Supplementary Material S3

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Mathematical formulas for calculation of the GoldSim Pipe parameters

1. Derivation of a model for one "Pipe" compartment

1.1. Cross section and flow cross section of the "Pipe" component - formulas for calculation

Equation (1) – calculation of the pore volume of the "Pipe" component from the specified parameters of the cross section, length and porosity

$$Vp_{pipe} = \eta_{SP} \cdot V_{pipe} = \eta_{SP} \cdot S_{pipe} \cdot L_{pipe} \tag{1}$$

Where:

L_{pipe} length of the "Pipe" component (Length parameter), [L]
S_{pipe} cross section of the "Pipe" component (Area parameter), [L^2]
V_{pipe} volume of the "Pipe" component, [L^3]
Vp_{pipe} pore volume of the "Pipe" component, [L^3]
η_{SP} porosity of the filling material (Porosity Solid in "Pipe"), [-]

Equation (2) – calculation of the flow area of the "Pipe" component from the specified parameters of the cross section, length and porosity (substitution of equation (1))

$$SF_{pipe} = \frac{Vp_{pipe}}{L_{pipe}} = \frac{\eta_{SP} \cdot S_{pipe} \cdot L_{pipe}}{L_{pipe}} = \eta_{SP} \cdot S_{pipe} \tag{2}$$

Where:

L_{pipe} length of the "Pipe" component (Length parameter), [L]
S_{pipe} cross section of the "Pipe" component (Area parameter), [L^2]
SF_{pipe} flow cross section of the "Pipe" component (Area parameter, if there is no infill medium), [L^2]
Vp_{pipe} pore volume of the "Pipe" component, [L^3]
η_{SP} porosity of the filling material (Porosity Solid in "Pipe"), [-]

1.2.Transport time through the component "Pipe" - formulas for calculation

Equation (3) – flow through the "Pipe" component = sum of all outflows from the component

$$Q_{pipe} = \sum_i Qo_{pipe,i} \tag{3}$$

Where:

Q_{pipe} fluid flow through the "Pipe" component, [$L^3 \cdot T^{-1}$]

$Qo_{pipe,i}$ *i*-th outflow of carrier fluid from the "Pipe" component, [$L^3 \cdot T^{-1}$]

Equation (4) – flow time through the component "Pipe"

$$T_{pipe} = \frac{Vp_{pipe}}{Q_{pipe}} = \frac{SF_{pipe} \cdot L_{pipe}}{Q_{pipe}} \tag{4}$$

Where:

L_{pipe} length of the "Pipe" component (Length parameter), [L]

Q_{pipe} fluid flow through the "Pipe" component, [$L^3 \cdot T^{-1}$]

SF_{pipe} flow cross section of the "Pipe" component (Area parameter, if there is no infill medium), [L^2]

$T_{pipe,i}$ flow time through the component "Pipe", [$L^3 \cdot T^{-1}$]

Vp_{pipe} pore volume of the "Pipe" component, [L^3]

Equation (5) – calculation of the flow area of the "Pipe" component from the entered parameters of the cross section, length and porosity, adjustment of the equation (4) and inserting the equation (3)

$$SF_{pipe} = \frac{T_{pipe} \cdot Q_{pipe}}{L_{pipe}} = \frac{T_{pipe} \cdot \sum_i Qo_{pipe,i}}{L_{pipe}} \tag{5}$$

Where:

L_{pipe} length of the "Pipe" component (Length parameter), [L]

$Qop_{ipe,i}$ *i*-th outflow of carrier fluid from the "Pipe" component, [$L^3 \cdot T^{-1}$]

Q_{pipe} fluid flow through the "Pipe" component, [$L^3 \cdot T^{-1}$]

SF_{pipe} flow cross section of the "Pipe" component (Area parameter, if there is no infill medium), [L^2]

$T_{pipe,i}$ flow time through the component "Pipe", $L^3 \cdot T^{-1}$

Equation (6) – calculation of cross section of "Pipe" component, compilation of equations (2) and (5)

$$S_{pipe} = \frac{SF_{pipe}}{\eta_{SP}} = \frac{T_{pipe} \cdot Q_{pipe}}{\eta_{SP} \cdot L_{pipe}} = \frac{T_{pipe} \cdot \sum_i Qo_{pipe,i}}{\eta_{SP} \cdot L_{pipe}} \tag{6}$$

Where:

L_{pipe} length of the "Pipe" component (Length parameter), [L]

$Qo_{pipe,i}$ *i*-th outflow of carrier fluid from the "Pipe" component, [$L^3 \cdot T^{-1}$]

Q_{pipe} fluid flow through the "Pipe" component, [$L^3 \cdot T^{-1}$]

SF_{pipe} flow cross section of the "Pipe" component (Area parameter, if there is no infill medium), [L^2]

S_{pipe} cross section of the "Pipe" component (Area parameter), [L^2]

η_{SP} porosity of the filling material (Porosity Solid in "Pipe"), [-]

2. Derivation of a model for two cascade "Pipe"

Equation (7): total length

$$L = L_1 + L_2 \;\Rightarrow\; L_1 = L - L_2 \;\Leftrightarrow\; L_2 = L - L_1 \tag{7}$$

Where:

L total (transport) length, [L]
L_1 length of the first (cascade) "Pipe" component, [L]
L_2 length of the second (cascade) "Pipe" component, [L]

Equation (8): total transport time

$$T = T_1 + T_2 \;\Rightarrow\; T_1 = T - T_2 \;\Leftrightarrow\; T_2 = T - T_1 \tag{8}$$

Where:

T total transport time, [T]
T_1 transport time by the first (cascade) "Pipe" component, [T]
T_2 transport time by the second (cascade) "Pipe" component, [T]

Equation (9) – transport time by the first part (first component "Pipe")

$$T_1 = \frac{L_1 \cdot \eta_1 \cdot S_1}{Q_0} \;\Rightarrow\; L_1 = \frac{T_1 \cdot Q_0}{\eta_1 \cdot S_1} \tag{9}$$

Where:

L_1 length of the first (cascade) "Pipe" component, [L]
Q_1 flow through the first (cascade) "Pipe" component, [$L^3 \cdot T^{-1}$]
S_1 cross section first (cascade) "Pipe" component, [L^2]
T_1 transport time by the first (cascade) "Pipe" component, [T]
η_1 porosity of the filling material of the first (cascade) component "Pipe", [-]

Equation (10) – transport time the second part (the second component "Pipe")

$$T_2 = \frac{L_2 \cdot \eta_2 \cdot S_2}{Q_L} \;\Rightarrow\; L_2 = \frac{T_2 \cdot Q_L}{\eta_2 \cdot S_2} \tag{10}$$

Where:

L_2 length of the second (cascade) "Pipe" component, [L]
Q_2 flow through the second (cascade) component "Pipe", [$L^3 \cdot T^{-1}$]
S_2 cross section of second (cascade) "Pipe" component, [L^2]
T_2 transport time by the second (cascade) "Pipe" component, [T]
η_2 porosity of the filling material of the second (cascade) component "Pipe", [-]

