Double shrinkage estimator

Following review about the double shrinkage estimator is due to Zhao [1].

Assume that there exists a statistic S_e^2 independent of w_e that contains information of σ_e^2. It follows to assume in general that $S_e^2|\sigma_e^2 \sim \sigma_e^2 \frac{\chi^2_{d_e}}{d_e}$ where d_e represents the degrees of freedom corresponding to the e^{th} statistic S_e^2. Now approximate $\log\left(\frac{\chi^2_{d_e}}{d_e}\right)$ to follow $N(m_e, \sigma^2_{ch,e})$ distribution with mean $m_e = E\left(\log\left(\frac{\chi^2_{d_e}}{d_e}\right)\right) = \psi(\frac{d_e}{2}) - \log(\frac{d_e}{2})$ and variance $\sigma^2_{ch,e} = V\left(\log\frac{\chi^2_{d_e}}{d_e}\right) = \frac{d}{dx}\psi(\frac{d_e}{2})$, where $\psi(x) = \frac{d}{dx}\log(\Gamma(x))$ is known as the digamma function.

Assumption of $S_e^2|\sigma_e^2 \sim \sigma_e^2 \frac{\chi^2_{d_e}}{d_e}$ and the approximation of $\log\left(\frac{\chi^2_{d_e}}{d_e}\right)$ to follow $N(m_e, \sigma^2_{ch,e})$ distribution follows that

$$\log(S_e^2)|\log(\sigma_e^2) \sim N(m_e + \log(\sigma_e^2), \sigma^2_{ch,e}). \tag{S1}$$

Furthermore, this model assumes that $\log(\sigma_e^2)$ is a normal random variable with unknown mean μ_v and variance τ_v^2. Thus,

$$\log(\sigma_e^2) \sim N(\mu_v, \tau_v^2). \tag{S2}$$

Combining the information from equations 1 and 2 provides the following equation for $\log(\sigma_e^2)|\log(S_e^2)$ as

$$\log(\sigma_e^2)|\log(S_e^2) \sim N\Big(M_{v,e}(\log(S_e^2) - m_e) + (1 - M_{v,e})\mu_v,\ M_{v,e}\,\sigma^2_{ch,e}\Big), \tag{S3}$$

where $M_{v,e} = \tau_v^2/(\tau_v^2 + \sigma^2_{ch,e})$. Thus, the shrinkage variance estimate of σ_e^2 can be obtained as the posterior mean from equation 3 as

$$\sigma_e^2 = \exp\Big(M_{v,e}(\log(S_e^2) - m_e) + (1 - M_{v,e})\mu_v\Big) \tag{S4}$$

under the assumption that both μ_v and τ_v^2 are known. However, μ_v and τ_v^2 are assumed unknown in the model.

Now focus on obtaining an empirical Bayes estimator $\hat{\sigma}_e^2$ using the estimated $\hat{\mu}_v$ and $\hat{\tau}_v^2$ from the data. It can be obtained from equation 1 that

$$\log(S_e^2) - m_e|\log(\sigma_e^2) \sim N(\log(\sigma_e^2), \sigma^2_{ch,e}).$$

It is known from equation 2 that

$$\log(\sigma_e^2) \sim N(\mu_v, \tau_v^2).$$

Thus, using conditional expectation and conditional variance, it can be shown that $E(\log(S_e^2) - m_e) = \mu_v$ and $E(\log(S_e^2) - m_e)^2 = \mu_v^2 + \tau_v^2 + \sigma_{ch,e}^2$. Hence, the model estimates μ_v by $\hat{\mu}_v = \frac{1}{n}\sum_{e=1}^{n}(\log(S_e^2) - m_e)$ and τ_v^2 by $\hat{\tau}_v^2 = \Big(\frac{1}{n}\Big(\sum_{e=1}^{n}(\log(S_e^2) - m_e)^2 - \hat{\mu}_v^2 - \sigma_{ch,e}^2\Big)\Big)_+$, where $+$ indicates the positive part estimator in standard notation. Hence, $M_{v,e}$ can be estimated as $\hat{M}_{v,e} = \hat{\tau}_v^2/(\hat{\tau}_v^2 + \sigma_{ch,e}^2)$ and the empirical Bayes estimator of σ_e^2 can be derived as

$$\hat{\sigma}_e^2 = \exp\Big(\hat{M}_{v,e}(\log(S_e^2) - m_e) + (1 - \hat{M}_{v,e})\hat{\mu}_v\Big).$$

Next step is to estimate μ and τ^2 of the assumed distribution of θ_e for $e = 1, 2, 3, ..., n$. Revise that the model begins with assuming $w_e|\theta_e, \sigma_e^2 \sim N(\theta_e, \sigma_e^2)$ and $\theta_e \sim N(\mu, \tau^2)$ for $e = 1, 2, 3, ..., n$. It can be shown following the same procedure discussed above for estimating μ_v and τ_v^2 that

$$\mathrm{E}(w_e|\sigma_e^2) = \mu \text{ and} \tag{S5}$$

$$\mathrm{E}(w_e - \mu)^2|\sigma_e^2 = \sigma_e^2 + \tau^2. \tag{S6}$$

From equation 5, μ can be estimated by the weighted average as $\hat{\mu} = \sum_{e=1}^{n}\frac{w_e/\hat{\sigma}_e^2}{\sum_{e=1}^{n}1/\hat{\sigma}_e^2}$. Equation 6 can be used to estimate τ^2 as $\hat{\tau}^2 = \Big(\frac{\sum_{e=1}^{n}(w_e-\hat{\mu})^2-\hat{\sigma}_e^2}{n}\Big)_+$. According to Zhao [1], this estimator $\hat{\tau}^2$ can be inconsistent for τ^2 as $n \to \infty$. Therefore, following estimator $\hat{\tau}^2 = \Big(\frac{\sum_{e=1}^{n}(w_e-\hat{\mu})^2-S_e^2\exp(-m_e-\sigma_{ch,e}^2/2)}{n}\Big)_+$ is suggested in the model. Two estimators of $\hat{\tau}^2$ and $\hat{\sigma}_e^2$ are used to derive double shrinkage estimator of θ_e as discussed in the paper.

Reference

[1] Zhigen Zhao. "Double shrinkage empirical Bayesian estimation for unknown and unequal variances". In: *Statistics and Its Interface* 3.4 (2010), pp. 533–541.