetilde{\beta}_i = \beta_{2,i}/\eta$ and $\widetilde{\gamma}_i = \gamma_{2,i}/\eta$, we have:

$$
\begin{aligned}
0 = {} & g_2(-a+\delta, a-\delta) - g_0(-a+\delta, a-\delta) - Pg_0(-a+\delta, a-\delta) \\
& + \alpha_{2,5} g_5(a-\delta, a+\delta) + \alpha_{2,3} g_3(a-\delta, a+\delta) + \alpha_{2,1} g_1(a-\delta, a+\delta) \\
& \qquad + \alpha_{2,4} g_4(a-\delta, a+\delta) + \alpha_{2,2} g_2(a-\delta, a+\delta) \\
& \qquad - \left(16P\delta^3 + 3a\delta^4 - a^5 + 16\delta^3 - 8a^2\delta^3 + 6a^3\delta^2\right) \zeta g_0(a-\delta, a+\delta) \\
& - \alpha_{2,5} g_5(-a-\delta, -a+\delta) - \alpha_{2,3} g_3(-a-\delta, -a+\delta) - \alpha_{2,1} g_1(-a-\delta, -a+\delta) \\
& \qquad + \alpha_{2,4} g_4(-a-\delta, -a+\delta) + \alpha_{2,2} g_2(-a-\delta, -a+\delta) \\
& \qquad - \left(16P\delta^3 + 3a\delta^4 - a^5 + 16\delta^3 - 8a^2\delta^3 + 6a^3\delta^2\right) \zeta g_0(-a-\delta, -a+\delta)\Big) \\
& + \left(a^2 - 1 - P\right)\left(g_0(a+\delta, b-\delta) + g_0(-b+\delta, -a-\delta)\right) \\
& + \left(a^2 - 1 - P\right)\Big(\widetilde{\beta}_4 g_4(b-\delta, b+\delta) + \widetilde{\beta}_2 g_2(b-\delta, b+\delta) + \widetilde{\beta}_0 g_0(b-\delta, b+\delta) \\
& \qquad\qquad + \widetilde{\beta}_3 g_3(b-\delta, b+\delta) + \widetilde{\beta}_1 g_1(b-\delta, b+\delta)\Big) \\
& + \left(a^2 - 1 - P\right)\Big(\widetilde{\beta}_4 g_4(-b-\delta, -b+\delta) + \widetilde{\beta}_2 g_2(-b-\delta, -b+\delta) + \widetilde{\beta}_0 g_0(-b-\delta, -b+\delta) \\
& \qquad\qquad - \widetilde{\beta}_3 g_3(-b-\delta, -b+\delta) - \widetilde{\beta}_1 g_1(-b-\delta, -b+\delta)\Big) \\
& + \frac{a^2 - 1 - P}{c-b}\left(c g_0(b+\delta, c-\delta) - g_1(b+\delta, c-\delta)\right. \\
& \qquad\qquad \left. + c g_0(-c+\delta, -b-\delta) + g_1(-c+\delta, -b-\delta)\right) \\
& + \left(a^2 - 1 - P\right)\Big(\widetilde{\gamma}_4 g_4(c-\delta, c+\delta) + \widetilde{\gamma}_2 g_2(c-\delta, c+\delta) + \widetilde{\gamma}_0 g_0(c-\delta, c+\delta) \\
& \qquad\qquad + \widetilde{\gamma}_3 g_3(c-\delta, c+\delta) + \widetilde{\gamma}_1 g_1(c-\delta, c+\delta)\Big) \\
& + \left(a^2 - 1 - P\right)\Big(\widetilde{\gamma}_4 g_4(-c-\delta, -c+\delta) + \widetilde{\gamma}_2 g_2(-c-\delta, -c+\delta) + \widetilde{\gamma}_0 g_0(-c-\delta, -c+\delta) \\
& \qquad\qquad - \widetilde{\gamma}_3 g_3(-c-\delta, -c+\delta) - \widetilde{\gamma}_1 g_1(-c-\delta, -c+\delta)\Big)
\end{aligned}
$$

Solving for P, we get $P = \text{numerator}/\text{denominator}$, where:

numerator =

$$
\begin{aligned}
& g_2(-a+\delta, a-\delta) - g_0(-a+\delta, a-\delta) \\
& \quad + \alpha_{2,5} g_5(a-\delta, a+\delta) + \alpha_{2,3} g_3(a-\delta, a+\delta) + \alpha_{2,1} g_1(a-\delta, a+\delta) \\
& \qquad + \alpha_{2,4} g_4(a-\delta, a+\delta) + \alpha_{2,2} g_2(a-\delta, a+\delta) \\
& \qquad - \left(3a\delta^4 - a^5 + 16\delta^3 - 8a^2\delta^3 + 6a^3\delta^2\right) \zeta g_0(a-\delta, a+\delta) \\
& \quad - \alpha_{2,5} g_5(-a-\delta, -a+\delta) - \alpha_{2,3} g_3(-a-\delta, -a+\delta) - \alpha_{2,1} g_1(-a-\delta, -a+\delta) \\
& \qquad + \alpha_{2,4} g_4(-a-\delta, -a+\delta) + \alpha_{2,2} g_2(-a-\delta, -a+\delta) \\
& \qquad - \left(3a\delta^4 - a^5 + 16\delta^3 - 8a^2\delta^3 + 6a^3\delta^2\right) \zeta g_0(-a-\delta, -a+\delta)\Big) \\
& \quad + \left(a^2-1\right)\left(g_0(a+\delta, b-\delta) + g_0(-b+\delta, -a-\delta)\right) \\
& \quad + \left(a^2-1\right)\Big(\widetilde{\beta}_4 g_4(b-\delta, b+\delta) + \widetilde{\beta}_2 g_2(b-\delta, b+\delta) + \widetilde{\beta}_0 g_0(b-\delta, b+\delta) \\
& \qquad\qquad + \widetilde{\beta}_3 g_3(b-\delta, b+\delta) + \widetilde{\beta}_1 g_1(b-\delta, b+\delta)\Big) \\
& \quad + \left(a^2-1\right)\Big(\widetilde{\beta}_4 g_4(-b-\delta, -b+\delta) + \widetilde{\beta}_2 g_2(-b-\delta, -b+\delta) + \widetilde{\beta}_0 g_0(-b-\delta, -b+\delta) \\
& \qquad\qquad - \widetilde{\beta}_3 g_3(-b-\delta, -b+\delta) - \widetilde{\beta}_1 g_1(-b-\delta, -b+\delta)\Big) \\
& \quad + \frac{a^2-1}{c-b}\left(c g_0(b+\delta, c-\delta) - g_1(b+\delta, c-\delta)\right. \\
& \qquad\qquad \left. + c g_0(-c+\delta, -b-\delta) + g_1(-c+\delta, -b-\delta)\right) \\
& \quad + \left(a^2-1\right)\Big(\widetilde{\gamma}_4 g_4(c-\delta, c+\delta) + \widetilde{\gamma}_2 g_2(c-\delta, c+\delta) + \widetilde{\gamma}_0 g_0(c-\delta, c+\delta) \\
& \qquad\qquad + \widetilde{\gamma}_3 g_3(c-\delta, c+\delta) + \widetilde{\gamma}_1 g_1(c-\delta, c+\delta)\Big) \\
& \quad + \left(a^2-1\right)\Big(\widetilde{\gamma}_4 g_4(-c-\delta, -c+\delta) + \widetilde{\gamma}_2 g_2(-c-\delta, -c+\delta) + \widetilde{\gamma}_0 g_0(-c-\delta, -c+\delta) \\
& \qquad\qquad - \widetilde{\gamma}_3 g_3(-c-\delta, -c+\delta) - \widetilde{\gamma}_1 g_1(-c-\delta, -c+\delta)\Big)
\end{aligned}
$$

