Supplementary S1

Linear Representation

Utilizing the Taylor series expansion, $sin(x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!}$, we get

$$\begin{aligned}
f(z;b,\alpha,\lambda,\underline{\phi}) &= 2\pi b\alpha\lambda g(z;\underline{\phi})\overline{G}^{\lambda-1}(z;\underline{\phi})\frac{[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha-1}}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha+1}} \\
&\times \left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^2\right\}^{b-1} \\
&\times \sum_{j=0}^{\infty}\frac{(-1)^j\left(\frac{\pi}{2}\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^2\right\}^{b}\right)^{2j+1}}{(2j+1)!} \\
&= \sum_{j=0}^{\infty}\frac{2\pi b\alpha\lambda(0.5\pi)^{2j+1}(-1)^j}{(2j+1)!}g(z;\underline{\phi})\overline{G}^{\lambda-1}(z;\underline{\phi})\frac{[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha-1}}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha+1}} \\
&\times \left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^2\right\}^{2b(j+1)-1}.
\end{aligned}$$

Applying the generalized binomial series expansion,

$$\begin{aligned}
\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^2\right\}^{2b(j+1)-1} &= \sum_{k=0}^{n}(-1)^k\binom{2b(j+1)-1}{k} \\
&\times \left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2k},
\end{aligned}$$

we get

$$\begin{aligned}
f(z;b,\alpha,\lambda,\underline{\phi}) &= \sum_{j,k=0}^{\infty}\binom{2b(j+1)-1}{k}\frac{2\pi b\alpha\lambda(0.5\pi)^{2j+1}(-1)^{j+k}}{(2j+1)!}g(z;\underline{\phi}) \\
&\times \overline{G}^{\lambda-1}(z;\underline{\phi})\frac{[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha-1}}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha+1}}\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2k+1}.
\end{aligned}$$

Now, employing the generalized binomial series expansion,

$$\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2k+1} = \sum_{l=0}^{\infty}(-1)^l\binom{2k+1}{l}\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha l},$$

we get

$$f(z;b,\alpha,\lambda,\underline{\phi}) = \sum_{j,k,l=0}^{\infty}\binom{2b(j+1)-1}{k}\binom{2k+1}{l}\frac{2\pi b\alpha\lambda(0.5\pi)^{2j+1}(-1)^{i+j+k}}{(2j+1)!} \times g(z;\underline{\phi})\overline{G}^{\lambda-1}(z;\underline{\phi})\frac{[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha(l+1)-1}}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha(l+1)+1}}.$$

Employing the generalized binomial series expansions,

$$[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{-\alpha(l+1)-1} = \sum_{m=0}^{\infty}\binom{-\alpha(l+1)-1}{m}\overline{G}^{\lambda m}(z;\underline{\phi})$$

and

$$[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha(l+1)-1} = \sum_{p=0}^{\infty}(-1)^p\binom{\alpha(l+1)-1}{p}\overline{G}^{\lambda p}(z;\underline{\phi}),$$

we have

$$\begin{aligned} f(z;b,\alpha,\lambda,\underline{\phi}) &= \sum_{j,k,l,m,p=0}^{\infty}\binom{2b(j+1)-1}{k}\binom{2k+1}{l}\binom{-\alpha(l+1)-1}{m} \\ &\times \binom{\alpha(l+1)-1}{p}\frac{2\pi b\alpha\lambda(0.5\pi)^{2j+1}(-1)^{j+k+m+p}}{(2j+1)!} \\ &\times g(z;\underline{\phi})\overline{G}^{\lambda(m+p+1)-1}(z;\underline{\phi}). \end{aligned}$$

