Supplementary Materials for

The Spiral Spectrum of a Laguerre–Gaussian Beam Carrying the Cross-Phase Propagating in Weak-to-Strong Atmospheric Turbulence

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Theoretical derivation of the propagation formula for a Laguerre–Gaussian beam carrying the cross-phase in atmospheric turbulence

We consider the example of a Laguerre-Gaussian (LG) beam carrying the cross-phase with a radial mode index p=0 and an azimuthal mode index l, whose electric field in the source plane has the following form:

$$E_0(\mathbf{r}) = Cr^{|l|}\exp\left(-\frac{r^2}{\omega_0^2}\right)\exp(il\varphi)\exp(iuxy), \tag{S1}$$

where $C=\sqrt{\frac{2}{\pi\omega_0^2|l|!}}\left(\frac{\sqrt{2}}{\omega_0}\right)^{|l|}$ denotes the constant coefficient, $\boldsymbol{r}=(x, y)$ and $\varphi = \arctan(y/x)$ are the radial coordinate and azimuthal angle coordinate, respectively. ω_0 is the initial beam width. The last term $\exp(iuxy)$ is CP structure, where the quantity u stands for the CP strength factor.

Based on the extended Huygens–Fresnel integral, the mutual coherent function of an LG beam carrying the cross-phase in the receiving plane after propagating in atmospheric turbulence can be written as follows [1]:

$$\begin{aligned}\Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z) = &\frac{1}{\lambda^2 z^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_0(\mathbf{r}_1)E_0^*(\mathbf{r}_2)\exp\left[\frac{ik}{2z}(\boldsymbol{\rho}_1-\mathbf{r}_1)^2-\frac{ik}{2z}(\boldsymbol{\rho}_2-\mathbf{r}_2)^2\right],\\ &\times\left\langle\exp\left[\psi(r_1,\rho_1,z)+\psi^*(r_2,\rho_2,z)\right]\right\rangle d^2\mathbf{r}_1 d^2\mathbf{r}_2\end{aligned} \tag{S2}$$

where $\boldsymbol{\rho}_i=(\rho_{ix}, \rho_{iy})$ (i =1, 2) denotes the transverse position vectors in the receiving plane (z>0). λ and the asterisk stand for the wavelength of the light beam and the complex conjugate, respectively. The angle brackets represent the ensemble averaging over the turbulence fluctuations, where the term ψ ($\mathbf{r}$, $\boldsymbol{\rho}$, z) is turbulence-induced complex perturbations of a spherical wave propagating from ($\mathbf{r}$, 0) to ($\boldsymbol{\rho}$, z).

To generalize our calculation results, we assume that the turbulence is anisotropic and obeys the Kolmogorov spectrum when calculating Eq. (S2). According to the turbulence spectrum model with the power law index α=11/3 proposed in reference, the second-order statistics of the complex phase perturbation may be represented as follows:

$$\left\langle\exp\left[\psi(r_1,\rho_1,z)+\psi^*(r_2,\rho_2,z)\right]\right\rangle = \exp\left[-T\mu_z\left(\frac{\rho_{xd}^2}{\mu_x^2}+\frac{\rho_{yd}^2}{\mu_y^2}+\frac{x_d^2}{\mu_x^2}+\frac{y_d^2}{\mu_y^2}+\frac{\rho_{xd}x_d}{\mu_x^2}+\frac{\rho_{yd}y_d}{\mu_y^2}\right)\right], \tag{S3}$$

with

$$\begin{aligned}&\boldsymbol{\rho}_d = \boldsymbol{\rho}_1-\boldsymbol{\rho}_2 = (\rho_{xd},\rho_{yd}) = (\rho_{1x}-\rho_{2x},\rho_{1y}-\rho_{2y})\\ &\mathbf{r}_d = \mathbf{r}_1-\mathbf{r}_2 = (x_d,y_d) = (x_1-x_2,y_1-y_2)\\ &T = 0.0033\pi^2k^2zC_n^2\left[\eta\kappa_m^{-5/3}\exp(\kappa_0^2/\kappa_m^2)\Gamma_1(1/6,\kappa_0^2/\kappa_m^2)-2\kappa_0^{1/3}\right],\\ &\eta = 2\kappa_0^2+5/3\kappa_m^2, \kappa_0 = 2\pi/L_0, \kappa_m = 5.92/l_0\end{aligned} \tag{S4}$$

where k is the wave number and C_n^2 denotes the structure constant of turbulence with unit m$^{-2/3}$. L_0 and l_0 stand for the outer and inner scale of the turbulence, respectively. The symbol Γ_1 is the incomplete Gamma function. Three anisotropic factors μ_x, μ_y and μ_z were introduced in anisotropic turbulence that was pertinent to the size of eddies along the x, y and z directions [2].