2.1. Model for dividing the lengths of two cascade "Pipes" for previously known porosities

Equation (11): deriving a relation for calculating the length of the first part, substituting eq.(8) and (10) to eq. (9)

$$L_1 = \frac{T_1 \cdot Q_0}{\eta_1 \cdot S_1} = \frac{(T - T_2) \cdot Q_0}{\eta_1 \cdot S_1} =$$

$$= \frac{\left(T - \frac{L_2 \cdot \eta_2 \cdot S_2}{Q_L}\right) \cdot Q_0}{\eta_1 \cdot S_1} = \frac{(T \cdot Q_L - L_2 \cdot \eta_2 \cdot S_2) \cdot Q_0}{\eta_1 \cdot S_1 \cdot Q_L} =$$

$$= \frac{(T \cdot Q_L - (L - L_1) \cdot \eta_2 \cdot S_2) \cdot Q_0}{\eta_1 \cdot S_1 \cdot Q_L} = \frac{(T \cdot Q_L + (L_1 - L) \cdot \eta_2 \cdot S_2) \cdot Q_0}{\eta_1 \cdot S_1 \cdot Q_L} =$$

$$= \frac{(T \cdot Q_L - L \cdot \eta_2 \cdot S_2) \cdot Q_0}{\eta_1 \cdot S_1 \cdot Q_L} + L_1 \cdot \frac{\eta_2 \cdot S_2 \cdot Q_0}{\eta_1 \cdot S_1 \cdot Q_L}$$

$$\Rightarrow \tag{11}$$

$$L_1 \cdot \left(1 - \frac{\eta_2 \cdot S_2 \cdot Q_0}{\eta_1 \cdot S_1 \cdot Q_L}\right) = L_1 \cdot \frac{\eta_1 \cdot S_1 \cdot Q_L - \eta_2 \cdot S_2 \cdot Q_0}{\eta_1 \cdot S_1 \cdot Q_L} = \frac{(T \cdot Q_L - L \cdot \eta_2 \cdot S_2) \cdot Q_0}{\eta_1 \cdot S_1 \cdot Q_L}$$

$$\Rightarrow$$

$$L_1 = \frac{(T \cdot Q_L - L \cdot \eta_2 \cdot S_2) \cdot Q_0}{\eta_1 \cdot S_1 \cdot Q_L} \cdot \frac{\eta_1 \cdot S_1 \cdot Q_L}{\eta_1 \cdot S_1 \cdot Q_L - \eta_2 \cdot S_2 \cdot Q_0} = \frac{(T \cdot Q_L - L \cdot \eta_2 \cdot S_2) \cdot Q_0}{\eta_1 \cdot S_1 \cdot Q_L - \eta_2 \cdot S_2 \cdot Q_0}$$

$$\boxed{L_1 = Q_0 \cdot \frac{T \cdot Q_L - L \cdot \eta_2 \cdot S_2}{\eta_1 \cdot S_1 \cdot Q_L - \eta_2 \cdot S_2 \cdot Q_0}}$$

Where:

L	total length, [L]
L_1	length of the first (cascade) "Pipe" component, [L]
L_2	length of the second (cascade) "Pipe" component, [L]
Q_0	inflow, i.e. flow through the first (cascade) "Pipe" component, also flow at a distance of 0 from the source, [$L^3 \cdot T^{-1}$]
Q_L	outflow, i.e. flow through the second (cascade) component "Pipe", also flow at a distance of L from the source, [$L^3 \cdot T^{-1}$]
S_1	cross section first (cascade) "Pipe" component, [L^2]
S_2	cross section of second (cascade) "Pipe" component, [L^2]
T	total flow time through the "Pipe" component, [T]
T_1	transport time by the first (cascade) "Pipe" component, [T]
T_2	transport time by the second (cascade) "Pipe" component, [T]
η_1	porosity of the filling material of the first (cascade) "Pipe" component, [-]
η_2	porosity of the filling material of the second (cascade) "Pipe" component, [-]

Equation (12): deriving a relation for calculating the length of the second part, substituting eq.(8) and (10) to eq. (9)

$$L_2 = \frac{T_2 \cdot Q_L}{\eta_2 \cdot S_2} = \frac{(T - T_1) \cdot Q_L}{\eta_2 \cdot S_2} =$$

$$= \frac{\left(T - \frac{L_1 \cdot \eta_1 \cdot S_1}{Q_0}\right) \cdot Q_L}{\eta_2 \cdot S_2} = \frac{(T \cdot Q_0 - L_1 \cdot \eta_1 \cdot S_1) \cdot Q_L}{\eta_2 \cdot S_2 \cdot Q_0} =$$

$$= \frac{(T \cdot Q_0 - (L - L_2) \cdot \eta_1 \cdot S_1) \cdot Q_L}{\eta_2 \cdot S_2 \cdot Q_0} = \frac{(T \cdot Q_0 + (L_2 - L) \cdot \eta_1 \cdot S_1) \cdot Q_L}{\eta_2 \cdot S_2 \cdot Q_0} =$$

$$= \frac{(T \cdot Q_0 - L \cdot \eta_1 \cdot S_1) \cdot Q_L}{\eta_2 \cdot S_2 \cdot Q_0} + L_2 \cdot \frac{\eta_1 \cdot S_1 \cdot Q_L}{\eta_2 \cdot S_2 \cdot Q_0}$$

$$\Rightarrow \tag{12}$$

$$L_2 \cdot \left(1 - \frac{\eta_1 \cdot S_1 \cdot Q_L}{\eta_2 \cdot S_2 \cdot Q_0}\right) = L_2 \cdot \frac{\eta_2 \cdot S_2 \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L}{\eta_2 \cdot S_2 \cdot Q_0} = \frac{(T \cdot Q_0 - L \cdot \eta_1 \cdot S_1) \cdot Q_L}{\eta_2 \cdot S_2 \cdot Q_0}$$

$$\Rightarrow$$

$$L_2 = \frac{(T \cdot Q_0 - L \cdot \eta_1 \cdot S_1) \cdot Q_L}{\eta_2 \cdot S_2 \cdot Q_0} \cdot \frac{\eta_2 \cdot S_2 \cdot Q_0}{\eta_2 \cdot S_2 \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L} = \frac{(T \cdot Q_0 - L \cdot \eta_1 \cdot S_1) \cdot Q_L}{\eta_2 \cdot S_2 \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L}$$

$$\boxed{L_2 = Q_L \cdot \frac{T \cdot Q_0 - L \cdot \eta_1 \cdot S_1}{\eta_2 \cdot S_2 \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L}}$$

Where:

L	total length, [L]
L_1	length of the first (cascade) “Pipe” component, [L]
L_2	length of the second (cascade) “Pipe” component, [L]
Q_0	inflow, i.e. flow through the first (cascade) “Pipe” component, also flow at a distance of 0 from the source, [$L^3 \cdot T^{-1}$]
Q_L	outflow, i.e. flow through the second (cascade) component “Pipe”, also flow at a distance of L from the source, [$L^3 \cdot T^{-1}$]
S_1	cross section first (cascade) “Pipe” component, [L^2]
S_2	cross section of second (cascade) “Pipe” component, [L^2]
T	total flow time through the “Pipe” component, [T]
T_1	transport time by the first (cascade) “Pipe” component, [T]
T_2	transport time by the second (cascade) “Pipe” component, [T]
η_1	porosity of the filling material of the first (cascade) “Pipe” component, [-]
η_2	porosity of the filling material of the second (cascade) “Pipe” component, [-]

Equation (13): verifying the correctness of relations (11) and (12) by substituting them into eq. (7)

$$L = L_1 + L_2 =$$

$$= Q_0 \cdot \frac{T \cdot Q_L - L \cdot \eta_2 \cdot S_2}{\eta_1 \cdot S_1 \cdot Q_L - \eta_2 \cdot S_2 \cdot Q_0} + Q_L \cdot \frac{T \cdot Q_0 - L \cdot \eta_1 \cdot S_1}{\eta_2 \cdot S_2 \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L} =$$

$$= \frac{Q_0 \cdot L \cdot \eta_2 \cdot S_2 \cancel{- Q_0 \cdot T \cdot Q_L}}{\eta_2 \cdot S_2 \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L} + \frac{\cancel{Q_L \cdot T \cdot Q_0} - Q_L \cdot L \cdot \eta_1 \cdot S_1}{\eta_2 \cdot S_2 \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L} = \tag{13}$$

$$= L \cdot \frac{Q_0 \cdot \eta_2 \cdot S_2 - Q_L \cdot \eta_1 \cdot S_1}{\eta_2 \cdot S_2 \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L} = L$$