Furthermore, with

$$\begin{aligned} \overline{G}^{\lambda(m+p+1)-1}(z;\underline{\phi}) &= [1-G(z;\underline{\phi})]^{\lambda(m+p+1)-1} \\ &= \sum_{q=0}^{\infty}(-1)^q\binom{\lambda(m+p+1)-1}{q}G^q(z;\underline{\phi}), \end{aligned}$$

we get

$$\begin{aligned} f(z;b,\alpha,\lambda,\underline{\phi}) &= \sum_{j,k,l,m,p,q=0}^{\infty}\binom{2b(j+1)-1}{k}\binom{2k+1}{l}\binom{-\alpha(l+1)-1}{m} \\ &\times \binom{\alpha(l+1)-1}{p}\binom{\lambda(m+p+1)-1}{q} \\ &\times \frac{2\pi b\alpha\lambda(0.5\pi)^{2j+1}(-1)^{i+j+k+m+p+q}}{(2j+1)!}\left(\frac{q+1}{q+1}\right)g(z;\underline{\phi})G^q(z;\underline{\phi}) \\ &= \sum_{q=0}^{\infty}\varepsilon_{q+1}g_{q+1}(z;\underline{\phi}), \end{aligned}$$

where $g_{q+1}(z;\underline{\phi}) = (q+1)g(z;\underline{\phi})G^q(z;\underline{\phi})$ is the exponentiated-G (Exp-G) distribution with parameter $(q+1)$, and

$$\begin{aligned}\varepsilon_{q+1} &= \sum_{j,k,l,m,p=0}^{\infty} \binom{2b(j+1)-1}{k}\binom{2k+1}{l}\binom{-\alpha(k+1)-1}{m} \\ &\times \binom{\alpha(k+1)-1}{p}\binom{\lambda(m+p+1)-1}{q}\frac{1}{(q+1)} \\ &\times \frac{2\pi b\alpha\lambda(0.5\pi)^{2j+1}(-1)^{i+j+k+m+p}}{(2j+1)!}.\end{aligned}$$

Order Statistics

The pdf of the ith order statistics from the Cos-TL-EHL-G FoD is defined as

$$f_{i:n}(z) = \frac{1}{B(i,n-i+1)}f(z)\sum_{r=0}^{n-i}(-1)^r\binom{n-i}{r}[F(z;\underline{\phi})]^{r+i-1}. \tag{S1}$$

Using the pdf and cdf of the Cos-TL-EHL-G FoD, we write

$$\begin{aligned}f(z)[F(z)]^{r+i-1} &= 2\pi b\alpha\lambda g(z;\underline{\phi})\overline{G}^{\lambda-1}(z;\underline{\phi})\frac{[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha-1}}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha+1}} \\ &\times \left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^2\right\}^{b-1} \\ &\times sin(B)\,[1-cos(B)]^{r+i-1},\end{aligned}$$

where $B = \left(\frac{\pi}{2}\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^2\right\}^{b}\right)$. Applying the generalized binomial series expansion, $(1-x)^n = \sum_{q=0}^{\infty}(-1)^q\binom{n}{q}x^q$, $|x|<1$, we have

$$\begin{aligned}[1-cos(B)]^{r+i-1} &= \sum_{s=0}^{\infty}(-1)^s\binom{r+i-1}{s}[Cos(B)]^s \\ &= \sum_{s=0}^{\infty}(-1)^s\binom{r+i-1}{s}[1-sin^2(B)]^{\frac{s}{2}} \\ &= \sum_{s,t=0}^{\infty}(-1)^{s+t}\binom{r+i-1}{s}\binom{s/2}{t}sin^{2t}(B),\end{aligned}$$

so that

$$
\begin{aligned}
f(z)[F(z)]^{r+i-1} &= \sum_{s,t=0}^{\infty}(-1)^{s+t}\binom{r+i-1}{s}\binom{s/2}{t} \\
&\times \quad 2\pi b\alpha\lambda g(z;\underline{\phi})\overline{G}^{\lambda-1}(z;\underline{\phi})\frac{[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha-1}}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha+1}} \\
&\times \quad \left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2}\right\}^{b-1} \\
&\times \quad sin^{2t+1}(B).
\end{aligned}
$$

Applying Taylor series expansion of $sin^{2t+1}(B) = \sum_{j=0}^{\infty}\binom{2t+1}{2j+1}\frac{(-1)^j B^{2j+1}}{(2j+1)!}$, we get