denominator =

$$
\begin{aligned}
& g_0(-a+\delta, a-\delta) + g_0(a-\delta, a+\delta) + g_0(-a-\delta, -a+\delta) \\
& \quad + g_0(a+\delta, b-\delta) + g_0(-b+\delta, -a-\delta) \\
& \quad + \widetilde{\beta}_4 g_4(b-\delta, b+\delta) + \widetilde{\beta}_2 g_2(b-\delta, b+\delta) + \widetilde{\beta}_0 g_0(b-\delta, b+\delta) \\
& \qquad + \widetilde{\beta}_3 g_3(b-\delta, b+\delta) + \widetilde{\beta}_1 g_1(b-\delta, b+\delta) \\
& \quad + \widetilde{\beta}_4 g_4(-b-\delta, -b+\delta) + \widetilde{\beta}_2 g_2(-b-\delta, -b+\delta) + \widetilde{\beta}_0 g_0(-b-\delta, -b+\delta) \\
& \qquad - \widetilde{\beta}_3 g_3(-b-\delta, -b+\delta) - \widetilde{\beta}_1 g_1(-b-\delta, -b+\delta) \\
& \quad + \frac{1}{c-b}\left(c g_0(b+\delta, c-\delta) - g_1(b+\delta, c-\delta) + c g_0(-c+\delta, -b-\delta) + g_1(-c+\delta, -b-\delta)\right) \\
& \quad + \widetilde{\gamma}_4 g_4(c-\delta, c+\delta) + \widetilde{\gamma}_2 g_2(c-\delta, c+\delta) + \widetilde{\gamma}_0 g_0(c-\delta, c+\delta) \\
& \qquad + \widetilde{\gamma}_3 g_3(c-\delta, c+\delta) + \widetilde{\gamma}_1 g_1(c-\delta, c+\delta) \\
& \quad + \widetilde{\gamma}_4 g_4(-c-\delta, -c+\delta) + \widetilde{\gamma}_2 g_2(-c-\delta, -c+\delta) + \widetilde{\gamma}_0 g_0(-c-\delta, -c+\delta) \\
& \qquad - \widetilde{\gamma}_3 g_3(-c-\delta, -c+\delta) - \widetilde{\gamma}_1 g_1(-c-\delta, -c+\delta)
\end{aligned}
$$

5.4 Finding a root, $T[\psi, G]$

Once again we seek to find $T[\psi, G] = (\mu, \sigma)^T$ that satisfies (**??**). We may again write the equation in terms of y and dy, as for the three-part redescender, and for the smoothed three-part redescender we have that

$$
\int_{-\infty}^{\infty} \Psi\left(\frac{x-\mu}{\sigma}\right) dG(x) = \begin{bmatrix} (1-\epsilon)\sigma\omega_1\left(\frac{0-\mu}{\sigma}, \frac{1}{\sigma}\right) + \frac{\epsilon\sigma}{\alpha}\omega_1\left(\frac{\mu_2-\mu}{\sigma}, \frac{\alpha}{\sigma}\right) \\ (1-\epsilon)\sigma\omega_2\left(\frac{0-\mu}{\sigma}, \frac{1}{\sigma}\right) + \frac{\epsilon\sigma}{\alpha}\omega_2\left(\frac{\mu_2-\mu}{\sigma}, \frac{\alpha}{\sigma}\right) \end{bmatrix}
$$

where

$$
\begin{aligned}
\omega_1(u,v) = {} & \mathrm{S}_h^1\left(-a+\delta, a-\delta, u, v\right) \\
& + \alpha_{1,4}\,\mathrm{S}_h^4\left(a-\delta, a+\delta, u, v\right) + \alpha_{1,2}\,\mathrm{S}_h^2\left(a-\delta, a+\delta, u, v\right) + \alpha_{1,0} h_0(a-\delta, a+\delta, u, v) \\
& \quad + \alpha_{1,3}\,\mathrm{S}_h^3\left(a-\delta, a+\delta, u, v\right) + \alpha_{1,1}\,\mathrm{S}_h^1\left(a-\delta, a+\delta, u, v\right) \\
& \quad - \alpha_{1,4}\,\mathrm{S}_h^4\left(-a-\delta, -a+\delta, u, v\right) - \alpha_{1,2}\,\mathrm{S}_h^2\left(-a-\delta, -a+\delta, u, v\right) \\
& \quad - \alpha_{1,0} h_0(-a-\delta, -a+\delta, u, v) \\
& \quad + \alpha_{1,3}\,\mathrm{S}_h^3\left(-a-\delta, -a+\delta, u, v\right) + \alpha_{1,1}\,\mathrm{S}_h^1\left(-a-\delta, -a+\delta, u, v\right) \\
& + a\left(h_0(a+\delta, b-\delta, u, v) - h_0(-b+\delta, -a-\delta, u, v)\right) \\
& + \beta_{1,4}\,\mathrm{S}_h^4\left(b-\delta, b+\delta, u, v\right) + \beta_{1,2}\,\mathrm{S}_h^2\left(b-\delta, b+\delta, u, v\right) + \beta_{1,0} h_0(b-\delta, b+\delta, u, v) \\
& \quad + \beta_{1,3}\,\mathrm{S}_h^3\left(b-\delta, b+\delta, u, v\right) + \beta_{1,1}\,\mathrm{S}_h^1\left(b-\delta, b+\delta, u, v\right) \\
& \quad - \beta_{1,4}\,\mathrm{S}_h^4\left(-b-\delta, -b+\delta, u, v\right) - \beta_{1,2}\,\mathrm{S}_h^2\left(-b-\delta, -b+\delta, u, v\right) \\
& \quad - \beta_{1,0} h_0(-b-\delta, -b+\delta, u, v) \\
& \quad + \beta_{1,3}\,\mathrm{S}_h^3\left(-b-\delta, -b+\delta, u, v\right) + \beta_{1,1}\,\mathrm{S}_h^1\left(-b-\delta, -b+\delta, u, v\right) \\
& + \frac{a}{c-b}\Big(c h_0(b+\delta, c-\delta, u, v) - \mathrm{S}_h^1\left(b+\delta, c-\delta, u, v\right) \\
& \qquad\qquad - c h_0(-c+\delta, -b-\delta, u, v) - \mathrm{S}_h^1\left(-c+\delta, -b-\delta, u, v\right)\Big) \\
& + \gamma_{1,4}\,\mathrm{S}_h^4\left(c-\delta, c+\delta, u, v\right) + \gamma_{1,2}\,\mathrm{S}_h^2\left(c-\delta, c+\delta, u, v\right) + \gamma_{1,0} h_0(c-\delta, c+\delta, u, v) \\
& \quad + \gamma_{1,3}\,\mathrm{S}_h^3\left(c-\delta, c+\delta, u, v\right) + \gamma_{1,1}\,\mathrm{S}_h^1\left(c-\delta, c+\delta, u, v\right) \\
& \quad - \gamma_{1,4}\,\mathrm{S}_h^4\left(-c-\delta, -c+\delta, u, v\right) - \gamma_{1,2}\,\mathrm{S}_h^2\left(-c-\delta, -c+\delta, u, v\right) \\
& \quad - \gamma_{1,0} h_0(-c-\delta, -c+\delta, u, v) \\
& \quad + \gamma_{1,3}\,\mathrm{S}_h^3\left(-c-\delta, -c+\delta, u, v\right) \gamma_{1,1}\,\mathrm{S}_h^1\left(-c-\delta, -c+\delta, u, v\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\omega_2(u,v) = \\
&\quad \mathrm{S}_h^2\left(-a+\delta, a-\delta, u, v\right) - (1+P)h_0(-a+\delta, a-\delta, u, v) \\
&\qquad + \alpha_{2,5}\,\mathrm{S}_h^5\left(a-\delta, a+\delta, u, v\right) + \alpha_{2,3}\,\mathrm{S}_h^3\left(a-\delta, a+\delta, u, v\right) + \alpha_{2,1}\,\mathrm{S}_h^1\left(a-\delta, a+\delta, u, v\right) \\
&\qquad\quad + \alpha_{2,4}\,\mathrm{S}_h^4\left(a-\delta, a+\delta, u, v\right) + \alpha_{2,2}\,\mathrm{S}_h^2\left(a-\delta, a+\delta, u, v\right) + \alpha_{2,0}h_0(a-\delta, a+\delta, u, v) \\
&\qquad\quad - \alpha_{2,5}\,\mathrm{S}_h^5\left(-a-\delta, -a+\delta, u, v\right) - \alpha_{2,3}\,\mathrm{S}_h^3\left(-a-\delta, -a+\delta, u, v\right) \\
&\qquad\quad - \alpha_{2,1}\,\mathrm{S}_h^1\left(-a-\delta, -a+\delta, u, v\right) \\
&\qquad\quad + \alpha_{2,4}\,\mathrm{S}_h^4\left(-a-\delta, -a+\delta, u, v\right) + \alpha_{2,2}\,\mathrm{S}_h^2\left(-a-\delta, -a+\delta, u, v\right) \\
&\qquad\quad + \alpha_{2,0}h_0(-a-\delta, -a+\delta, u, v) \\
&\qquad + \eta\left(h_0(a+\delta, b-\delta, u, v) + h_0(-b+\delta, -a-\delta, u, v)\right) \\
&\qquad + \beta_{2,4}\,\mathrm{S}_h^4\left(b-\delta, b+\delta, u, v\right) + \beta_{2,2}\,\mathrm{S}_h^2\left(b-\delta, b+\delta, u, v\right) + \beta_{2,0}h_0(b-\delta, b+\delta, u, v) \\
&\qquad\quad + \beta_{2,3}\,\mathrm{S}_h^3\left(b-\delta, b+\delta, u, v\right) + \beta_{2,1}\,\mathrm{S}_h^1\left(b-\delta, b+\delta, u, v\right) \\
&\qquad\quad + \beta_{2,4}\,\mathrm{S}_h^4\left(-b-\delta, -b+\delta, u, v\right) + \beta_{2,2}\,\mathrm{S}_h^2\left(-b-\delta, -b+\delta, u, v\right) \\
&\qquad\quad + \beta_{2,0}h_0(-b-\delta, -b+\delta, u, v) \\
&\qquad\quad - \beta_{2,3}\,\mathrm{S}_h^3\left(-b-\delta, -b+\delta, u, v\right) - \beta_{2,1}\,\mathrm{S}_h^1\left(-b-\delta, -b+\delta, u, v\right) \\
&\qquad + \frac{\eta}{c-b}\Big(ch_0(b+\delta, c-\delta, u, v) - \mathrm{S}_h^1\left(b+\delta, c-\delta, u, v\right) \\
&\qquad\qquad\qquad + ch_0(-c+\delta, -b-\delta, u, v) + \mathrm{S}_h^1\left(-c+\delta, -b-\delta, u, v\right)\Big) \\
&\qquad + \gamma_{2,4}\,\mathrm{S}_h^4\left(c-\delta, c+\delta, u, v\right) + \gamma_{2,2}\,\mathrm{S}_h^2\left(c-\delta, c+\delta, u, v\right) + \gamma_{2,0}h_0(c-\delta, c+\delta, u, v) \\
&\qquad\quad + \gamma_{2,3}\,\mathrm{S}_h^3\left(c-\delta, c+\delta, u, v\right) + \gamma_{2,1}\,\mathrm{S}_h^1\left(c-\delta, c+\delta, u, v\right) \\
&\qquad\quad + \gamma_{2,4}\,\mathrm{S}_h^4\left(-c-\delta, -c+\delta, u, v\right) + \gamma_{2,2}\,\mathrm{S}_h^2\left(-c-\delta, -c+\delta, u, v\right) \\
&\qquad\quad + \gamma_{2,0}h_0(-c-\delta, -c+\delta, u, v) \\
&\qquad\quad - \gamma_{2,3}\,\mathrm{S}_h^3\left(-c-\delta, -c+\delta, u, v\right) - \gamma_{2,1}\,\mathrm{S}_h^1\left(-c-\delta, -c+\delta, u, v\right)
\end{aligned}
$$