Where:

L	total length, [L]
L_1	length of the first (cascade) “Pipe” component, [L]
L_2	length of the second (cascade) “Pipe” component, [L]
Q_0	inflow, i.e. flow through the first (cascade) “Pipe” component, also flow at a distance of 0 from the source, [$L^3 \cdot T^{-1}$]
Q_L	outflow, i.e. flow through the second (cascade) component “Pipe”, also flow at a distance of L from the source, [$L^3 \cdot T^{-1}$]
S_1	cross section first (cascade) “Pipe” component, [L^2]
S_2	cross section of second (cascade) “Pipe” component, [L^2]
T	total flow time through the “Pipe” component, [T]
η_1	porosity of the filling material of the first (cascade) “Pipe” component, [-]
η_2	porosity of the filling material of the second (cascade) “Pipe” component, [-]

2.2. Model for calculating the length distribution of two cascade "Pipes" with the calculation of the porosity of the second "Pipe" so that the total mean porosity is maintained

Equation (14) : derivation of the formula calculation of the porosity of the second "Pipe", from the lengths of both "Pipes" and from the known porosity of the first "Pipe":

$$\eta_1 = \eta_{min}$$

$$\eta = \frac{V_p}{V} = \frac{L_1 \cdot \eta_1 \cdot S_1 + L_2 \cdot \eta_2 \cdot S_2}{L_1 \cdot S_1 + L_2 \cdot S_2} \Rightarrow$$

$$\Rightarrow \eta \cdot (L_1 \cdot S_1 + L_2 \cdot S_2) = L_1 \cdot \eta_1 \cdot S_1 + L_2 \cdot \eta_2 \cdot S_2 \Rightarrow \tag{14}$$

$$\boxed{\Rightarrow \eta_2 = \frac{\eta \cdot (L_1 \cdot S_1 + L_2 \cdot S_2) - L_1 \cdot \eta_1 \cdot S_1}{L_2 \cdot S_2} = \frac{(L - L_2) \cdot (\eta \cdot S_1 - \eta_1 \cdot S_1) + \eta \cdot L_2 \cdot S_2}{L_2 \cdot S_2}}$$

Where:

L	total length, [L]
L_1	length of the first (cascade) "Pipe" component, [L]
L_2	length of the second (cascade) "Pipe" component, [L]
S_1	cross section first (cascade) "Pipe" component, [L^2]
S_2	cross section of second (cascade) "Pipe" component, [L^2]
V	volume of transport pathway, [L^3]
V_p	pore volume of transport pathway, [L^3]
η	total mean porosity of the transport path, [-]
η_1	porosity of the filling material of the first (cascade) "Pipe" component, [-]
η_2	porosity of the filling material of the second (cascade) "Pipe" component, [-]
η_{min}	minimal porosity of the transport pathway, [-]

Equation (15): derivation of a relation for calculating the length of the first part, substitution of eq. (14) to eq. (11), it is not easy to deal with:

$$L_1 = Q_0 \cdot \frac{T \cdot Q_L - L \cdot \eta_2 \cdot S_2}{\eta_1 \cdot S_1 \cdot Q_L - \eta_2 \cdot S_2 \cdot Q_0} =$$

$$= Q_0 \cdot \frac{T \cdot Q_L - L \cdot S_2 \cdot \frac{\eta \cdot (L_1 \cdot S_1 + L_2 \cdot S_2) - L_1 \cdot \eta_1 \cdot S_1}{L_2 \cdot S_2}}{\eta_1 \cdot S_1 \cdot Q_L - S_2 \cdot Q_0 \cdot \frac{\eta \cdot (L_1 \cdot S_1 + L_2 \cdot S_2) - L_1 \cdot \eta_1 \cdot S_1}{L_2 \cdot S_2}} =$$

$$= Q_0 \cdot \frac{T \cdot Q_L \cdot L_2 - L \cdot (\eta \cdot (L_1 \cdot S_1 + L_2 \cdot S_2) - L_1 \cdot \eta_1 \cdot S_1)}{\eta_1 \cdot S_1 \cdot Q_L \cdot L_2 - Q_0 \cdot (\eta \cdot (L_1 \cdot S_1 + L_2 \cdot S_2) - L_1 \cdot \eta_1 \cdot S_1)} =$$

$$= Q_0 \cdot \frac{T \cdot Q_L \cdot (L - L_1) - L \cdot (\eta \cdot (L_1 \cdot S_1 + (L - L_1) \cdot S_2) - L_1 \cdot \eta_1 \cdot S_1)}{\eta_1 \cdot S_1 \cdot Q_L \cdot (L - L_1) - Q_0 \cdot (\eta \cdot (L_1 \cdot S_1 + (L - L_1) \cdot S_2) - L_1 \cdot \eta_1 \cdot S_1)} =$$

$$= Q_0 \cdot \frac{-L_1 \cdot T \cdot Q_L + T \cdot Q_L \cdot L - L \cdot \eta \cdot L_1 \cdot S_1 - L^2 \cdot \eta \cdot S_2 + L \cdot \eta \cdot L_1 \cdot S_2 + L \cdot L_1 \cdot \eta_1 \cdot S_1}{\eta_1 \cdot S_1 \cdot Q_L \cdot L - \eta_1 \cdot S_1 \cdot Q_L \cdot L_1 - Q_0 \cdot \eta \cdot L_1 \cdot S_1 - Q_0 \cdot \eta \cdot L \cdot S_2 + Q_0 \cdot \eta \cdot L_1 \cdot S_2 + Q_0 \cdot L_1 \cdot \eta_1 \cdot S_1} = \tag{15}$$

$$= Q_0 \cdot \frac{L_1 \cdot (-T \cdot Q_L - L \cdot \eta \cdot S_1 + L \cdot \eta \cdot S_2 - L \cdot \eta_1 \cdot S_1) + L \cdot (T \cdot Q_L - L \cdot \eta \cdot S_2)}{L_1 \cdot (-\eta \cdot S_1 \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L + Q_0 \cdot \eta \cdot S_2 + \eta_1 \cdot S_1 \cdot Q_0) + L \cdot (\eta_1 \cdot S_1 \cdot Q_L - \eta \cdot S_2 \cdot Q_0)} \Rightarrow$$

$$\Rightarrow L_1 \cdot \big(L_1 \cdot (-\eta \cdot S_1 \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L + Q_0 \cdot \eta \cdot S_2 + \eta_1 \cdot S_1 \cdot Q_0) + L \cdot (\eta_1 \cdot S_1 \cdot Q_L - \eta \cdot S_2 \cdot Q_0)\big)$$
$$= Q_0 \cdot \big(L_1 \cdot (-T \cdot Q_L - L \cdot \eta \cdot S_1 + L \cdot \eta \cdot S_2 - L \cdot \eta_1 \cdot S_1) + L \cdot (T \cdot Q_L - L \cdot \eta \cdot S_2)\big) \Rightarrow$$

$$\Rightarrow L_1^2 \cdot (-\eta \cdot S_1 \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L + Q_0 \cdot \eta \cdot S_2 + \eta_1 \cdot S_1 \cdot Q_0) + L_1 \cdot L \cdot (\eta_1 \cdot S_1 \cdot Q_L - \eta \cdot S_2 \cdot Q_0)$$
$$= L_1 \cdot Q_0 \cdot (-T \cdot Q_L - L \cdot \eta \cdot S_1 + L \cdot \eta \cdot S_2 - L \cdot \eta_1 \cdot S_1) - L \cdot Q_0 \cdot (T \cdot Q_L - L \cdot \eta \cdot S_2) \Rightarrow$$

$$\Rightarrow L_1^2 \cdot (-\eta \cdot S_1 \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L + Q_0 \cdot \eta \cdot S_2 + \eta_1 \cdot S_1 \cdot Q_0) + L_1 \cdot (L \cdot (\eta_1 \cdot S_1 \cdot Q_L - \eta \cdot S_2 \cdot Q_0) + Q_0 \cdot$$
$$(T \cdot Q_L + L \cdot \eta \cdot S_1 - L \cdot \eta \cdot S_2 + L \cdot \eta_1 \cdot S_1)\,) + L \cdot Q_0 \cdot (T \cdot Q_L - L \cdot \eta \cdot S_2) = 0$$

