fluids

Supplementary Material

Supplementary Material: Defect Theory Derivation of the Outer Wall of Taylor-Couette Cell Flow

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Using the defect theory, Panton [1] matches the approximately constant angular momentum in the bulk of the TC flow to that of the wall layers.

$$\langle \gamma \rangle = r \cdot \langle V_\theta \rangle \quad \text{(S)}$$

In the outer bulk region, the angular momentum in his theory is expressed as:

$$\langle \gamma \rangle = \langle \gamma_o \rangle + \langle \frac{u_\tau}{V_i} \rangle \langle \gamma_1 \rangle \quad \text{(S1)}$$

where $V_i = \Omega \cdot r_i$ is the inner rotor velocity (measuring bob)

$\langle \gamma_o \rangle$ is the constant angular velocity; $\langle \gamma_1 \rangle$ is a first-order correction

V_θ is the circumferential velocity, and a time average is indicated by $\langle \quad \rangle$

In the limit of $\mathrm{Re} \longrightarrow \infty$, the ratio of the rotor velocity $\langle \frac{u_\tau}{V_i} \rangle \to 0$.

Therefore, it is the introduction of the defect term $\langle \gamma_1 \rangle$ that facilitates matching with the wall layer.

On the inner wall, the angular momentum decreases as $(r_i V_i) - \langle \gamma \rangle$,

where the $(r_i V_i)$ is the angular momentum, which is induced by wall motion.

Upon scaling with friction velocity u_τ and the viscous length scale δ_ν , ascaled angular momentum $\langle \gamma^+ \rangle$ can be defined as a function of the scaled wall distance $y^+ = \frac{(r-r_i)}{\delta_\nu}$ as follows:

$$\gamma^+{}_{(y^+)} = \frac{r_i V_i - \langle \gamma \rangle}{r_i \cdot u_\tau} \quad \text{(S2)}$$

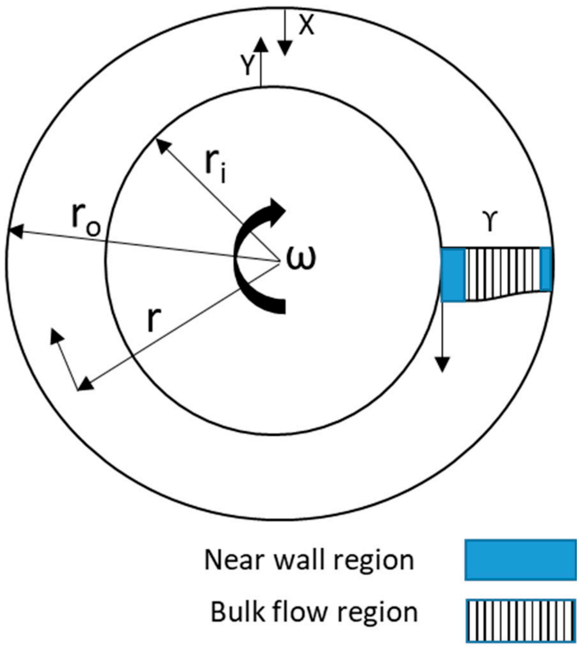

Figure S1. Taylor—Couette flow with the inner cylinder rotating and the outer fixed cylinder

As Panton [1] suggested, the most important results come from mathematically matching the angular momentum laws of the core and wall regions. There is an overlap region where both the wall representation $\gamma^+{}_{(y^+)}$ and the core representation $\Gamma_{(Y)}$ are valid. Where $\Gamma_o = \frac{\langle \gamma_o \rangle}{(r_i \cdot V_i)}$, and $\Gamma_{1(Y)} = \frac{\langle \gamma_1 \rangle}{(r_i \cdot V_i)}$.

Matching these values in the overlap region leads to:

$$(\gamma^+)\left(\frac{u_\tau}{V_i}\right) = 1 - \Gamma_o - \left(\frac{u_\tau}{V_i}\right) \cdot \Gamma_{1(Y)} \quad \text{(S3)}$$

where Γ_1 is a function of the outer variable $Y = \frac{(r-r_i)}{d}$ while γ^+ is a function of

$$y^{+} = \frac{(r - r_i)}{\delta_\nu}$$

Proceeding according to Panton [1], differentiating both sides of Eq. (3) by Y in the overlap region and using the relation $\frac{y^{+}}{Y} = \frac{\Delta r}{\delta_\nu}$ results in the relation

$$y^{+}\frac{d\gamma^{+}}{dy^{+}} = -Y\frac{d\Gamma_1}{dY} \equiv \frac{1}{\alpha}$$

where α is a constant independent of Y and y^{+}.