$$
\begin{aligned}
f(z)[F(z)]^{r+i-1} &= \sum_{s,t,j=0}^{\infty}(-1)^{s+t}\binom{r+i-1}{s}\binom{s/2}{t}\binom{2t+1}{2j+1} \\
&\times \quad 2\pi b\alpha\lambda g(z;\underline{\phi})\overline{G}^{\lambda-1}(z;\underline{\phi})\frac{[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha-1}}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha+1}} \\
&\times \quad \left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2}\right\}^{b-1} \\
&\times \quad \frac{(-1)^j\left(\frac{\pi}{2}\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2}\right\}^{b}\right)^{2j+1}}{(2j+1)!} \\
&= \sum_{s,t,j=0}^{\infty}\frac{2\pi b\alpha\lambda(0.5\pi)^{2j+1}(-1)^{s+t+j}}{(2j+1)!}\binom{r+i-1}{s}\binom{s/2}{t}\binom{2t+1}{2j+1} \\
&\times \quad g(z;\underline{\phi})\overline{G}^{\lambda-1}(z;\underline{\phi})\frac{[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha-1}}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha+1}} \\
&\times \quad \left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2}\right\}^{2b(j+1)-1}.
\end{aligned}
$$

Applying the generalized binomial series expansion,

$$\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2}\right\}^{2b(j+1)-1} = \sum_{k=0}^{n}(-1)^{k}\binom{2b(j+1)-1}{k} \times \left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2k},$$

we get

$$f(z)[F(z)]^{r+i-1} = \sum_{s,t,j=0}^{\infty}\frac{2\pi b\alpha\lambda(0.5\pi)^{2j+1}(-1)^{s+t+j+k}}{(2j+1)!}\binom{r+i-1}{s}\binom{s/2}{t}\binom{2t+1}{2j+1} \times \binom{2b(j+1)-1}{k}g(z;\underline{\phi})\overline{G}^{\lambda-1}(z;\underline{\phi})\frac{[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha-1}}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha+1}} \times \left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2k+1}.$$

Further, applying the generalized binomial series expansion,

$$\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2k+1} = \sum_{l=0}^{\infty}(-1)^{l}\binom{2k+1}{l}\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha l},$$

we get

$$f(z)[F(z)]^{r+i-1} = \sum_{s,t,j,k,l=0}^{\infty}\frac{2\pi b\alpha\lambda(0.5\pi)^{2j+1}(-1)^{s+t+j+k}}{(2j+1)!}\binom{r+i-1}{s}\binom{s/2}{t}\binom{2t+1}{2j+1} \times \binom{2b(j+1)-1}{k}\binom{2k+1}{l}g(z;\underline{\phi})\overline{G}^{\lambda-1}(z;\underline{\phi})\frac{[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha(l+1)-1}}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha(l+1)+1}}.$$

Using the generalized binomial series expansion

$$[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{-\alpha(l+1)-1} = \sum_{m=0}^{\infty}\binom{-\alpha(l+1)-1}{m}\overline{G}^{\lambda m}(z;\underline{\phi})$$

and

$$[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\alpha(l+1)-1} = \sum_{p=0}^{\infty}(-1)^{p}\binom{\alpha(l+1)-1}{p}\overline{G}^{\lambda p}(z;\underline{\phi}),$$

we obtain

$$\begin{aligned} f(z)[F(z)]^{r+i-1} &= \sum_{s,t,j,k,l,m,p=0}^{\infty} \frac{2\pi b\alpha\lambda(0.5\pi)^{2j+1}(-1)^{s+t+j+k+p}}{(2j+1)!}\binom{r+i-1}{s}\binom{s/2}{t} \\ &\times \binom{2t+1}{2j+1}\binom{2b(j+1)-1}{k}\binom{2k+1}{l}\binom{-\alpha(l+1)-1}{m} \\ &\times \binom{\alpha(l+1)-1}{p} g(z;\underline{\phi})\overline{G}^{\lambda(m+p+1)-1}(z;\underline{\phi}). \end{aligned}$$

Furthermore, with

$$\begin{aligned} \overline{G}^{\lambda(m+p+1)-1}(z;\underline{\phi}) &= [1-G(z;\underline{\phi})]^{\lambda(m+p+1)-1} \\ &= \sum_{q=0}^{\infty}(-1)^q\binom{\lambda(m+p+1)-1}{q}G^q(z;\underline{\phi}) \end{aligned}$$