Where:

L	total length, [L]
L_1	length of the first (cascade) "Pipe" component, [L]
L_2	length of the second (cascade) "Pipe" component, [L]
Q_0	inflow, i.e. flow through the first (cascade) "Pipe" component, also flow at a distance of *0* from the source, [$L^3 \cdot T^{-1}$]
Q_L	outflow, i.e. flow through the second (cascade) component "Pipe", also flow at a distance of *L* from the source, [$L^3 \cdot T^{-1}$]
S_1	cross section first (cascade) "Pipe" component, [L^2]
S_2	cross section of second (cascade) "Pipe" component, [L^2]
T	total flow time through the "Pipe" component, [T]
V	volume of transport pathway, [L^3]
V_p	pore volume of transport pathway, [L^3]
η	total mean porosity of the transport path [-]
η_1	porosity of the filling material of the first (cascade) "Pipe" component, [-]
η_2	porosity of the filling material of the second (cascade) "Pipe" component, [-]

Equation (16): length "Pipe" 2 for optimization according to porosity

$$L_2 = Q_L \cdot \frac{T \cdot Q_0 - L \cdot \eta_1 \cdot S_1}{\eta_2 \cdot S_2 \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L} =$$

$$= Q_L \cdot \frac{T \cdot Q_0 - L \cdot \eta_1 \cdot S_1}{\frac{\eta \cdot (L_1 \cdot S_1 + L_2 \cdot S_2) - L_1 \cdot \eta_1 \cdot S_1}{L_2 \cdot S_2} \cdot S_2 \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L} =$$

$$\cancel{L_2} \cdot 1 = Q_L \cdot \frac{(T \cdot Q_0 - L \cdot \eta_1 \cdot S_1) \cdot \cancel{L_2 \cdot S_2}}{(\eta \cdot (L_1 \cdot S_1 + L_2 \cdot S_2) - L_1 \cdot \eta_1 \cdot S_1) \cdot \cancel{S_2} \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L \cdot L_2 \cdot \cancel{S_2}} \Rightarrow \tag{16}$$

$$\Rightarrow (\eta \cdot (L_1 \cdot S_1 + L_2 \cdot S_2) - L_1 \cdot \eta_1 \cdot S_1) \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L \cdot L_2 = Q_L \cdot (T \cdot Q_0 - L \cdot \eta_1 \cdot S_1) \Rightarrow$$

$$\Rightarrow \left(\eta \cdot \left((L - L_2) \cdot S_1 + L_2 \cdot S_2\right) - (L - L_2) \cdot \eta_1 \cdot S_1\right) \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L \cdot L_2 = Q_L \cdot (T \cdot Q_0 - L \cdot \eta_1 \cdot S_1) \Rightarrow$$

$$\Rightarrow L_2 \cdot \left((-\eta \cdot S_1 + S_2 + \eta_1 \cdot S_1) \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L\right) + (\eta \cdot L - L \cdot \eta_1 \cdot S_1) \cdot Q_0 = Q_L \cdot (T \cdot Q_0 - L \cdot \eta_1 \cdot S_1) \Rightarrow$$

$$\boxed{\Rightarrow L_2 = \frac{Q_L \cdot (T \cdot Q_0 - L \cdot \eta_1 \cdot S_1) + (\eta_1 - \eta) \cdot L \cdot S_1 \cdot Q_0}{(-\eta \cdot S_1 + S_2 + \eta_1 \cdot S_1) \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L}}$$

Where:

L	total length, [L]
L_1	length of the first (cascade) “Pipe” component, [L]
L_2	length of the second (cascade) “Pipe” component, [L]
Q_0	inflow, i.e. flow through the first (cascade) “Pipe” component, also flow at a distance of 0 from the source, [$L^3 \cdot T^{-1}$]
Q_L	outflow, i.e. flow through the second (cascade) component “Pipe”, also flow at a distance of L from the source, [$L^3 \cdot T^{-1}$]
S_1	cross section first (cascade) “Pipe”, component [L^2]
S_2	cross section of second (cascade) “Pipe” component, [L^2]
T	total flow time through the “Pipe” component, [T]
V	volume of transport pathway, [L^3]
V_p	pore volume of transport pathway, [L^3]
η	total mean porosity of the transport path, [-]
η_1	porosity of the filling material of the first (cascade) “Pipe” component, [-]
η_2	porosity of the filling material of the second (cascade) “Pipe” component, [-]

2.3. Model for calculating the length distribution of two cascade "Pipes" with the calculation of the porosity of the first "Pipe" so that the total mean porosity is maintained

Equation (17): derivation of the formula calculation of the porosity of the first "Pipe", from the lengths of both "Pipe" and the known porosity of the second "Pipe":

$$\eta_2 = \eta_{max}$$

$$\eta = \frac{V_p}{V} = \frac{L_1 \cdot \eta_1 \cdot S_1 + L_2 \cdot \eta_2 \cdot S_2}{L_1 \cdot S_1 + L_2 \cdot S_2} \Rightarrow$$

$$\Rightarrow \eta \cdot (L_1 \cdot S_1 + L_2 \cdot S_2) = L_1 \cdot \eta_1 \cdot S_1 + L_2 \cdot \eta_2 \cdot S_2 \Rightarrow$$

$$\Rightarrow \eta_1 = \frac{\eta \cdot (L_1 \cdot S_1 + L_2 \cdot S_2) - \eta_2 \cdot L_2 \cdot S_2}{L_1 \cdot S_1} \tag{17}$$

$$\eta_1 = \frac{\eta \cdot (L_1 \cdot S_1 + (L - L_1) \cdot S_2) - \eta_2 \cdot (L - L_1) \cdot S_2}{L_1 \cdot S_1} =$$
$$= \frac{L_1 \cdot (\eta \cdot (S_1 - S_2) + \eta_2 \cdot S_2) + L \cdot (\eta - \eta_2) \cdot S_2}{L_1 \cdot S_1} =$$
$$= \frac{\eta \cdot \big((L - L_2) \cdot S_1 + L_2 \cdot S_2\big) - \eta_2 \cdot L_2 \cdot S_2}{(L - L_2) \cdot S_1} =$$

$$\boxed{\eta_1 = \frac{L_2 \cdot (\eta \cdot (-S_1 + S_2) - \eta_2 \cdot S_2) + L \cdot \eta \cdot S_1}{(L - L_2) \cdot S_1} = \frac{(L - L_1) \cdot (\eta \cdot (-S_1 + S_2) - \eta_2 \cdot S_2) + L \cdot \eta \cdot S_1}{L_1 \cdot S_1}}$$

Where:

L	total length, [L]
L_1	length of the first (cascade) "Pipe" component, [L]
L_2	length of the second (cascade) "Pipe" component, [L]
S_1	cross section first (cascade) "Pipe" component, [L^2]
S_2	cross section of second (cascade) "Pipe" component, [L^2]
V	volume of transport pathway, [L^3]
V_p	pore volume of transport pathway, [L^3]
η	total mean porosity of the transport path, [-]
η_1	porosity of the filling material of the first (cascade) "Pipe" component, [-]
η_2	porosity of the filling material of the second (cascade) "Pipe" component, [-]
η_{max}	maximal porosity of the transport pathway, [-]

Equation (18): deriving a relation for calculating the length of the first part, substituting eq. (17) to eq. (11)

$$L_1 = Q_0 \cdot \frac{T \cdot Q_L - L \cdot \eta_2 \cdot S_2}{\eta_1 \cdot S_1 \cdot Q_L - \eta_2 \cdot S_2 \cdot Q_0} =$$

$$L_1 = Q_0 \cdot \frac{T \cdot Q_L - L \cdot \eta_2 \cdot S_2}{\frac{L_1 \cdot (\eta \cdot (S_1 - S_2) + \eta_2 \cdot S_2) + L \cdot (\eta - \eta_2) \cdot S_2}{L_1 \cdot S_1} \cdot S_1 \cdot Q_L - \eta_2 \cdot S_2 \cdot Q_0} =$$

$$L_1 = Q_0 \cdot L_1 \cdot S_1 \cdot \frac{T \cdot Q_L - L \cdot \eta_2 \cdot S_2}{L_1 \cdot (\eta \cdot (S_1 - S_2) + \eta_2 \cdot S_2) + L \cdot (\eta - \eta_2) \cdot S_2 \cdot S_1 \cdot Q_L - L_1 \cdot S_1 \cdot \eta_2 \cdot S_2 \cdot Q_0} \Longrightarrow \tag{18}$$

$$(L_1 \cdot (\eta \cdot (S_1 - S_2) + \eta_2 \cdot S_2) + L \cdot (\eta - \eta_2) \cdot S_2) \cdot S_1 \cdot Q_L - L_1 \cdot S_1 \cdot \eta_2 \cdot S_2 \cdot Q_0 = Q_0 \cdot S_1 \cdot (T \cdot Q_L - L \cdot \eta_2 \cdot S_2)$$
$$\Longrightarrow$$

$$L_1 \cdot \big((\eta \cdot (S_1 - S_2) + \eta_2 \cdot S_2) \cdot S_1 \cdot Q_L - S_1 \cdot \eta_2 \cdot S_2 \cdot Q_0\big)$$
$$= Q_0 \cdot S_1 \cdot (T \cdot Q_L - L \cdot \eta_2 \cdot S_2) - L \cdot (\eta - \eta_2) \cdot S_2 \cdot S_1 \cdot Q_L \Longrightarrow$$

$$\boxed{L_1 = \frac{Q_0 \cdot S_1 \cdot (T \cdot Q_L - L \cdot \eta_2 \cdot S_2) - L \cdot (\eta - \eta_2) \cdot S_2 \cdot S_1 \cdot Q_L}{(\eta \cdot (S_1 - S_2) + \eta_2 \cdot S_2) \cdot S_1 \cdot Q_L - S_1 \cdot \eta_2 \cdot S_2 \cdot Q_0}}$$

Where:

L	total length, [L]
L_1	length of the first (cascade) “Pipe” component, [L]
L_2	length of the second (cascade) “Pipe” component, [L]
Q_0	inflow, i.e. flow through the first (cascade) “Pipe” component, also flow at a distance of *0* from the source, [$L^3 \cdot T^{-1}$]
Q_L	outflow, i.e. flow through the second (cascade) component “Pipe”, also flow at a distance of *L* from the source, [$L^3 \cdot T^{-1}$]
S_1	cross section first (cascade) “Pipe” component, [L^2]
S_2	cross section of second (cascade) “Pipe” component, [L^2]
T	total flow time through the “Pipe” component, [T]
V	volume of transport pathway, [L^3]
V_p	pore volume of transport pathway, [L^3]
η	total mean porosity of the transport path, [-]
η_1	porosity of the filling material of the first (cascade) “Pipe” component, [-]
η_2	porosity of the filling material of the second (cascade) “Pipe” component, [-]

Equation (19): deriving a relation for calculating the length of the first part, substituting eq. (17) to eq. (12), cannot be easily solved

$$L_2 = Q_L \cdot \frac{T \cdot Q_0 - L \cdot \eta_1 \cdot S_1}{\eta_2 \cdot S_2 \cdot Q_0 - \eta_1 \cdot S_1 \cdot Q_L} =$$

$$L_2 = Q_L \cdot \frac{T \cdot Q_0 - L \cdot \frac{L_2 \cdot (\eta \cdot (-S_1 + S_2) - \eta_2 \cdot S_2) + L \cdot \eta \cdot S_1}{(L - L_2) \cdot \cancel{S_1}} \cdot \cancel{S_1}}{\eta_2 \cdot S_2 \cdot Q_0 - \frac{L_2 \cdot (\eta \cdot (-S_1 + S_2) - \eta_2 \cdot S_2) + L \cdot \eta \cdot S_1}{(L - L_2) \cdot \cancel{S_1}} \cdot \cancel{S_1} \cdot Q_L} =$$

$$L_2 = Q_L \cdot \frac{(L - L_2) \cdot T \cdot Q_0 - L \cdot (L_2 \cdot (\eta \cdot (-S_1 + S_2) - \eta_2 \cdot S_2) + L \cdot \eta \cdot S_1)}{(L - L_2) \cdot \eta_2 \cdot S_2 \cdot Q_0 - (L_2 \cdot (\eta \cdot (-S_1 + S_2) - \eta_2 \cdot S_2) + L \cdot \eta \cdot S_1) \cdot Q_L} \Rightarrow \tag{19}$$

$$L_2 \cdot \big((L - L_2) \cdot \eta_2 \cdot S_2 \cdot Q_0 - (L_2 \cdot (\eta \cdot (-S_1 + S_2) - \eta_2 \cdot S_2) + L \cdot \eta \cdot S_1) \cdot Q_L\big) = Q_L \cdot \big((L - L_2) \cdot T \cdot Q_0 - L \cdot (L_2 \cdot (\eta \cdot (-S_1 + S_2) - \eta_2 \cdot S_2) + L \cdot \eta \cdot S_1)\big) \Rightarrow$$

$$\Rightarrow L_2 \cdot \big((L - L_2) \cdot \eta_2 \cdot S_2 \cdot Q_0 - (L_2 \cdot (\eta \cdot (-S_1 + S_2) - \eta_2 \cdot S_2) + L \cdot \eta \cdot S_1) \cdot Q_L\big) - Q_L \cdot \big((L - L_2) \cdot T \cdot Q_0 - L \cdot (L_2 \cdot (\eta \cdot (-S_1 + S_2) - \eta_2 \cdot S_2) + L \cdot \eta \cdot S_1)\big) = 0 =$$

$$= {L_2}^2 \cdot (-\eta_2 \cdot S_2 \cdot Q_0 - (\eta \cdot (-S_1 + S_2) - \eta_2 \cdot S_2) \cdot Q_L) + L_2 \cdot \big(L \cdot \eta_2 \cdot S_2 \cdot Q_0 + L \cdot \eta \cdot S_1 \cdot Q_L + Q_L \cdot T \cdot Q_0 + Q_L \cdot L \cdot (\eta \cdot (-S_1 + S_2) - \eta_2 \cdot S_2)\big) + 1 \cdot (-Q_L \cdot L \cdot T \cdot Q_0 - Q_L \cdot L^2 \cdot \eta \cdot S_1) = 0$$

Where:

L	total length, [L]
L_1	length of the first (cascade) “Pipe” component, [L]
L_2	length of the second (cascade) “Pipe” component, [L]
Q_0	inflow, i.e. flow through the first (cascade) “Pipe” component, also flow at a distance of 0 from the source, [$L^3 \cdot T^{-1}$]
Q_L	outflow, i.e. flow through the second (cascade) component “Pipe”, also flow at a distance of L from the source, [$L^3 \cdot T^{-1}$]
S_1	cross section first (cascade) “Pipe” component, [L^2]
S_2	cross section of second (cascade) “Pipe” component, [L^2]
T	total flow time through the “Pipe” component, [T]
V	volume of transport pathway, [L^3]
V_p	pore volume of transport pathway, [L^3]
η	total mean porosity of the transport path, [-]
η_1	porosity of the filling material of the first (cascade) “Pipe” component, [-]
η_2	porosity of the filling material of the second (cascade) “Pipe” component, [-]

3. Derivation of a model for three cascade "Pipe"

3.1. Total length (balance)

Equation (20): total length (balance)

$$L_1 + L_2 + L_3 = L \tag{20}$$

Where:

L total length, [L]
L_1 length of the first (cascade) "Pipe" component, [L]
L_2 length of the second (cascade) "Pipe" component, [L]
L_3 length of the third (cascade) "Pipe" component, [L]

3.2. Model for calculation of cross section middle "Pipe"

Using postprocessing, outflow areas were determined: outflow sources (S_1) and contaminated ouflow cross section of the model (S_3), which are used for the first and the third "Pipes" as value of "Area" parameter. Now it is necessary to specify the cross section of the middle "Pipe". From postprocessing, in addition to data on areas S_1 and S_3, we also have data on the total length (L) and the total volume of the transport path (V). We start from the assumption that in the first approach the transport path has the shape of two consecutive truncated cones with a common base S_2. The relation for calculating the volume of a truncated cone is described by equation (21), according to (Bartsch 2002). We will consider the height of each of the two cones to be half the length of the transport path, see equation (22).

Equation (21): cone frustum volume, according to (Bartsch 2002):

$$V = \frac{1}{3} \cdot v \cdot \left(S_1 + S_2 + \sqrt{S_1 \cdot S_2}\right) \tag{21}$$

Where:

V truncated cone volume, [L^3]
S_1 area of the first base, [L^2]
S_2 area of the second base, [L^2]
v truncated cone height, [L]

Equation (22): the height of both parts (both truncated cones) as half the length of the transport path:

$$v_1 = v_2 = \frac{1}{2} \cdot L \tag{22}$$

Where:

L transport pathway length, [L]
v_1 height of the first truncated cone (length of the first half of the transport path), [L]
v_2 height of the second truncated cone (length of the second half of the transport path), [L]

Equation (23): auxiliary relation for substitution per square root of the surface of the common base of both parts (both truncated cones):

$$L_{S2} = \sqrt{S_2} \tag{23}$$

Where:

L_{S2} substitution for the square root area of the common base of both truncated cones, [L]

S_2 area of the common base of both truncated cones (cross section of the second cascade “Pipe”), [L^2]

Equation (24): deriving a relation for calculating the cross section of the second cascade “Pipe”, using equation (21), substituting equation (22) and using substitution (23):

$$V = V_1 + V_2 = \frac{1}{3} \cdot v_1 \cdot \left(S_1 + S_2 + \sqrt{S_1 \cdot S_2}\right) + \frac{1}{3} \cdot v_2 \cdot \left(S_3 + S_2 + \sqrt{S_3 \cdot S_2}\right) =$$

$$= \frac{1}{3} \cdot \frac{1}{2} \cdot L \cdot \left(S_1 + S_2 + \sqrt{S_1 \cdot S_2}\right) + \frac{1}{3} \cdot \frac{1}{2} \cdot L \cdot \left(S_3 + S_2 + \sqrt{S_3 \cdot S_2}\right) =$$

$$= \frac{L}{6} \cdot \left(S_1 + 2 \cdot S_2 + S_3 + \sqrt{S_2} \cdot \left(\sqrt{S_1} + \sqrt{S_3}\right)\right) =$$

$$= \frac{L}{6} \cdot \left(S_1 + 2 \cdot L_{S2}^2 + S_3 + L_{S2} \cdot \left(\sqrt{S_1} + \sqrt{S_3}\right)\right) =$$

$$= V = \frac{L}{6} \cdot \left(2 \cdot L_{S2}^2 + L_{S2} \cdot \left(\sqrt{S_1} + \sqrt{S_3}\right) + S_1 + S_3\right) \tag{24}$$

$$\Rightarrow$$

$$2 \cdot L_{S2}^2 + \left(\sqrt{S_1} + \sqrt{S_3}\right) \cdot L_{S2} + S_1 + S_3 - \frac{6 \cdot V}{L} = 0$$

$$\Rightarrow$$

$$L_{S2} = \frac{-\left(\sqrt{S_1} + \sqrt{S_3}\right) + \sqrt{\left(\sqrt{S_1} + \sqrt{S_3}\right)^2 - 4 \cdot 2 \cdot \left(S_1 + S_3 - \frac{6 \cdot V}{L}\right)}}{2 \cdot 2}$$

$$\Rightarrow$$

$$\boxed{S_2 = \frac{1}{16} \cdot \left(-\left(\sqrt{S_1} + \sqrt{S_3}\right) + \sqrt{\left(\sqrt{S_1} + \sqrt{S_3}\right)^2 - 8 \cdot \left(S_1 + S_3 - \frac{6 \cdot V}{L}\right)}\right)^2}$$

Where:

L	transport pathway length, [L]
L_{S2}	substitution for the square root area of the common base of both truncated cones, [L], see equation (23)
S_1	cross section first (cascade) “Pipe” component (i.e. the outflow area of the source), [L^2]
S_2	area of the common base of both truncated cones (cross section of the second cascade “Pipe”), [L^2]
S_3	cross section third (cascade) “Pipe” component (i.e. the outflow area of the model), [L^2]
V	volume of transport pathway, [L^3]
V_1	the volume of the first part of the transport path (the first truncated cone), [L^3]
v_1	height of the first truncated cone (length of the first half of the transport path), [L]
V_2	the volume of the second part of the transport path (the second truncated cone), [L^3]
v_2	height of the second truncated cone (length of the second half of the transport path), [L]

3.2.1. Other options for determining the cross section of the middle “Pipe”

Equation (25): other options for determining the cross section of the middle “Pipe”:

$S = \frac{V}{L}$	(25)
$S_1 < S < S_3 \Longrightarrow S_2 = S$	
$S > max(S_1; S_3) \Rightarrow S_2 = S \cdot p$	
$S < min(S_1; S_3) \Rightarrow S_2 = \frac{S}{p}$	
$L_1 = L_2 = L_3 = \frac{L}{3}$	
$V = V_1 + V_2 + V_3 = \frac{L}{3} \cdot (S_1 + S_2 + S_3) \Longrightarrow$	
$S_2 = \frac{3 \cdot V}{L} - S_1 - S_3$	

Where:

L total length of the transport pathway, [L]
L_1 length of the first (cascade) “Pipe” component, [L]
L_2 length of the second (cascade) “Pipe” component, [L]
L_3 length of the third (cascade) “Pipe” component, [L]
V total volume of the transport pathway, [L^3]
V_1 volume of the first (cascade) component “Pipe”, [L^3]
V_2 volume of the second (cascade) component “Pipe”, [L^3]
V_3 volume of the third (cascade) component “Pipe”, [L^3]
S mean cross section of the transport pathway, [L^2]
S_1 cross section first (cascade) “Pipe” component, [L^2]
S_2 cross section of second (cascade) “Pipe” component, [L^2]
S_3 cross section third (cascade) “Pipe” component, [L^2]
p multiple, [-]