5.5 The p_{11} element

Note that

$$\begin{aligned}\left(-ax^4 + bx^3 - cx^2 + dx - e\right)^2 &= \left(ax^4 - bx^3 + cx^2 - dx + e\right)^2 \\ &= a^2x^8 + (-2ab)x^7 + \left(b^2 + 2ac\right)x^6 + (-2ad - 2bc)\,x^5 + \left(c^2 + 2ae + 2bd\right)x^4 \\ &\quad + (-2be - 2cd)\,x^3 + \left(d^2 + 2ce\right)x^2 + (-2de)\,x + e^2\end{aligned}$$

then let

$$\begin{aligned}w(r) = r_{1,4}^2x^8 &+ \left(r_{1,3}^2 + 2r_{1,4}r_{1,2}\right)x^6 + \left(r_{1,2}^2 + 2r_{1,4}r_{1,0} + 2r_{1,3}r_{1,1}\right)x^4 \\ &+ \left(r_{1,1}^2 + 2r_{1,2}r_{1,0}\right)x^2 + r_{1,0}^2 \\ &+ \left(2r_{1,4}r_{1,3}x^7 + \left(2r_{1,4}r_{1,1} + 2r_{1,3}r_{1,2}\right)x^5 + \right. \\ &\quad \left. \left(2r_{1,3}r_{1,0} + 2r_{1,2}r_{1,1}\right)x^3 + 2r_{1,1}r_{1,0}x\right)\operatorname{sign}(x)\end{aligned} \tag{3}$$

where $r \in \{\{\alpha_{1,j}\}, \{\beta_{1,j}\}, \{\gamma_{1,j}\}\}$ represents a set of coefficients for the location ψ_1 function.

Then we have

$$\psi_1^2(x) = \begin{cases} x^2 & |x| \le (a-\delta) \\ w(\alpha) & (a-\delta) < |x| \le (a+\delta) \\ a^2 & (a+\delta) < |x| \le (b-\delta) \\ w(\beta) & (b-\delta) < |x| \le (b+\delta) \\ \left(\frac{a}{c-b}\right)^2 \left(c^2 - 2cx\operatorname{sign}(x) + x^2\right) & (b+\delta) \le |x| \le (c-\delta) \\ w(\gamma) & (c-\delta) < |x| \le (c+\delta) \\ 0 & \text{otherwise} \end{cases}$$

and, as derived before,

$$p_{11} = \int \left[\psi_1\left(\frac{x-\mu}{\sigma}\right)\right]^2 dG(x) = (1-\epsilon)\,\sigma\omega_{11}\left(\frac{0-\mu}{\sigma}, \frac{1}{\sigma}\right) + \frac{\epsilon\sigma}{\alpha}\omega_{11}\left(\frac{\mu_2-\mu}{\sigma}, \frac{\alpha}{\sigma}\right)$$

this time where

$$
\begin{aligned}
&\omega_{11}(u,v) = \\
&\quad \mathrm{S}_h^2\left(-a+\delta, a-\delta, u, v\right) \\
&\qquad + a^2 h_0\left(a+\delta, b-\delta, u, v\right) + a^2 h_0\left(-b+\delta, -a-\delta, u, v\right) \\
&\qquad + \frac{a^2}{(c-b)^2}\left[c^2 h_0\left(b+\delta, c-\delta, u, v\right) - 2c\,\mathrm{S}_h^1\left(b+\delta, c-\delta, u, v\right) + \mathrm{S}_h^2\left(b+\delta, c-\delta, u, v\right)\right] \\
&\qquad + \frac{a^2}{(c-b)^2}\left[c^2 h_0\left(-c+\delta, -b-\delta, u, v\right) + 2c\,\mathrm{S}_h^1\left(-c+\delta, -b-\delta, u, v\right)\right. \\
&\qquad\qquad \left. + \mathrm{S}_h^2\left(-c+\delta, -b-\delta, u, v\right)\right] \\
&\qquad + \tilde{w}_{11}(\alpha, a, u, v) + \tilde{w}_{11}(\beta, b, u, v) + \tilde{w}_{11}(\gamma, c, u, v)
\end{aligned}
$$