we get

$$\begin{aligned} f(z)[F(z)]^{r+i-1} &= \sum_{s,t,j,k,l,m,p,q=0}^{\infty} \frac{2\pi b\alpha\lambda(0.5\pi)^{2j+1}(-1)^{s+t+j+k+p+q}}{(2j+1)!}\binom{r+i-1}{s} \\ &\times \binom{s/2}{t}\binom{2t+1}{2j+1}\binom{2b(j+1)-1}{k}\binom{2k+1}{l}\binom{-\alpha(l+1)-1}{m} \\ &\times \binom{\alpha(l+1)-1}{p}\binom{\lambda(m+p+1)-1}{q}\left(\frac{q+1}{q+1}\right) g(z;\underline{\phi})G^q(z;\underline{\phi}) \\ &= \sum_{q=0}^{\infty}\eta_{q+1}g_{q+1}(z;\underline{\phi}), \end{aligned} \tag{S2}$$

where $g_{q+1}(z;\underline{\phi}) = (q+1)g(z;\underline{\phi})G^q(z;\underline{\phi})$ is the Exp-G distribution with parameter $(q+1)$, and

$$\begin{aligned} \eta_{q+1} &= \sum_{s,t,j,k,l,m,p=0}^{\infty} \frac{2\pi b\alpha\lambda(0.5\pi)^{2j+1}(-1)^{s+t+j+k+p}}{(2j+1)!(q+1)}\binom{r+i-1}{s} \\ &\times \binom{s/2}{t}\binom{2t+1}{2j+1}\binom{2b(j+1)-1}{k}\binom{2k+1}{l}\binom{-\alpha(l+1)-1}{m} \\ &\times \binom{\alpha(l+1)-1}{p}\binom{\lambda(m+p+1)-1}{q}. \end{aligned} \tag{S3}$$

Therefore, substituting Equation (S2) into (S1), we obtain

$$f_{i:n}(z) = \frac{1}{B(i,n-i+1)}\sum_{q=0}^{\infty}\sum_{r=0}^{n-i}(-1)^r\binom{n-i}{r}\eta_{q+1}g_{q+1}(z;\underline{\phi}). \tag{S4}$$

Uncertainty Measure

If $Z \sim$ Cos-TL-EHL-G FoD, the Rényi of order θ is defined as

$$I_R(\theta) = \frac{1}{1-\theta} \log \left(\int_0^{\infty} [f(z)]^{\theta} \, dz \right), \tag{S5}$$

for $\theta > 0$ and $\theta \neq 1$. Using the pdf of Cos-TL-EHL-G FoD, we write

$$\begin{aligned} f^{\theta}(z) &= (2\pi b\alpha\lambda)^{\theta} g^{\theta}(z;\underline{\phi}) \overline{G}^{\theta(\lambda-1)}(z;\underline{\phi}) \frac{[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\theta(\alpha-1)}}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{\theta(\alpha+1)}} \\ &\times \left[1 - \left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})} \right)^{\alpha} \right]^{\theta} \left\{ 1 - \left[1 - \left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})} \right)^{\alpha} \right]^{2} \right\}^{\theta(b-1)} \\ &\times \left[sin\left(\frac{\pi}{2} \Omega \right) \right]^{\theta}, \end{aligned} \tag{S6}$$

where $\Omega = \left\{ 1 - \left[1 - \left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})} \right)^{\alpha} \right]^{2} \right\}^{b}$. Let $W(\Omega) = sin^{\theta}(\frac{\pi}{2}\Omega)$, then the Taylor series expansion of

$$sin^{\theta}\left(\frac{\pi}{2} \Omega \right) = \sum_{q=0}^{\infty} \sum_{j=0}^{q} (-1)^{q} a_q \binom{q}{j} \Omega^{j},$$

where $a_q = \frac{(-1)^q W^q(1)}{q!}$ and $W^q(1)$ is the qth derivative of $W(\cdot)$, (Nanga, 2023). Therefore,