3.3.Volume balance (as a function of lengths)

Equation(26): volume balance

$$V_1 + V_2 + V_3 = V$$

$$V_x = L_x \cdot S_x \tag{26}$$

$$\boxed{L_1 \cdot S_1 + L_2 \cdot S_2 + L_3 \cdot S_3 = V}$$

Where:

L	total length of the transport pathway, [L]
L_1	length of the first (cascade) “Pipe” component, [L]
L_2	length of the second (cascade) “Pipe” component, [L]
L_3	length of the third (cascade) “Pipe” component, [L]
L_x	length *x* components “Pipe”, [L]
S_1	cross section first (cascade) “Pipe” component, [L^2]
S_2	cross section of second (cascade) “Pipe” component, [L^2]
S_3	cross section third (cascade) “Pipe” component, [L^2]
S_x	cross section general components “Pipe”, [L^2]
V	total volume of the transport pathway, [L^3]
V_1	volume of the first (cascade) “Pipe” component, [L^3]
V_2	volume of the second (cascade) “Pipe” component, [L^3]
V_3	volume of the third (cascade) “Pipe” component, [L^3]
V_x	volume of the general “Pipe” component [L^3]

3.4.Options for calculating the flow through the middle component "Pipe"

Equation (27): various options for calculating the flow through the second (middle) cascade component "Pipe":

$Q_1 < Q_2 < Q_3$	(27)
$Q_2 = \sqrt{Q_1 \cdot Q_3}$	
$Q_{x+1} = Q_x \cdot k$	
$Q_2 = Q_1 \cdot k = \sqrt{Q_1 \cdot Q_3} \Longrightarrow$	
$k = \frac{\sqrt{Q_1 \cdot Q_3}}{Q_1} = \sqrt{\frac{Q_3}{Q_1}} = \sqrt{r} \Longrightarrow$	
$Q_2 = \sqrt{r} \cdot Q_1$	

Where:

Q_1 flow through the first (cascade) "Pipe" component, i.e. inflow into the geosphere, [$L^3 \cdot T$]

Q_2 flow through the second (cascade) "Pipe" component, [$L^3 \cdot T$]

Q_3 flow through the third (cascade) "Pipe" component, i.e. outflow from the geosphere, [$L^3 \cdot T$]

Q_x flow through general "Pipe" component, [$L^3 \cdot T$]

k flow ratio of the following two "Pipes", [-]

r total dilution in the geosphere [-]

3.5.Porosity balance - calculation of the porosity of the middle "Pipe"

Equation (28): Porosity balance, porosity calculation of the middle "Pipe":

$$Vp_1 + Vp_2 + Vp_3 = Vp$$

$$Vp_x = V_x \cdot \eta_x$$

$$L_1 \cdot S_1 \cdot \eta_1 + L_2 \cdot S_2 \cdot \eta_2 + L_3 \cdot S_3 \cdot \eta_3 = V \cdot \eta \implies \tag{28}$$

$$L_2 \cdot \eta_2 = \frac{V \cdot \eta - L_1 \cdot S_1 \cdot \eta_1 - L_3 \cdot S_3 \cdot \eta_3}{S_2} \implies$$

$$\boxed{\eta_2 = \frac{V \cdot \eta - L_1 \cdot S_1 \cdot \eta_1 - L_3 \cdot S_3 \cdot \eta_3}{L_2 \cdot S_2}}$$

Where:

L_1	length of the first (cascade) "Pipe" component, [L]
L_2	length of the second (cascade) "Pipe" component, [L]
L_3	length of the third (cascade) "Pipe" component, [L]
S_1	cross section first (cascade) "Pipe" component, [L^2]
S_2	cross section of second (cascade) "Pipe" component, [L^2]
S_3	cross section third (cascade) "Pipe" component, [L^2]
S_x	cross section general) "Pipe" component, [L^2]
Vp	total pore volume of transport pathway, [L^3]
Vp_1	pore volume of the first (cascade) "Pipe" component, [L^3]
Vp_2	pore volume of the second (cascade) "Pipe" component, [L^3]
Vp_3	pore volume of the third (cascade) "Pipe" component, [L^3]
Vp_x	pore volume of the general "Pipe" component, [L^3]
V_x	volume of the general "Pipe" component, [L^3]
η	total porosity of the transport path, [-]
η_1	porosity of the first (cascade) "Pipe" component, [-]
η_2	porosity of the second (cascade) "Pipe" component, [-]
η_3	porosity of the third (cascade) "Pipe" component, [-]
η_x	porosity of the general "Pipe" component, [-]

3.6. Relationship between lengths of individual "Pipes" and total transport time - (transport time balance)

Equation (29): derivation of the relationship between the lengths of individual "Pipes" and the total flow time, behind the porosity of the second "Pipe" the relationship from equation is established (28)

$$T_1 + T_2 + T_3 = T$$

$$T_x = \frac{Vp_x}{Q_x} = \frac{L_x \cdot S_x \cdot \eta_x}{Q_x}$$

$$\frac{L_1 \cdot S_1 \cdot \eta_1}{Q_1} + \frac{L_2 \cdot S_2 \cdot \eta_2}{Q_2} + \frac{L_3 \cdot S_3 \cdot \eta_3}{Q_3} = T = \frac{L_1 \cdot S_1 \cdot \eta_1}{Q_1} + \frac{S_2}{Q_2} \cdot L_2 \cdot \eta_2 + \frac{L_3 \cdot S_3 \cdot \eta_3}{Q_3} \tag{29}$$

substitution of equation (28) $\Longrightarrow$

$$\frac{L_1 \cdot S_1 \cdot \eta_1}{Q_1} + \frac{\cancel{S_2}}{Q_2} \cdot \frac{V \cdot \eta - L_1 \cdot S_1 \cdot \eta_1 - L_3 \cdot S_3 \cdot \eta_3}{\cancel{S_2}} + \frac{L_3 \cdot S_3 \cdot \eta_3}{Q_3} = T \Longrightarrow$$

$$\boxed{L_1 \cdot S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right) + L_2 \cdot 0 + L_3 \cdot S_3 \cdot \eta_3 \cdot \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right) = T - \frac{V \cdot \eta}{Q_2}}$$

Where:

L_1	length of the first (cascade) "Pipe" component, [L]
L_2	length of the second (cascade) "Pipe" component, [L]
L_3	length of the third (cascade) "Pipe" component, [L]
Q_1	flow through the first (cascade) "Pipe" component, [$L^3 \cdot T$]
Q_2	flow through the second (cascade) "Pipe" component, [$L^3 \cdot T$]
Q_3	flow of the third (cascade) "Pipe" component, [$L^3 \cdot T$]
Q_x	flow through general "Pipe" component, [$L^3 \cdot T$]
S_1	cross section first (cascade) "Pipe" component, [L^2]
S_2	cross section of second (cascade) "Pipe" component, [L^2]
S_3	cross section third (cascade) "Pipe" component, [L^2]
S_x	cross section general "Pipe" components, [L^2]
T	total flow time by transport pathway, [T]
T_1	transport time by the first (cascade) "Pipe" component, [T]
T_2	transport time by the second (cascade) "Pipe" component, [T]
T_3	transport time third (cascade) "Pipe" component, [T]
T_x	transport time general "Pipe" component, [T]
Vp_x	pore volume of the general "Pipe" component, [L^3]
η_1	porosity of the first (cascade) "Pipe" component, [-]
η_2	porosity of the second (cascade) "Pipe" component, [-]
η_3	porosity of the third (cascade) "Pipe" component, [-]
η_x	porosity of the general "Pipe" component, [-]

3.7. Overall model for calculating the lengths of three cascaded "Pipe" - matrix form

By compiling equations (20), (26) and (29) we obtain an overall (matrix) relation for calculating the lengths of individual "Pipes" (L_1 to L_3), see equation (30).