with

$$
\begin{aligned}
&\tilde{w}_{11}(r,t,u,v) = \\
&\quad r_{1,4}^2\,\mathrm{S}_h^8\left(t-\delta, t+\delta, u, v\right) + \left(r_{1,3}^2 + 2r_{1,4}r_{1,2}\right)\mathrm{S}_h^6\left(t-\delta, t+\delta, u, v\right) \\
&\qquad + \left(r_{1,2}^2 + 2r_{1,4}r_{1,0} + 2r_{1,3}r_{1,1}\right)\mathrm{S}_h^4\left(t-\delta, t+\delta, u, v\right) \\
&\qquad + \left(r_{1,1}^2 + 2r_{1,2}r_{1,0}\right)\mathrm{S}_h^2\left(t-\delta, t+\delta, u, v\right) + r_{1,0}^2 h_0(t-\delta, t+\delta, u, v) \\
&\qquad + 2r_{1,4}r_{1,3}\,\mathrm{S}_h^7\left(t-\delta, t+\delta, u, v\right) + \left(2r_{1,4}r_{1,1} + 2r_{1,3}r_{1,2}\right)\mathrm{S}_h^5\left(t-\delta, t+\delta, u, v\right) \\
&\qquad + \left(2r_{1,3}r_{1,0} + 2r_{1,2}r_{1,1}\right)\mathrm{S}_h^3\left(t-\delta, t+\delta, u, v\right) + 2r_{1,1}r_{1,0}\,\mathrm{S}_h^1\left(t-\delta, t+\delta, u, v\right) \\
&\qquad + r_{1,4}^2\,\mathrm{S}_h^8\left(-t-\delta, -t+\delta, u, v\right) + \left(r_{1,3}^2 + 2r_{1,4}r_{1,2}\right)\mathrm{S}_h^6\left(-t-\delta, -t+\delta, u, v\right) \\
&\qquad + \left(r_{1,2}^2 + 2r_{1,4}r_{1,0} + 2r_{1,3}r_{1,1}\right)\mathrm{S}_h^4\left(-t-\delta, -t+\delta, u, v\right) \\
&\qquad + \left(r_{1,1}^2 + 2r_{1,2}r_{1,0}\right)\mathrm{S}_h^2\left(-t-\delta, -t+\delta, u, v\right) + r_{1,0}^2 h_0(-t-\delta, -t+\delta, u, v) \\
&\qquad - 2r_{1,4}r_{1,3}\,\mathrm{S}_h^7\left(-t-\delta, -t+\delta, u, v\right) - \left(2r_{1,4}r_{1,1} + 2r_{1,3}r_{1,2}\right)\mathrm{S}_h^5\left(-t-\delta, -t+\delta, u, v\right) \\
&\qquad - \left(2r_{1,3}r_{1,0} + 2r_{1,2}r_{1,1}\right)\mathrm{S}_h^3\left(-t-\delta, -t+\delta, u, v\right) - 2r_{1,1}r_{1,0}\,\mathrm{S}_h^1\left(-t-\delta, -t+\delta, u, v\right)
\end{aligned}
$$

5.6 The p_{22} element

Note that

$$\begin{aligned}
&\left(-ax^5 + bx^4 - cx^3 + dx^2 - ex + f\right)^2 \\
&\quad = a^2x^{10} + (-2ab)\,x^9 + \left(b^2 + 2ac\right)x^8 + (-2ad - 2bc)\,x^7 + \left(c^2 + 2bd + 2ae\right)x^6 \\
&\quad\quad + (-2af - 2be - 2cd)\,x^5 + \left(d^2 + 2ce + 2bf\right)x^4 + (-2de - 2cf)\,x^3 \\
&\quad\quad + \left(e^2 + 2df\right)x^2 + (-2ef)\,x + f^2
\end{aligned}$$

and then let

$$\begin{aligned}
w_a(x) &= \alpha_{2,5}^2 x^{10} + \left(\alpha_{2,4}^2 + 2\alpha_{2,5}\alpha_{2,3}\right)x^8 + \left(\alpha_{2,3}^2 + 2\alpha_{2,4}\alpha_{2,2} + 2\alpha_{2,5}\alpha_{2,1}\right)x^6 \\
&\quad + \left(\alpha_{2,2}^2 + 2\alpha_{2,3}\alpha_{2,1} + 2\alpha_{2,4}\alpha_{2,0}\right)x^4 + \left(\alpha_{2,1}^2 + 2\alpha_{2,2}\alpha_{2,0}\right)x^2 + \alpha_{2,0}^2 \\
&\quad + \left[(2\alpha_{2,5}\alpha_{2,4})\,x^9 + (2\alpha_{2,5}\alpha_{2,2} + 2\alpha_{2,4}\alpha_{2,3})\,x^7 + (2\alpha_{2,5}\alpha_{2,0} + 2\alpha_{2,4}\alpha_{2,1} + 2\alpha_{2,3}\alpha_{2,2})\,x^5\right. \\
&\quad\quad \left. + (2\alpha_{2,2}\alpha_{2,1} + 2\alpha_{2,3}\alpha_{2,0})\,x^3 + 2\alpha_{2,0}x\right]\operatorname{sign}(x) \qquad (4)
\end{aligned}$$

With (3) and (4), we can write

$$\psi_2^2(x) = \begin{cases}
x^4 - 2(1+P)x^2 + (1+P)^2 & |x| \le (a-\delta) \\
w_a(x) & (a-\delta) \le |x| \le (a+\delta) \\
\eta^2 & (a+\delta) \le |x| \le (b-\delta) \\
w(\beta) & (b-\delta) \le |x| \le (b+\delta) \\
\left(\frac{\eta}{c-b}\right)^2 (c^2 - 2cx\operatorname{sign}(x) + x^2) & (b+\delta) \le x \le (c-\delta) \\
w(\gamma) & (c-\delta) \le |x| \le (c+\delta) \\
0 & \text{otherwise}
\end{cases}$$

and

$$p_{22} = \int \left[\psi_2\left(\frac{x-\mu}{\sigma}\right)\right]^2 dG(x) = (1-\epsilon)\,\sigma\omega_{22}\left(\frac{0-\mu}{\sigma}, \frac{1}{\sigma}\right) + \frac{\epsilon\sigma}{\alpha}\omega_{22}\left(\frac{\mu_2-\mu}{\sigma}, \frac{\alpha}{\sigma}\right)$$