$$\begin{aligned} f^{\theta}(z) &= (2\pi b\alpha\lambda)^{\theta} g^{\theta}(z;\underline{\phi}) \overline{G}^{\theta(\lambda-1)}(z;\underline{\phi}) \frac{[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\theta(\alpha-1)}}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{\theta(\alpha+1)}} \\ &\times \left[1 - \left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})} \right)^{\alpha} \right]^{\theta} \left\{ 1 - \left[1 - \left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})} \right)^{\alpha} \right]^{2} \right\}^{\theta(b-1)} \\ &\times \sum_{q=0}^{\infty} \sum_{j=0}^{q} (-1)^{q} a_q \binom{q}{j} \left\{ 1 - \left[1 - \left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})} \right)^{\alpha} \right]^{2} \right\}^{bj} \\ &= \sum_{q=0}^{\infty} \sum_{j=0}^{q} (2\pi b\alpha\lambda)^{\theta} (-1)^{q} a_q \binom{q}{j} g^{\theta}(z;\underline{\phi}) \overline{G}^{\theta(\lambda-1)}(z;\underline{\phi}) \frac{[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\theta(\alpha-1)}}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{\theta(\alpha+1)}} \\ &\times \left[1 - \left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})} \right)^{\alpha} \right]^{\theta} \left\{ 1 - \left[1 - \left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})} \right)^{\alpha} \right]^{2} \right\}^{\theta(b-1)+bj}. \end{aligned}$$

Applying the binomial series expansion,

$$\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2}\right\}^{\theta(b-1)+bj} = \sum_{k=0}^{\infty}(-1)^{k}\binom{\theta(b-1)+bj}{k} \times \left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2k}$$

we get

$$f^{\theta}(z) = \sum_{q,k=0}^{\infty}\sum_{j=0}^{q}(2\pi b\alpha\lambda)^{\theta}(-1)^{q+k}a_{q}\binom{q}{j}\binom{\theta(b-1)+bj}{k}g^{\theta}(z;\underline{\phi})\overline{G}^{\theta(\lambda-1)}(z;\underline{\phi}) \times \frac{[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\theta(\alpha-1)}}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{\theta(\alpha+1)}}\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{\theta+2k}.$$

Further, applying the binomial series expansion

$$\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{\theta+2k} = \sum_{l=0}^{\infty}(-1)^{l}\binom{\theta+2k}{l}\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha l},$$

we get

$$f^{\theta}(z) = \sum_{q,k,l=0}^{\infty}\sum_{j=0}^{q}(2\pi b\alpha\lambda)^{\theta}(-1)^{q+k+l}a_{q}\binom{q}{j}\binom{\theta(b-1)+bj}{k}\binom{\theta+2k}{l} \times g^{\theta}(z;\underline{\phi})\overline{G}^{\theta(\lambda-1)}(z;\underline{\phi})\frac{[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\theta(\alpha-1)+\alpha l}}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{\theta(\alpha+1)+\alpha l}}.$$

Using the generalized binomial series expansion

$$[1+\overline{G}^{\lambda}(z;\underline{\phi})]^{-\theta(\alpha+1)-\alpha l} = \sum_{m=0}^{\infty}\binom{-\theta(\alpha+1)-\alpha l}{m}\overline{G}^{\lambda m}(z;\underline{\phi})$$

and

$$[1-\overline{G}^{\lambda}(z;\underline{\phi})]^{\theta(\alpha-1)+\alpha l} = \sum_{p=0}^{\infty}(-1)^{p}\binom{\theta(\alpha-1)+\alpha l}{p}\overline{G}^{\lambda p}(z;\underline{\phi}),$$

we have

$$f^{\theta}(z) = \sum_{q,k,l,m,p=0}^{\infty}\sum_{j=0}^{q}(2\pi b\alpha\lambda)^{\theta}(-1)^{q+k+l+p}a_{q}\binom{q}{j}\binom{\theta(b-1)+bj}{k}\binom{\theta+2k}{l} \times \binom{-\theta(\alpha+1)-\alpha l}{m}\binom{\theta(\alpha-1)+\alpha l}{p}g^{\theta}(z;\underline{\phi})\overline{G}^{\theta(\lambda-1)+\lambda m+\lambda p}(z;\underline{\phi}).$$

Furthermore,

$$\begin{aligned}\overline{G}^{\theta(\lambda-1)+\lambda m+\lambda p}(z;\underline{\phi}) &= [1-G(z;\underline{\phi})]^{\theta(\lambda-1)+\lambda m+\lambda p} \\ &= \sum_{s=0}^{\infty}(-1)^s\binom{\theta(\lambda-1)+\lambda m+\lambda p}{s}G^s(z;\underline{\phi}),\end{aligned}$$

yields

$$\begin{aligned}f^{\theta}(z) &= \sum_{q,k,l,m,p,s=0}^{\infty}\sum_{j=0}^{q}(2\pi b\alpha\lambda)^{\theta}(-1)^{q+k+l+p+s}a_q\binom{q}{j}\binom{\theta(b-1)+bj}{k}\binom{\theta+2k}{l} \\ &\times \binom{-\theta(\alpha+1)-\alpha l}{m}\binom{\theta(\alpha-1)+\alpha l}{p}\binom{\theta(\lambda-1)+\lambda m+\lambda p}{s} \\ &\times g^{\theta}(z;\underline{\phi})G^s(z;\underline{\phi}).\end{aligned}$$