By substituting Eq. (S1) and Eq. (S3) into Eq. (S2), we can rewrite Eq. (S2) as follows:

$$\begin{aligned}\Gamma(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,z) = &\frac{1}{\lambda^2z^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|C|^2 r_1^{|l|}r_2^{|l|}\exp\left[il(\varphi_1-\varphi_2)\right]\exp\left[-\frac{\mathbf{r}_1^2+\mathbf{r}_2^2}{\omega_0^2}\right]\exp(iux_1y_1-iux_2y_2)\\ &\times\exp\left[\frac{ik}{2z}(\boldsymbol{\rho}_1-\mathbf{r}_1)^2-\frac{ik}{2z}(\boldsymbol{\rho}_2-\mathbf{r}_2)^2\right]\exp\left[-T\mu_z\left(\frac{\rho_{xd}^2}{\mu_x^2}+\frac{\rho_{yd}^2}{\mu_y^2}+\frac{x_d^2}{\mu_x^2}+\frac{y_d^2}{\mu_y^2}+\frac{\rho_{xd}x_d}{\mu_x^2}+\frac{\rho_{yd}y_d}{\mu_y^2}\right)\right]d^2\mathbf{r}_1d^2\mathbf{r}_2\end{aligned} ,\tag{S5}$$

To evaluate Eq. (S5), we use the binomial expansion theorem as follows:

$$\left(x_1+iy_1\right)^l=\sum_{c1=0}^{l}\frac{l!i^{c_1}}{c_1!\left(l-c_1\right)!}x_1^{l-c_1}y_1^{c_1},\quad \left(x_2-iy_2\right)^l=\sum_{c2=0}^{l}\frac{l!\left(-i\right)^{c_2}}{c_2!\left(l-c_2\right)!}x_2^{l-c_2}y_2^{c_2},\quad l>0\,. \tag{S6}$$

By substituting Eq. (S6) into Eq. (S5), we obtain the formula:

$$\begin{aligned}\Gamma(\rho_1,\rho_2,z)=&\frac{|C|^2}{\lambda^2z^2}\sum_{c_1=0}^{l}\sum_{c_2=0}^{l}\frac{l!i^{c_1}}{c_1!(l-c_1)!}\frac{l!(-i)^{c_2}}{c_2!(l-c_2)!}\exp\left[\frac{ik}{2z}\left(\boldsymbol{\rho}_1^2-\boldsymbol{\rho}_2^2\right)\right]\exp\left[-a\rho_{xd}^2-b\rho_{yd}^2\right]\\&\times\int_{-\infty}^{\infty}x_2^{l-c_2}\exp\left[-N_1^*x_2^2\right]\exp\left[\frac{ik}{z}\rho_{x2}x_2+a\rho_{xd}x_2-iuy_2x_2\right]dx_2\\&\times\int_{-\infty}^{\infty}y_2^{c_2}\exp\left[-N_2^*y_2^2\right]\exp\left[\frac{ik}{z}\rho_{y2}y_2+b\rho_{yd}y_2\right]dy_2\\&\times\int_{-\infty}^{\infty}y_1^{c_1}\exp\left[-N_2y_1^2\right]\exp\left[\left(-\frac{ik}{z}\rho_{y1}+2by_2-b\rho_{yd}\right)y_1\right]dy_1\\&\times\int_{-\infty}^{\infty}x_1^{l-c_1}\exp\left[-N_1x_1^2\right]\exp\left[\left(-\frac{ik}{z}\rho_{x1}+2ax_2-a\rho_{xd}+iuy_1\right)x_1\right]dx_1\end{aligned}\quad, \tag{S7}$$

with

$$a=\frac{T\mu_z}{\mu_x^2},b=\frac{T\mu_z}{\mu_y^2},N_1=\left(\frac{1}{\omega_0^2}-\frac{ik}{2z}+a\right),N_2=\left(\frac{1}{\omega_0^2}-\frac{ik}{2z}+b\right), \tag{S8}$$

To evaluate Eq. (S7), we complete the operation with the help of the following equations:

$$\begin{aligned}&\int_{-\infty}^{\infty}x^{\alpha}\exp\left[-\left(x-\beta\right)^2\right]dx=\left(2i\right)^{-\alpha}\sqrt{\pi}H_{\alpha}\left(i\beta\right)\\&H_{\alpha}\left(x+\beta\right)=\frac{1}{2^{\alpha/2}}\sum_{p=0}^{\alpha}\binom{\alpha}{p}H_p\left(\sqrt{2}x\right)H_{\alpha-p}\left(\sqrt{2}\beta\right),\\&H_n\left(x_1\right)=\sum_{m=0}^{[n/2]}(-1)^m\frac{n!}{m!\left(n-2m\right)!}\left(2x_1\right)^{n-2m}\end{aligned} \tag{S9}$$