this time where

$$
\begin{aligned}
&\omega_{22}\,(u,v) = \\
&\quad \mathrm{S}_h^4\,(-a+\delta,a-\delta,u,v) - 2(1+P)\,\mathrm{S}_h^2\,(-a+\delta,a-\delta,u,v) \\
&\qquad + (1+P)^2 h_0(-a+\delta,a-\delta,u,v) \\
&\qquad + \eta^2\,(h_0(a+\delta,b-\delta,u,v) + h_0(-b+\delta,-a-\delta,u,v)) \\
&\qquad + \left(\frac{\eta}{c-b}\right)^2 \big(c^2 h_0(b+\delta,c-\delta,u,v) - 2c\,\mathrm{S}_h^1\,(b+\delta,c-\delta,u,v) + \mathrm{S}_h^2\,(b+\delta,c-\delta,u,v) \\
&\qquad\qquad\qquad + c^2 h_0(-c+\delta,-b-\delta,u,v) + 2c\,\mathrm{S}_h^1\,(-c+\delta,-b-\delta,u,v) \\
&\qquad\qquad\qquad + \mathrm{S}_h^2\,(-c+\delta,-b-\delta,u,v)\big) \\
&\qquad + \alpha_{2,5}^2\,\mathrm{S}_h^{10}\,(a-\delta,a+\delta,u,v) + \left(\alpha_{2,4}^2 + 2\alpha_{2,5}\alpha_{2,3}\right)\mathrm{S}_h^8\,(a-\delta,a+\delta,u,v) \\
&\qquad\quad + \left(\alpha_{2,3}^2 + 2\alpha_{2,4}\alpha_{2,2} + 2\alpha_{2,5}\alpha_{2,1}\right)\mathrm{S}_h^6\,(a-\delta,a+\delta,u,v) \\
&\qquad\quad + \left(\alpha_{2,2}^2 + 2\alpha_{2,3}\alpha_{2,1} + 2\alpha_{2,4}\alpha_{2,0}\right)\mathrm{S}_h^4\,(a-\delta,a+\delta,u,v) \\
&\qquad\quad + \left(\alpha_{2,1}^2 + 2\alpha_{2,2}\alpha_{2,0}\right)\mathrm{S}_h^2\,(a-\delta,a+\delta,u,v) + \alpha_{2,0}^2 h_0(a-\delta,a+\delta,u,v) \\
&\qquad\quad + (2\alpha_{2,5}\alpha_{2,4})\,\mathrm{S}_h^9\,(a-\delta,a+\delta,u,v) + (2\alpha_{2,5}\alpha_{2,2} + 2\alpha_{2,4}\alpha_{2,3})\,\mathrm{S}_h^7\,(a-\delta,a+\delta,u,v) \\
&\qquad\quad + (2\alpha_{2,5}\alpha_{2,0} + 2\alpha_{2,4}\alpha_{2,1} + 2\alpha_{2,3}\alpha_{2,2})\,\mathrm{S}_h^5\,(a-\delta,a+\delta,u,v) \\
&\qquad\quad + (2\alpha_{2,2}\alpha_{2,1} + 2\alpha_{2,3}\alpha_{2,0})\,\mathrm{S}_h^3\,(a-\delta,a+\delta,u,v) + 2\alpha_{2,1}\alpha_{2,0}\,\mathrm{S}_h^1\,(a-\delta,a+\delta,u,v) \\
&\qquad + \alpha_{2,5}^2\,\mathrm{S}_h^{10}\,(-a-\delta,-a+\delta,u,v) + \left(\alpha_{2,4}^2 + 2\alpha_{2,5}\alpha_{2,3}\right)\mathrm{S}_h^8\,(-a-\delta,-a+\delta,u,v) \\
&\qquad\quad + \left(\alpha_{2,3}^2 + 2\alpha_{2,4}\alpha_{2,2} + 2\alpha_{2,5}\alpha_{2,1}\right)\mathrm{S}_h^6\,(-a-\delta,-a+\delta,u,v) \\
&\qquad\quad + \left(\alpha_{2,2}^2 + 2\alpha_{2,3}\alpha_{2,1} + 2\alpha_{2,4}\alpha_{2,0}\right)\mathrm{S}_h^4\,(-a-\delta,-a+\delta,u,v) \\
&\qquad\quad + \left(\alpha_{2,1}^2 + 2\alpha_{2,2}\alpha_{2,0}\right)\mathrm{S}_h^2\,(-a-\delta,-a+\delta,u,v) + \alpha_{2,0}^2 h_0(-a-\delta,-a+\delta,u,v) \\
&\qquad\quad - (2\alpha_{2,5}\alpha_{2,4})\,\mathrm{S}_h^9\,(-a-\delta,-a+\delta,u,v) \\
&\qquad\quad - (2\alpha_{2,5}\alpha_{2,2} + 2\alpha_{2,4}\alpha_{2,3})\,\mathrm{S}_h^7\,(-a-\delta,-a+\delta,u,v) \\
&\qquad\quad - (2\alpha_{2,5}\alpha_{2,0} + 2\alpha_{2,4}\alpha_{2,1} + 2\alpha_{2,3}\alpha_{2,2})\,\mathrm{S}_h^5\,(-a-\delta,-a+\delta,u,v) \\
&\qquad\quad - (2\alpha_{2,2}\alpha_{2,1} + 2\alpha_{2,3}\alpha_{2,0})\,\mathrm{S}_h^3\,(-a-\delta,-a+\delta,u,v) \\
&\qquad\quad - 2\alpha_{2,1}\alpha_{2,0}\,\mathrm{S}_h^1\,(-a-\delta,-a+\delta,u,v) \\
&\qquad + \tilde{w}_{11}(\beta,b,u,v) + \tilde{w}_{11}(\gamma,c,u,v)
\end{aligned}
$$

5.7 The p_{12}/p_{21} elements

Now with the polynomial expansions

$$\begin{aligned}
&\left(-ax^4 + bx^3 - cx^2 + dx - e\right)\left(sx^4 - tx^3 + ux^2 - vx + w\right) \\
&\quad = -asx^8 - (au + bt + cs)\,x^6 - (aw + bv + cu + dt + es)\,x^4 - (cw + dv + eu)\,x^2 - ew \\
&\qquad + (at + bs)\,x^7 + (av + bu + ct + ds)\,x^5 + (bw + cv + du + et)\,x^3 + (dw + ev)\,x
\end{aligned}$$

and

$$\begin{aligned}
&\left(-ax^4 + bx^3 - cx^2 + dx - e\right)\left(-rx^5 + sx^4 - tx^3 + ux^2 - vx + w\right) \\
&\quad = -(as + br)\,x^8 - (au + bt + cs + dr)\,x^6 \\
&\qquad - (aw + bv + cu + dt + es)\,x^4 - (cw + dv + eu)\,x^2 - ew \\
&\qquad + arx^9 + (at + bs + cr)\,x^7 + (av + bu + ct + ds + er)\,x^5 \\
&\qquad + (bw + cv + du + et)\,x^3 + (dw + ev)\,x
\end{aligned}$$

we let

$$\begin{aligned}
w_a(x) = \big[&(\alpha_{1,4}\alpha_{2,4} + \alpha_{1,3}\alpha_{2,5})\,x^8 + (\alpha_{1,4}\alpha_{2,2} + \alpha_{1,3}\alpha_{2,3} + \alpha_{1,2}\alpha_{2,4} + \alpha_{1,1}\alpha_{2,5})\,x^6 \\
&+ (\alpha_{1,4}\alpha_{2,0} + \alpha_{1,3}\alpha_{2,1} + \alpha_{1,2}\alpha_{2,2} + \alpha_{1,1}\alpha_{2,3} + \alpha_{1,0}\alpha_{2,4})\,x^4 \\
&+ (\alpha_{1,2}\alpha_{2,0} + \alpha_{1,1}\alpha_{2,1} + \alpha_{1,0}\alpha_{2,2})\,x^2 + \alpha_{1,0}\alpha_{2,0}\big]\,\mathrm{sign}\,(x) \\
&+ \alpha_{1,4}\alpha_{2,5}x^9 + (\alpha_{1,4}\alpha_{2,3} + \alpha_{1,3}\alpha_{2,4} + \alpha_{1,2}\alpha_{2,5})\,x^7 \\
&+ (\alpha_{1,4}\alpha_{2,1} + \alpha_{1,3}\alpha_{2,2} + \alpha_{1,2}\alpha_{2,3} + \alpha_{1,1}\alpha_{2,4} + \alpha_{1,0}\alpha_{2,5})\,x^5 \\
&+ (\alpha_{1,3}\alpha_{2,0} + \alpha_{1,2}\alpha_{2,1} + \alpha_{1,1}\alpha_{2,2} + \alpha_{1,0}\alpha_{2,3})\,x^3 + (\alpha_{1,1}\alpha_{2,0} + \alpha_{1,0}\alpha_{2,1})\,x
\end{aligned}$$

$$\begin{aligned}
w_b(r,x) = \big[&r_{1,4}r_{2,4}x^8 + (r_{1,4}r_{2,2} + r_{1,3}r_{2,3} + r_{1,2}r_{2,4})\,x^6 \\
&+ (r_{1,4}r_{2,0} + r_{1,3}r_{2,1} + r_{1,2}r_{2,2} + r_{1,1}r_{2,3} + r_{1,0}r_{2,4})\,x^4 \\
&+ (r_{1,2}r_{2,0} + r_{1,1}r_{2,1} + r_{1,0}r_{2,2})\,x^2 + r_{1,0}r_{2,0}\big]\,\mathrm{sign}\,(x) \\
&+ (r_{1,4}r_{2,3} + r_{1,3}r_{2,4})\,x^7 + (r_{1,4}r_{2,1} + r_{1,3}r_{2,2} + r_{1,2}r_{2,3} + r_{1,1}r_{2,4})\,x^5 \\
&+ (r_{1,3}r_{2,0} + r_{1,2}r_{2,1} + r_{1,1}r_{2,2} + r_{1,0}r_{2,3})\,x^3 + (r_{1,1}r_{2,0} + r_{1,0}r_{2,1})\,x
\end{aligned}$$

where $r \in \{\{\beta_{i,j}\}, \{\gamma_{i,j}\}\}$ represents the $\{\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j}\}$ coefficients for the location and scale ψ_i functions.