Therefore, the Rényi of the Cos-TL-EHL-G FoD is

$$I_R(\theta)=\frac{1}{1-\theta}\log\left(\sum_{s=0}^{\infty}\Phi_{s+1}e^{\{(1-\theta)I_{REG}\}}\right), \tag{S7}$$

where

$$\begin{aligned}\Phi_{s+1} &= \sum_{q,k,l,m,p=0}^{\infty}\sum_{j=0}^{q}(2\pi b\alpha\lambda)^{\theta}(-1)^{q+k+l+p+q}a_q\binom{q}{j}\binom{\theta(b-1)+bj}{k} \\ &\times \binom{-\theta(\alpha+1)-\alpha l}{m}\binom{\theta+2k}{l}\binom{\theta(\alpha-1)+\alpha l}{p} \\ &\times \binom{\theta(\lambda-1)+\lambda m+\lambda p}{s}\frac{1}{\left(\frac{s}{\theta}+1\right)^{\theta}},\end{aligned} \tag{S8}$$

and

$$I_{REG}=\frac{1}{1-\theta}\log\left[\int_0^{\infty}\left(\left[\frac{s}{\theta}+1\right]g(z;\underline{\phi})G^{\frac{s}{\theta}}(z;\underline{\phi})\right)^{\theta}dz\right], \tag{S9}$$

is the Rényi entropy of the Exp-G distribution with parameter $\left[\frac{s}{\theta}+1\right]$.

Elements of the Score Vector

The elements of score vector are

$$\begin{aligned}\frac{\partial \ell}{\partial b} &= \frac{n}{b}+\sum_{i=1}^{n}\ln\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2}\right\}\\ &+ \sum_{i=1}^{n}\frac{\frac{\partial \ell}{\partial b}\left[sin\left(\frac{\pi}{2}\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2}\right\}^{b}\right)\right]}{\left[sin\left(\frac{\pi}{2}\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2}\right\}^{b}\right)\right]},\end{aligned}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \lambda} &= \frac{n}{\lambda}+\sum_{i=1}^{n}\ln\left[\overline{G}(z_i;\underline{\phi})\right]\\ &+ (\alpha-1)\sum_{i=1}^{n}\frac{\frac{\partial \ell}{\partial \lambda}[1-\overline{G}^{\lambda}(z;\underline{\phi})]}{[1-\overline{G}^{\lambda}(z;\underline{\phi})]}-(\alpha+1)\sum_{i=1}^{n}\frac{\frac{\partial \ell}{\partial \lambda}[1+\overline{G}^{\lambda}(z;\underline{\phi})]}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]}\\ &+ \sum_{i=1}^{n}\frac{\frac{\partial \ell}{\partial \lambda}\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]}{\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]}+(b-1)\sum_{i=1}^{n}\frac{\frac{\partial \ell}{\partial \lambda}\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2}\right\}}{\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2}\right\}}\\ &+ \sum_{i=1}^{n}\ln\left[sin\left(\frac{\pi}{2}\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^{2}\right\}^{b}\right)\right].\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} \ln[1-\overline{G}^{\lambda}(z;\underline{\phi})] - \sum_{i=1}^{n} \ln[1+\overline{G}^{\lambda}(z;\underline{\phi})] \\
&- \sum_{i=1}^{n} \frac{\left[\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha} \ln\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)\right]}{\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]} \\
&+ (b-1)\sum_{i=1}^{n} \frac{\frac{\partial \ell}{\partial \alpha}\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^2\right\}}{\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^2\right\}} \\
&+ \sum_{i=1}^{n} \frac{\frac{\partial \ell}{\partial \alpha}\left[sin\left(\frac{\pi}{2}\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^2\right\}^{b}\right)\right]}{\left[sin\left(\frac{\pi}{2}\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^2\right\}^{b}\right)\right]},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \ell}{\partial \underline{\phi}_k} &= \sum_{i=1}^{n} \frac{\frac{\partial \ell}{\partial \underline{\phi}_k}[g(z_i;\underline{\phi})]}{[g(z_i;\underline{\phi})]} + (\lambda-1)\sum_{i=1}^{n} \frac{\frac{\partial \ell}{\partial \underline{\phi}_k}\left[\overline{G}(z_i;\underline{\phi})\right]}{\left[\overline{G}(z_i;\underline{\phi})\right]} \\
&+ (\alpha-1)\sum_{i=1}^{n} \frac{\frac{\partial \ell}{\partial \underline{\phi}_k}[1-\overline{G}^{\lambda}(z;\underline{\phi})]}{[1-\overline{G}^{\lambda}(z;\underline{\phi})]} - (\alpha+1)\sum_{i=1}^{n} \frac{\frac{\partial \ell}{\partial \underline{\phi}_k}[1+\overline{G}^{\lambda}(z;\underline{\phi})]}{[1+\overline{G}^{\lambda}(z;\underline{\phi})]} \\
&+ \sum_{i=1}^{n} \frac{\frac{\partial \ell}{\partial \underline{\phi}_k}\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]}{\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]} + (b-1)\sum_{i=1}^{n} \frac{\frac{\partial \ell}{\partial \underline{\phi}_k}\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^2\right\}}{\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^2\right\}} \\
&+ \sum_{i=1}^{n} \frac{\frac{\partial \ell}{\partial \underline{\phi}_k}\left[sin\left(\frac{\pi}{2}\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^2\right\}^{b}\right)\right]}{\left[sin\left(\frac{\pi}{2}\left\{1-\left[1-\left(\frac{1-\overline{G}^{\lambda}(z;\underline{\phi})}{1+\overline{G}^{\lambda}(z;\underline{\phi})}\right)^{\alpha}\right]^2\right\}^{b}\right)\right]}.
\end{aligned}
$$