Panton [1] suggested that for a boundary layer in the Taylor Couette cell, α should reduce to the universal von Karman constant K in the zero curvature limit. Solving this expression results in the identical set of the equation of ϒ+ and Γ 1 that Panton (1992) obtains as:

$$\Gamma_1 = \frac{-1}{\alpha} \cdot \ln Y + C_1 \longrightarrow \quad Y \tag{S4}$$

$$\gamma^{+} = \frac{1}{\alpha} \cdot \ln(y^{+}) \qquad y^{+} \to \infty \tag{S5}$$

However, the substitution of Equations (S4) and (S5) into our modified form of Equation (S3) gives

$$(1 - \Gamma_1) = \frac{1}{\alpha} \cdot lnRe_\tau + C_1 + C_2 \tag{S6}$$

where $Re_\tau = \frac{u_\tau \cdot d}{\nu}$

Repeating the same calculation at the coated stationary outer wall surface in the X direction with:

$$\gamma^{+} = \frac{(r_o \cdot V_{Slip} - \langle\gamma\rangle)}{r_i \cdot u_\tau} \tag{S8}$$

First considering the existence of an overlap region where both representations are valid, Equation (S) can be substituted into Equation (S7). Then, using the modified theory of Srinivasan's work [2] by introducing a finite averaged slip velocity $\langle V_{slip} \rangle$ that is related to the local viscous stress at the outer CDC's wall by the Navier slip hypothesis as

$$\langle V_{Slip} \rangle = \left(\frac{dV_\theta}{dX} \right)_{X=0}$$

where b is the effective slip length due to the superhydrophobic coating and $\left(\frac{dV_\theta}{dX} \right)_{X=0}$ is the time-averaged velocity gradient at the wall.

The distance away from the outer (coated) wall expressed in wall units is

$$X^+ = \frac{(r_o - r)}{\delta_\nu} \tag{S8}$$

If the velocity in the viscous sublayer close to the outer wall is shifted by a constant value, according to Min & Kim [3], so that $\langle V_\theta^+ \rangle = X^+ + b^+$ and $\frac{d\langle V_\theta \rangle^+}{dX^+} = 1,$ the Navier slip hypothesis upon scaling reduces to :

$$V_{Slip}^+ = b^+ \tag{S9}$$

where $V_{Slip}^+ = \frac{V_{Slip}}{u_\tau}$ and $b^+ = \frac{b_{eff}}{\delta_\nu}$

There is an overlap region where both the wall representation $(\gamma)_{(X)}^+$ and the core representation $\Gamma(X)$ are valid.

Matching the values in this region yields:

$$(\gamma^+ + b^+) = 1 - \Gamma_o - \frac{u_\tau}{V_i} \tag{S10}$$

Repeating the argument given in the Y direction for the coated wall layer on the outer cylinder gives two more overlaps laws $(X = r_o - r), X = \frac{X}{d}$

$$X^{+} = x \cdot \frac{u_\tau}{\nu} = XRe_\tau \; and \; \gamma^{+} = \frac{\langle \gamma \rangle}{(r_o u_\tau)}$$

This leads to the following matched expression

$$\Gamma_1 = \frac{1}{\beta} ln(X) + C_3 \qquad X \to 0 \tag{S11}$$

$$\gamma^{+} = \frac{1}{\beta} ln(X^{+}) + C_4 \qquad X \to \infty \tag{S12}$$

Substitute (12) and (13) in eqn. (11) leads to another friction velocity relation

$$\Gamma_o \frac{V_i}{u_\tau} = \frac{1}{\beta} lnRe_\tau + C_3 + C_4 + b^{+} \tag{S13}$$

Finally, the Equations (S7) and (S14) are added together to eliminate Γ_o and using

$$\frac{V_i}{u_\tau} = \left(\frac{C_f}{2}\right)^{\frac{-1}{2}}$$

We obtain the modified skin friction law in the presence of slip that is used as Equation (S14)

$$\sqrt{\frac{2}{C_f}} = M \cdot ln(Re_\tau) + N + b^{+} \tag{S14}$$

where, $M = \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$ and $N = C_1 + C_2 + C_3 + C_4$ are constants that depend only on the curvature of the TC cell. These constants can be determined from friction measurements with non-coated surfaces and plotted in Prandtl-von Karman coordinates.

References

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