Then we have:

$$\psi_1(x)\,\psi_2(x) = \begin{cases} x^3 - (1+P)x & |x| \le (a-\delta) \\ w_a(x) & (a-\delta) < |x| \le (a+\delta) \\ a\eta\,\mathrm{sign}\,(x) & (a+\delta) < |x| \le (b-\delta) \\ w_b(\beta, x) & (b-\delta) < |x| \le (b+\delta) \\ \frac{a\eta\,\mathrm{sign}(x)}{(c-b)^2}\left(c^2 - 2cx\,\mathrm{sign}\,(x) + x^2\right) & (b+\delta) < |x| \le (c-\delta) \\ w_b(\gamma, x) & (c-\delta) < |x| \le (c+\delta) \\ 0 & \text{otherwise} \end{cases}$$

and, as before,

$$p_{12} = p_{21} = \int \psi_1\left(\frac{x-\mu}{\sigma}\right)\psi_2\left(\frac{x-\mu}{\sigma}\right)dG(x) = (1-\epsilon)\,\sigma\omega_{12}\left(\frac{0-\mu}{\sigma}, \frac{1}{\sigma}\right) + \frac{\epsilon\sigma}{\alpha}\omega_{12}\left(\frac{\mu_2-\mu}{\sigma}, \frac{\alpha}{\sigma}\right)$$

this time where

$$
\begin{aligned}
&\omega_{12}(u,v) = \\
&\quad \mathrm{S}_h^3\left(-a+\delta, a-\delta, u, v\right) - (1+P)\,\mathrm{S}_h^1\left(-a+\delta, a-\delta, u, v\right) \\
&\qquad + a\eta\left[h_0\left(a+\delta, b-\delta, u, v\right) - h_0\left(-b+\delta, -a-\delta, u, v\right)\right] \\
&\qquad + \frac{a\eta}{(c-b)^2}\left[c^2 h_0\left(b+\delta, c-\delta, u, v\right) - 2c\,\mathrm{S}_h^1\left(b+\delta, c-\delta, u, v\right) + \mathrm{S}_h^2\left(b+\delta, c-\delta, u, v\right)\right. \\
&\qquad\qquad\qquad - c^2 h_0\left(-c+\delta, -b-\delta, u, v\right) - 2c\,\mathrm{S}_h^1\left(-c+\delta, -b-\delta, u, v\right) \\
&\qquad\qquad\qquad \left. - \mathrm{S}_h^2\left(-c+\delta, -b-\delta, u, v\right)\right] \\
&\qquad + \left(\alpha_{1,4}\alpha_{2,4} + \alpha_{1,3}\alpha_{2,5}\right)\mathrm{S}_h^8\left(a-\delta, a-\delta, u, v\right) \\
&\qquad\quad + \left(\alpha_{1,4}\alpha_{2,2} + \alpha_{1,3}\alpha_{2,3} + \alpha_{1,2}\alpha_{2,4} + \alpha_{1,1}\alpha_{2,5}\right)\mathrm{S}_h^6\left(a-\delta, a+\delta, u, v\right) \\
&\qquad\quad + \left(\alpha_{1,4}\alpha_{2,0} + \alpha_{1,3}\alpha_{2,1} + \alpha_{1,2}\alpha_{2,2} + \alpha_{1,1}\alpha_{2,3} + \alpha_{1,0}\alpha_{2,4}\right)\mathrm{S}_h^4\left(a-\delta, a+\delta, u, v\right) \\
&\qquad\quad + \left(\alpha_{1,2}\alpha_{2,0} + \alpha_{1,1}\alpha_{2,1} + \alpha_{1,0}\alpha_{2,2}\right)\mathrm{S}_h^2\left(a-\delta, a+\delta, u, v\right) + \alpha_{1,0}\alpha_{2,0}h_0(a-\delta, a+\delta, u, v) \\
&\qquad + \alpha_{1,4}\alpha_{2,5}\,\mathrm{S}_h^9\left(a-\delta, a+\delta, u, v\right) + \left(\alpha_{1,4}\alpha_{2,3} + \alpha_{1,3}\alpha_{2,4} + \alpha_{1,2}\alpha_{2,5}\right)\mathrm{S}_h^7\left(a-\delta, a+\delta, u, v\right) \\
&\qquad\quad + \left(\alpha_{1,4}\alpha_{2,1} + \alpha_{1,3}\alpha_{2,2} + \alpha_{1,2}\alpha_{2,3} + \alpha_{1,1}\alpha_{2,4} + \alpha_{1,0}\alpha_{2,5}\right)\mathrm{S}_h^5\left(a-\delta, a+\delta, u, v\right) \\
&\qquad\quad + \left(\alpha_{1,3}\alpha_{2,0} + \alpha_{1,2}\alpha_{2,1} + \alpha_{1,1}\alpha_{2,2} + \alpha_{1,0}\alpha_{2,3}\right)\mathrm{S}_h^3\left(a-\delta, a+\delta, u, v\right) \\
&\qquad\quad + \left(\alpha_{1,1}\alpha_{2,0} + \alpha_{1,0}\alpha_{2,1}\right)\mathrm{S}_h^1\left(a-\delta, a+\delta, u, v\right) \\
&\qquad - \left(\alpha_{1,4}\alpha_{2,4} + \alpha_{1,3}\alpha_{2,5}\right)\mathrm{S}_h^8\left(-a-\delta, -a-\delta, u, v\right) \\
&\qquad\quad - \left(\alpha_{1,4}\alpha_{2,2} + \alpha_{1,3}\alpha_{2,3} + \alpha_{1,2}\alpha_{2,4} + \alpha_{1,1}\alpha_{2,5}\right)\mathrm{S}_h^6\left(-a-\delta, -a+\delta, u, v\right) \\
&\qquad\quad - \left(\alpha_{1,4}\alpha_{2,0} + \alpha_{1,3}\alpha_{2,1} + \alpha_{1,2}\alpha_{2,2} + \alpha_{1,1}\alpha_{2,3} + \alpha_{1,0}\alpha_{2,4}\right)\mathrm{S}_h^4\left(-a-\delta, -a+\delta, u, v\right) \\
&\qquad\quad - \left(\alpha_{1,2}\alpha_{2,0} + \alpha_{1,1}\alpha_{2,1} + \alpha_{1,0}\alpha_{2,2}\right)\mathrm{S}_h^2\left(-a-\delta, -a+\delta, u, v\right) \\
&\qquad\quad - \alpha_{1,0}\alpha_{2,0}h_0(-a-\delta, -a+\delta, u, v) \\
&\qquad + \alpha_{1,4}\alpha_{2,5}\,\mathrm{S}_h^9\left(-a-\delta, -a+\delta, u, v\right) \\
&\qquad\quad + \left(\alpha_{1,4}\alpha_{2,3} + \alpha_{1,3}\alpha_{2,4} + \alpha_{1,2}\alpha_{2,5}\right)\mathrm{S}_h^7\left(-a-\delta, -a+\delta, u, v\right) \\
&\qquad\quad + \left(\alpha_{1,4}\alpha_{2,1} + \alpha_{1,3}\alpha_{2,2} + \alpha_{1,2}\alpha_{2,3} + \alpha_{1,1}\alpha_{2,4} + \alpha_{1,0}\alpha_{2,5}\right)\mathrm{S}_h^5\left(-a-\delta, -a+\delta, u, v\right) \\
&\qquad\quad + \left(\alpha_{1,3}\alpha_{2,0} + \alpha_{1,2}\alpha_{2,1} + \alpha_{1,1}\alpha_{2,2} + \alpha_{1,0}\alpha_{2,3}\right)\mathrm{S}_h^3\left(-a-\delta, -a+\delta, u, v\right) \\
&\qquad\quad + \left(\alpha_{1,1}\alpha_{2,0} + \alpha_{1,0}\alpha_{2,1}\right)\mathrm{S}_h^1\left(-a-\delta, -a+\delta, u, v\right) \\
&\qquad + \tilde{w}_{12}(\beta, b, u, v) + \tilde{w}_{12}(\gamma, c, u, v)
\end{aligned}
$$