Competing Models

Topp-Leone-Weibull (TLW) distribution with pdf

$$f_{TLW}(z;b,\theta) = 2b\theta z^{\theta-1}e^{-2z^{\theta}}\left[1-e^{-2z^{\theta}}\right]^{b-1},$$

for $b,\theta>0$, cosine Topp–Leone Weibull (CosTLW) with pdf

$$\begin{aligned} f_{CosTLW}(z;b,\underline{\phi}) &= \pi b g(z;\underline{\phi})e^{-z^{\theta}}\left\{1-e^{-2z^{\theta}}\right\}^{b-1} \\ &\times \ sin\left(\frac{\pi}{2}\left\{1-e^{-2z^{\theta}}\right\}^{b}\right), \end{aligned}$$

for $b>0$, and parent parameter vector $\underline{\phi}$, Marshall-Olkin Cosine Topp-Leone Weibull (MoCosineTL) with pdf

$$\begin{aligned} f_{MOCosTLW}(x;b,\theta,\lambda,\beta) &= \frac{\theta\pi b\beta\lambda z^{\beta-1}e^{-2\lambda z^{\beta}}\left[1-e^{-2\lambda z^{\beta}}\right]^{b-1}}{\left[1-\overline{\theta}cos\left(0.5\pi\left[1-e^{-2\lambda z^{\beta}}\right]^{b}\right)\right]^{2}} \\ &\times \ sin\left(0.5\pi\left[1-e^{-2\lambda z^{\beta}}\right]^{b}\right), \end{aligned}$$

for $b,\theta,\lambda,\beta>0$, $\bar{\theta}=1-\theta$, exponentiated half logistic- Weibull Poisson (EHLWP) distribution with pdf

$$\begin{aligned} f_{EHLWP}(z;\theta,\delta,\beta,\alpha) &= \frac{2\alpha\delta\beta\theta z^{\alpha-1}e^{(-\beta+1)z^{\alpha}}(1-e^{-\beta z^{\alpha}})^{\delta-1}}{(1+e^{-\beta z^{\alpha}})^{\delta+1}} \\ &\times \ \frac{\exp\left(\theta\left[1-\left(\frac{1-e^{-\beta z^{\alpha}}}{1+e^{-\beta z^{\alpha}}}\right)^{\delta}\right]\right)}{\exp(\theta)-1}, \end{aligned}$$