After integration over variables x_1, x_2, y_1, y_2, we can obtain the final analytical expression for mutual coherent function of a LG beam carrying the cross-phase in the receiving plane as follows:

$$\begin{aligned}\Gamma(\rho_1,\rho_2,z)=&\frac{|C|^2}{\lambda^2z^2}\sum_{c_1=0}^{l}\sum_{c_2=0}^{l}\frac{l!i^{c_1}}{c_1!(l-c_1)!}\frac{l!(-i)^{c_2}}{c_2!(l-c_2)!}\exp\left[\frac{ik}{2z}\left(\boldsymbol{\rho}_1^2-\boldsymbol{\rho}_2^2\right)\right]\exp\left[-a\rho_{xd}^2-b\rho_{yd}^2\right]\\&\times\frac{\pi^2\left(2i\right)^{2m_1-l-q_1-n_1-n_2}}{2^{(l+p_1+q_1-2m_1+p_2+n_1)/2}}N_1^{(c_1-l-1)/2}N_3^{-(c_1+q_1-2m_1+1)/2}N_4^{-(n_1+1)/2}N_6^{-(n_2+1)/2}\exp\left[\frac{\Delta_{x1}^2}{4N_1}+\frac{\Delta_{y1}^2}{4N_3}+\frac{\Delta_{x2}^2}{4N_4}+\frac{\Delta_{y_2}^2}{4N_6}\right]\\&\times\sum_{p_1=0}^{l-c_1}\sum_{q_1=0}^{p_1}\sum_{m_1=0}^{[q_1/2]}\sum_{m_2=0}^{[(p_1-q_1)/2]}\sum_{p_2=0}^{c_1+q_1-2m_1}\sum_{q_2=0}^{p_2}\sum_{m_3=0}^{[q_2/2]}\sum_{m_4=0}^{[(p_2-q_2)/2]}\sum_{p_3=0}^{n_1}\sum_{m_5=0}^{[p_3/2]}\binom{l-c_1}{p_1}\binom{p_1}{q_1}\binom{c_1+q_1-2m_1}{p_2}\binom{p_2}{q_2}\binom{n_1}{p_3}\\&\times\frac{(q_1)!}{m_1!(q_1-2m_1)!}\frac{(p_1-q_1)!}{m_2!(p_1-q_1-2m_2)!}\frac{q_2!}{m_3!(q_2-2m_3)!}\frac{(p_2-q_2)!}{m_4!(p_2-q_2-2m_4)!}\frac{p_3!}{m_5!(p_3-2m_5)!}\\&\times\left(-\frac{2u}{\sqrt{N_1}}\right)^{q_1-2m_1}\left(\frac{i4a}{\sqrt{N_1}}\right)^{p_1-q_1-2m_2}\left(\frac{4ib}{\sqrt{N_3}}\right)^{q_2-2m_3}\left(-\frac{2ua}{N_1\sqrt{N_3}}\right)^{p_2-q_2-2m_4}\left(\frac{\sqrt{2}iN_5}{\sqrt{N_4}}\right)^{p_3-2m_5}\\&\times(-1)^{m_1+m_2+m_3+m_4+m_5}H_{l-c_1-p_1}\left(\frac{i\Delta_{x1}}{\sqrt{2N_1}}\right)H_{c_1+q_1-2m_1-p2}\left(\frac{i\Delta_{y1}}{2\sqrt{N_3}}\right)H_{n_1-p_3}\left(\frac{i\Delta_{x2}}{\sqrt{2N_4}}\right)H_{n_2}\left(\frac{i\Delta_{y2}}{2\sqrt{N_6}}\right)\end{aligned}\quad, \tag{S10}$$

with

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\begin{aligned}
&N_3 = N_2 + \frac{u^2}{4N_1}, N_4 = N_1^* - \frac{a^2}{N_1} + \frac{u^2 a^2}{4N_1^2 N_3}, \;\; N_5 = \frac{iuab}{N_1 N_3} - iu, N_6 = N_2^* - \frac{b^2}{N_3} - \frac{N_5^2}{4N_4} \\
&\Delta_{x1} = -\frac{ik}{z}\rho_{x1} - a\rho_{xd}, \Delta_{y1} = -\frac{ik}{z}\rho_{y1} - b\rho_{yd} + \frac{iu\Delta_{x1}}{2N_1} \\
&\Delta_{x2} = \frac{ik}{z}\rho_{x2} + a\rho_{xd} + \frac{\Delta_{x1} a}{N_1} + \frac{\Delta_{y1} iua}{2N_1 N_3}, \;\; \Delta_{y2} = \frac{ik}{z}\rho_{y2} + b\rho_{yd} + \frac{b\Delta_{y1}}{N_3} + \frac{N_5 \Delta_{x2}}{2N_4} \\
&n_1 = l - c_2 + p_1 - q_1 - 2m_2 + p_2 - q_2 - 2m_4, \;\; n_2 = c_2 + q_2 - 2m_3 + p_3 - 2m_5
\end{aligned}
\tag{S11}
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References

1. Andrews, L.C. and Phillips, R.L. *Laser Beam Propagation through Random Media*; SPIE Press: Bellingham, USA, 2005.
2. Zeng, J.; Liu, X.; Zhao, C.; Wang, F.; Gbur, G. and Cai, Y. Spiral spectrum of a Laguerre-Gaussian beam propagating in anisotropic non-Kolmogorov turbulent atmosphere along horizontal path. *Opt. Express* **2019**, 27, 25342-25356.