with

$$
\begin{aligned}
&\tilde{w}_{12}(r,t,u,v) = \\
&\quad r_{1,4}r_{2,4}\,\mathrm{S}_h^8\,(t-\delta,t+\delta,u,v) + (r_{1,4}r_{2,2}+r_{1,3}r_{2,3}+r_{1,2}r_{2,4})\,\mathrm{S}_h^6\,(t-\delta,t+\delta,u,v) \\
&\qquad + (r_{1,4}r_{2,0}+r_{1,3}r_{2,1}+r_{1,2}r_{2,2}+r_{1,1}r_{2,3}+r_{1,0}r_{2,4})\,\mathrm{S}_h^4\,(t-\delta,t+\delta,u,v) \\
&\qquad + (r_{1,2}r_{2,0}+r_{1,1}r_{2,1}+r_{1,0}r_{2,2})\,\mathrm{S}_h^2\,(t-\delta,t+\delta,u,v) + r_{1,0}r_{2,0}h_0(t-\delta,t+\delta,u,v) \\
&\quad + (r_{1,4}r_{2,3}+r_{1,3}r_{2,4})\,\mathrm{S}_h^7\,(t-\delta,t+\delta,u,v) \\
&\qquad + (r_{1,4}r_{2,1}+r_{1,3}r_{2,2}+r_{1,2}r_{2,3}+r_{1,1}r_{2,4})\,\mathrm{S}_h^5\,(t-\delta,t+\delta,u,v) \\
&\qquad + (r_{1,3}r_{2,0}+r_{1,2}r_{2,1}+r_{1,1}r_{2,2}+r_{1,0}r_{2,3})\,\mathrm{S}_h^3\,(t-\delta,t+\delta,u,v) \\
&\qquad + (r_{1,1}r_{2,0}+r_{1,0}r_{2,1})\,\mathrm{S}_h^1\,(t-\delta,t+\delta,u,v) \\
&\quad - r_{1,4}r_{2,4}\,\mathrm{S}_h^8\,(-t-\delta,-t+\delta,u,v) - (r_{1,4}r_{2,2}+r_{1,3}r_{2,3}+r_{1,2}r_{2,4})\,\mathrm{S}_h^6\,(-t-\delta,-t+\delta,u,v) \\
&\qquad - (r_{1,4}r_{2,0}+r_{1,3}r_{2,1}+r_{1,2}r_{2,2}+r_{1,1}r_{2,3}+r_{1,0}r_{2,4})\,\mathrm{S}_h^4\,(-t-\delta,-t+\delta,u,v) \\
&\qquad - (r_{1,2}r_{2,0}+r_{1,1}r_{2,1}+r_{1,0}r_{2,2})\,\mathrm{S}_h^2\,(-t-\delta,-t+\delta,u,v) \\
&\qquad - r_{1,0}r_{2,0}h_0(-t-\delta,-t+\delta,u,v) \\
&\quad + (r_{1,4}r_{2,3}+r_{1,3}r_{2,4})\,\mathrm{S}_h^7\,(-t-\delta,-t+\delta,u,v) \\
&\quad + (r_{1,4}r_{2,1}+r_{1,3}r_{2,2}+r_{1,2}r_{2,3}+r_{1,1}r_{2,4})\,\mathrm{S}_h^5\,(-t-\delta,-t+\delta,u,v) \\
&\qquad + (r_{1,3}r_{2,0}+r_{1,2}r_{2,1}+r_{1,1}r_{2,2}+r_{1,0}r_{2,3})\,\mathrm{S}_h^3\,(-t-\delta,-t+\delta,u,v) \\
&\qquad + (r_{1,1}r_{2,0}+r_{1,0}r_{2,1})\,\mathrm{S}_h^1\,(-t-\delta,-t+\delta,u,v)
\end{aligned}
$$

5.8 The m_{11} element

$$
m_{11} = -\frac{1}{\sigma}\int \psi_1'\left(\frac{x-\mu}{\sigma}\right)\,dG(x) = (\epsilon-1)\,\kappa_{11}\left(\frac{0-\mu}{\sigma},\frac{1}{\sigma}\right) - \frac{\epsilon}{\alpha}\kappa_{11}\left(\frac{\mu_2-\mu}{\sigma},\frac{\alpha}{\sigma}\right)
$$

where

$$
\begin{aligned}
\kappa_{11}(u,v) &= h_0\,(-a+\delta,a-\delta,u,v) - \frac{a}{c-b}\left[h_0\,(b+\delta,c-\delta,u,v) + h_0\,(-c+\delta,-b-\delta,u,v)\right] \\
&\quad + \tilde{k}_{11}(\alpha,a,u,v) + \tilde{k}_{11}(\beta,b,u,v) + \tilde{k}_{11}(\gamma,c,u,v)
\end{aligned}
$$

and

$$\begin{aligned}\tilde{k}_{11}(r,t,u,v) = {} & 4r_{1,4}\,\mathrm{S}_h^3\,(t-\delta,t+\delta,u,v) + 2r_{1,2}\,\mathrm{S}_h^1\,(t-\delta,t+\delta,u,v) \\ & + 3r_{1,3}\,\mathrm{S}_h^2\,(t-\delta,t+\delta,u,v) + r_{1,1}h_0(t-\delta,t+\delta,u,v) \\ & - 4r_{1,4}\,\mathrm{S}_h^3\,(-t-\delta,-t+\delta,u,v) - 2r_{1,2}\,\mathrm{S}_h^1\,(-t-\delta,-t+\delta,u,v) \\ & + 3r_{1,3}\,\mathrm{S}_h^2\,(-t-\delta,-t+\delta,u,v) + r_{1,1}h_0(-t-\delta,-t+\delta,u,v)\end{aligned}$$

where $r \in \{\{\alpha_{1,j}\},\{\beta_{1,j}\},\{\gamma_{1,j}\}\}$ represents a set of coefficients for the location ψ_1 function.

5.9 The m_{12} element

$$m_{12} = -\frac{1}{\sigma^2}\int (x-\mu)\psi_1'\left(\frac{x-\mu}{\sigma}\right)\,dG(x) = (\epsilon-1)\,\kappa_{12}\left(\frac{0-\mu}{\sigma},\frac{1}{\sigma}\right) - \frac{\epsilon}{\alpha}\kappa_{12}\left(\frac{\mu_2-\mu}{\sigma},\frac{\alpha}{\sigma}\right)$$

where

$$\begin{aligned}\kappa_{12}(u,v) = {} & \mathrm{S}_h^1\,(-a+\delta,a-\delta,u,v) - \frac{a}{c-b}\left[\mathrm{S}_h^1\,(b+\delta,c-\delta,u,v) + \mathrm{S}_h^1\,(-c+\delta,-b-\delta,u,v)\right] \\ & + \tilde{k}_{12}(\alpha,a,u,v) + \tilde{k}_{12}(\beta,b,u,v) + \tilde{k}_{12}(\gamma,c,u,v)\end{aligned}$$

and

$$\begin{aligned}\tilde{k}_{12}(r,t,u,v) = {} & 4r_{1,4}\,\mathrm{S}_h^4\,(t-\delta,t+\delta,u,v) + 2r_{1,2}\,\mathrm{S}_h^2\,(t-\delta,t+\delta,u,v) \\ & + 3r_{1,3}\,\mathrm{S}_h^3\,(t-\delta,t+\delta,u,v) + r_{1,1}\,\mathrm{S}_h^1\,(t-\delta,t+\delta,u,v) \\ & - 4r_{1,4}\,\mathrm{S}_h^4\,(-t-\delta,-t+\delta,u,v) - 2r_{1,2}\,\mathrm{S}_h^2\,(-t-\delta,-t+\delta,u,v) \\ & + 3r_{1,3}\,\mathrm{S}_h^3\,(-t-\delta,-t+\delta,u,v) + r_{1,1}\,\mathrm{S}_h^1\,(-t-\delta,-t+\delta,u,v)\end{aligned}$$

where $r \in \{\{\alpha_{1,j}\},\{\beta_{1,j}\},\{\gamma_{1,j}\}\}$ represents a set of coefficients for the location ψ_1 function.