for $\theta,\delta,\beta,\alpha>0$, Gamma Type II Exponentiated Half Logistic-Topp-Leone-Weibull with pdf

$$\begin{aligned} f(z;\sigma,\alpha,\vartheta,\lambda) &= \frac{4\alpha\vartheta}{\Gamma(\sigma)}\left(-\log\left[1-W_W(z;\alpha,\vartheta,\lambda)\right]\right)^{\sigma-1} \\ &\times \ \frac{\lambda z^{\lambda-1}\exp(-z^{\lambda})[1-\exp(-2z^{\lambda})]^{\vartheta-1}\exp(-z^{\lambda})}{(1+[1-\exp(-2z^{\lambda})]^{\vartheta})^{\alpha+1}} \\ &\times \ \left(1-[1-\exp(-2z^{\lambda})]^{\vartheta}\right)^{\alpha-1}, \end{aligned}$$

for $\sigma,\alpha,\vartheta,\lambda>0$, where $W_W(z;\alpha,\vartheta,\lambda)=\left(\frac{1-[1-\exp(-2z^{\lambda})]^{\vartheta}}{1+[1-\exp(-2z^{\lambda})]^{\vartheta}}\right)^{\alpha}$, Gompertz Topp-Leone Weibull (GomTLW) distribution with pdf

$$\begin{aligned} f_{GomTLW}(x;\lambda,b,a) &= 2bax^{a-1}\exp(-\lambda x^{a})[1-\exp(-2\lambda x^{a})]^{b-1} \\ &\times \ (1-[1-\exp(-2\lambda x^{a})]^{b})^{-\lambda-1} \\ &\times \ \exp\left(\frac{1}{\lambda}(1-[1-\exp(-2\lambda x^{a})]^{b})^{-\lambda}]\right), \end{aligned}$$

for $\lambda, b, a > 0$, exponentiated Weibull Lindley (EWL) distribution with pdf

$$\begin{aligned} f_{EWL}(z;\alpha,v,a,b) &= \frac{abv(1+\alpha)\alpha^2(1+z)\exp(\alpha z)}{(1+\alpha+\alpha z)^2} \\ &\times \exp\left(-a\left[\frac{(1+\alpha)\exp(\alpha z)}{1+\alpha+\alpha z}-1\right]^b\right) \\ &\times \left(1-\exp\left(-a\left[\frac{(1+\alpha)\exp(\alpha z)}{1+\alpha+\alpha z}-1\right]^b\right)\right)^{v-1} \\ &\times \left[\frac{(1+\alpha)\exp(\alpha z)}{1+\alpha+\alpha z}-1\right]^{b-1}, \end{aligned}$$

for $\alpha, v, a, b > 0$ and $z > 0$, and exponentiated half logistic odd Weibull-Topp-Leone-log-logistic (EHLOWTLL) with pdf

$$\begin{aligned} f_{EHLOWTLL}(x;b,\beta,\delta,c) &= \frac{4b\beta\delta c x^{c-1}(1+x^c)^{-3}[1-(1+x^c)]^{b\beta-1}}{(1-[1-(1+x^c)^{-2}]^b)^{\beta+1}} \\ &\times \exp(-t)(1+\exp(-t))^{-2}\left[\frac{1-\exp(-t)}{1+\exp(-t)}\right]^{\delta-1}, \end{aligned}$$

where $t = \left[\frac{[1-(1+x^c)^{-2}]^b}{1-[1-(1+x^c)]^b}\right]^\beta$, for $b, \beta, \delta, c > 0$.

Datasets

Water Flow Data

The data are: 29.19, 8.49, 7.37, 82.93, 44.18, 13.82, 22.28, 28.06, 6.84, 12.14, 153.78, 17.04, 13.47, 15.43, 30.36, 6.91, 22.12, 35.45, 44.66, 95.81, 6.18, 10.00, 58.39,24.05, 17.03, 38.65, 47.17, 27.99, 11.84, 9.60, 6.72, 13.74, 14.60, 9.65, 10.39, 60.14, 15.51,14.69, 16.44.

Relief Times Data

The actual relief revenue data (in 1000 million Egyptian pounds) are: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.