5.10 The m_{21} element

$$m_{21} = -\frac{1}{\sigma}\int \psi_2'\left(\frac{x-\mu}{\sigma}\right)\,dG(x) = (\epsilon-1)\,\kappa_{21}\left(\frac{0-\mu}{\sigma},\frac{1}{\sigma}\right) - \frac{\epsilon}{\alpha}\kappa_{21}\left(\frac{\mu_2-\mu}{\sigma},\frac{\alpha}{\sigma}\right)$$

where

$$
\begin{aligned}
&\kappa_{21}(u,v) = \\
&\quad 2\,\mathrm{S}_h^1\left(-a+\delta, a-\delta, u, v\right) + \frac{a^2-1-P}{c-b}\left[h_0\left(-c+\delta, -b-\delta, u, v\right) - h_0\left(b+\delta, c-\delta, u, v\right)\right] \\
&\qquad + 5\alpha_{2,5}\,\mathrm{S}_h^4\left(a-\delta, a+\delta, u, v\right) + 3\alpha_{2,3}\,\mathrm{S}_h^2\left(a-\delta, a+\delta, u, v\right) + \alpha_{2,1}h_0(a-\delta, a+\delta, u, v) \\
&\qquad\quad + 4\alpha_{2,4}\,\mathrm{S}_h^3\left(a-\delta, a+\delta, u, v\right) + 2\alpha_{2,2}\,\mathrm{S}_h^1\left(a-\delta, a+\delta, u, v\right) \\
&\qquad\quad - 5\alpha_{2,5}\,\mathrm{S}_h^4\left(-a-\delta, -a+\delta, u, v\right) - 3\alpha_{2,3}\,\mathrm{S}_h^2\left(-a-\delta, -a+\delta, u, v\right) \\
&\qquad\quad - \alpha_{2,1}h_0(-a-\delta, -a+\delta, u, v) \\
&\qquad\quad + 4\alpha_{2,4}\,\mathrm{S}_h^3\left(-a-\delta, -a+\delta, u, v\right) + 2\alpha_{2,2}\,\mathrm{S}_h^1\left(-a-\delta, -a+\delta, u, v\right) \\
&\qquad + \tilde{k}_{21}(\beta, b, u, v) + \tilde{k}_{21}(\gamma, c, u, v)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{k}_{21}(r,t,u,v) = {} & 4r_{2,4}\,\mathrm{S}_h^3\left(t-\delta, t+\delta, u, v\right) + 2r_{2,2}\,\mathrm{S}_h^1\left(t-\delta, t+\delta, u, v\right) \\
& + 3r_{2,3}\,\mathrm{S}_h^2\left(t-\delta, t+\delta, u, v\right) + r_{2,1}h_0(t-\delta, t+\delta, u, v) \\
& + 4r_{2,4}\,\mathrm{S}_h^3\left(-t-\delta, -t+\delta, u, v\right) + 2r_{2,2}\,\mathrm{S}_h^1\left(-t-\delta, -t+\delta, u, v\right) \\
& - 3r_{2,3}\,\mathrm{S}_h^2\left(-t-\delta, -t+\delta, u, v\right) - r_{2,1}h_0(-t-\delta, -t+\delta, u, v)
\end{aligned}
$$

where $r \in \{\{\alpha_{2,j}\}, \{\beta_{2,j}\}, \{\gamma_{2,j}\}\}$ represents a set of coefficients for the location ψ_2 function.

5.11 The m_{22} element

$$
m_{22} = -\frac{1}{\sigma^2}\int (x-\mu)\psi_2'\left(\frac{x-\mu}{\sigma}\right)\, dG(x) = (\epsilon-1)\,\kappa_{22}\left(\frac{0-\mu}{\sigma}, \frac{1}{\sigma}\right) - \frac{\epsilon}{\alpha}\kappa_{22}\left(\frac{\mu_2-\mu}{\sigma}, \frac{\alpha}{\sigma}\right)
$$

where

$$
\begin{aligned}
&\kappa_{22}(u,v) = \\
&\quad 2\,\mathrm{S}_h^2\,(-a+\delta, a-\delta, u, v) + \frac{a^2-1-P}{c-b}\left[\mathrm{S}_h^1\,(-c+\delta, -b-\delta, u, v) - \mathrm{S}_h^1\,(b+\delta, c-\delta, u, v)\right] \\
&\qquad + 5\alpha_{2,5}\,\mathrm{S}_h^5\,(a-\delta, a+\delta, u, v) + 3\alpha_{2,3}\,\mathrm{S}_h^3\,(a-\delta, a+\delta, u, v) + \alpha_{2,1}\,\mathrm{S}_h^1\,(a-\delta, a+\delta, u, v) \\
&\qquad\quad + 4\alpha_{2,4}\,\mathrm{S}_h^4\,(a-\delta, a+\delta, u, v) + 2\alpha_{2,2}\,\mathrm{S}_h^2\,(a-\delta, a+\delta, u, v) \\
&\qquad\quad - 5\alpha_{2,5}\,\mathrm{S}_h^5\,(-a-\delta, -a+\delta, u, v) - 3\alpha_{2,3}\,\mathrm{S}_h^3\,(-a-\delta, -a+\delta, u, v) \\
&\qquad\quad - \alpha_{2,1}\,\mathrm{S}_h^1\,(-a-\delta, -a+\delta, u, v) \\
&\qquad\quad + 4\alpha_{2,4}\,\mathrm{S}_h^4\,(-a-\delta, -a+\delta, u, v) + 2\alpha_{2,2}\,\mathrm{S}_h^2\,(-a-\delta, -a+\delta, u, v) \\
&\qquad + \tilde{k}_{22}(\beta, b, u, v) + \tilde{k}_{22}(\gamma, c, u, v)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{k}_{22}(r,t,u,v) = {} & 4r_{2,4}\,\mathrm{S}_h^4\,(t-\delta, t+\delta, u, v) + 2r_{2,2}\,\mathrm{S}_h^2\,(t-\delta, t+\delta, u, v) \\
& + 3r_{2,3}\,\mathrm{S}_h^3\,(t-\delta, t+\delta, u, v) + r_{2,1}\,\mathrm{S}_h^1\,(t-\delta, t+\delta, u, v) \\
& + 4r_{2,4}\,\mathrm{S}_h^4\,(-t-\delta, -t+\delta, u, v) + 2r_{2,2}\,\mathrm{S}_h^2\,(-t-\delta, -t+\delta, u, v) \\
& - 3r_{2,3}\,\mathrm{S}_h^3\,(-t-\delta, -t+\delta, u, v) - r_{2,1}\,\mathrm{S}_h^1\,(-t-\delta, -t+\delta, u, v)
\end{aligned}
$$

where $r \in \{\{\alpha_{2,j}\}, \{\beta_{2,j}\}, \{\gamma_{2,j}\}\}$ represents a set of coefficients for the location ψ_2 function.