Equation (30): general relationship for calculating the lengths of three cascade "Pipe" (matrix form), composed of equations (20), (26) and (29):

$$\begin{pmatrix} 1 & 1 & 1 \\ S_1 & S_2 & S_3 \\ S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right) & 0 & S_3 \cdot \eta_3 \cdot \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right) \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} L \\ V \\ T - \frac{V \cdot \eta}{Q_2} \end{pmatrix} \tag{30}$$

Where:

L	total length of the transport pathway [L]
L_1	length of the first (cascade) "Pipe" component [L]
L_2	length of the second (cascade) "Pipe" component [L]
L_3	length of the third (cascade) "Pipe" component [L]
Q_1	flow through the first (cascade) "Pipe" component [$L^3 \cdot T$]
Q_2	flow through the second (cascade) component "Pipe" [$L^3 \cdot T$]
Q_3	flow of the third (cascade) component "Pipe" [$L^3 \cdot T$]
S_1	cross section first (cascade) "Pipe" component, [L^2]
S_2	cross section of second (cascade) "Pipe" component, [L^2]
S_3	cross section third (cascade) "Pipe" component, [L^2]
T	total flow time by transport pathway [T]
V	total volume of the transport pathway [L^3]
η	total porosity of the transport path [-]
η_1	porosity of the first (cascade) "Pipe" component [-]
η_3	porosity of the third (cascade) component "Pipe" [-]

3.8. Analytical solution of the matrix model of three "Pipes" (for use in optimization with GoldSim software)

3.8.1. Relationship for the length of the first cascade "Pipe"

After expressing the length L_1 as a function of L_3, we use equation (29) and further adjust, see equation (31).

Equation (31): relation for the length of the first "Pipe" L_1 as a function of the length of the third "Pipe" L_3:

$$L_1 \cdot S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right) + L_2 \cdot 0 + L_3 \cdot S_3 \cdot \eta_3 \cdot \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right) = T - \frac{V \cdot \eta}{Q_2}$$

$$L_1 = \frac{T - \frac{V \cdot \eta}{Q_2} - L_3 \cdot S_3 \cdot \eta_3 \cdot \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right)}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} \tag{31}$$

$$\boxed{L_1 = -L_3 \cdot \frac{S_3 \cdot \eta_3 \cdot \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right)}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} + \frac{T - \frac{V \cdot \eta}{Q_2}}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)}}$$

Where:

L	total length of the transport pathway, [L]
L_1	length of the first (cascade) "Pipe" component, [L]
L_2	length of the second (cascade) "Pipe" component, [L]
L_3	length of the third (cascade) "Pipe" component, [L]
Q_1	flow through the first (cascade) "Pipe" component, [$L^3 \cdot T$]
Q_2	flow through the second (cascade) "Pipe" component, [$L^3 \cdot T$]
Q_3	flow of the third (cascade) "Pipe" component, [$L^3 \cdot T$]
S_1	cross section first (cascade) "Pipe" component, [L^2]
S_2	cross section of second (cascade) "Pipe" component, [L^2]
S_3	cross section third (cascade) "Pipe" component, [L^2]
T	total flow time by transport pathway [T]
V	total volume of the transport pathway [L^3]
η	total porosity of the transport path [-]
η_1	porosity of the first (cascade) "Pipe" component [-]
η_3	porosity of the third (cascade) component "Pipe" [-]

3.8.2. Relationship for the length of the second cascade "Pipe"

To express the length L_2 as a function of L_3, we use equation (20) into which we substitute equation (31), and then we obtain a relation for calculating the length of L_2 as a function of L_3, see equation (32).

Equation (32): deriving a relation for calculating the length L2 as a function of L3, substituting equation (31) into equation (20) and then adjusting:

$$L_1 + L_2 + L_3 = L$$

$$-L_3 \cdot \frac{S_3 \cdot \eta_3 \cdot \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right)}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} + \frac{T - \frac{V \cdot \eta}{Q_2}}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} + L_2 + L_3 = L$$

$$L_2 = L + L_3 \cdot \frac{S_3 \cdot \eta_3 \cdot \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right)}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} - \frac{T - \frac{V \cdot \eta}{Q_2}}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} - L_3 \tag{32}$$

$$\boxed{L_2 = L_3 \cdot \left(\frac{S_3 \cdot \eta_3 \cdot \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right)}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} - 1\right) + L - \frac{T - \frac{V \cdot \eta}{Q_2}}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)}}$$

Where:

L	total length of the transport pathway, [L]
L_1	length of the first (cascade) "Pipe" component, [L]
L_2	length of the second (cascade) "Pipe" component, [L]
L_3	length of the third (cascade) "Pipe" component, [L]
Q_1	flow through the first (cascade) "Pipe" component, [$L^3 \cdot T$]
Q_2	flow through the second (cascade) "Pipe" component, [$L^3 \cdot T$]
Q_3	flow of the third (cascade) "Pipe" component, [$L^3 \cdot T$]
S_1	cross section first (cascade) "Pipe" component, [L^2]
S_2	cross section of second (cascade) "Pipe" component, [L^2]
S_3	cross section third (cascade) "Pipe" component, [L^2]
T	total flow time by transport pathway, [T]
V	total volume of the transport pathway, [L^3]
η	total porosity of the transport path, [-]
η_1	porosity of the first (cascade) "Pipe" component, [-]
η_3	porosity of the third (cascade) "Pipe" component, [-]

3.8.3. Relationship for the length of the third cascade "Pipe"

Equation (33): the relation for calculating the length of the third "Pipe", formed by substituting equations (31) and (32) into equation (26) and next modifying:

$$L_1 \cdot S_1 + L_2 \cdot S_2 + L_3 \cdot S_3 = V$$

$$\left(-L_3 \cdot \frac{S_3 \cdot \eta_3 \cdot \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right)}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} + \frac{T - \frac{V \cdot \eta}{Q_2}}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)}\right) \cdot S_1 + \left(L_3 \cdot \left(\frac{S_3 \cdot \eta_3 \cdot \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right)}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} - 1\right) + L - \frac{T - \frac{V \cdot \eta}{Q_2}}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)}\right) \cdot S_2 + L_3 \cdot S_3 = V$$

$$L_3 \cdot \left(S_3 - \frac{S_3 \cdot \eta_3 \cdot \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right)}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} \cdot S_1 + \left(\frac{S_3 \cdot \eta_3 \cdot \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right)}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} - 1\right) \cdot S_2\right) = V - \frac{T - \frac{V \cdot \eta}{Q_2}}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} \cdot S_1 - \left(L - \frac{T - \frac{V \cdot \eta}{Q_2}}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)}\right) \cdot S_2$$

$$L_3 = \frac{V - \frac{T - \frac{V \cdot \eta}{Q_2}}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} \cdot S_1 - \left(L - \frac{T - \frac{V \cdot \eta}{Q_2}}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)}\right) \cdot S_2}{S_3 - \frac{S_3 \cdot \eta_3 \cdot \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right)}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} \cdot S_1 + \left(\frac{S_3 \cdot \eta_3 \cdot \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right)}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} - 1\right) \cdot S_2} \tag{33}$$

$$\boxed{L_3 = \frac{V - L \cdot S_2 + \frac{T - \frac{V \cdot \eta}{Q_2}}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} \cdot (S_2 - S_1)}{S_3 - S_2 + \frac{S_3 \cdot \eta_3 \cdot \left(\frac{1}{Q_3} - \frac{1}{Q_2}\right)}{S_1 \cdot \eta_1 \cdot \left(\frac{1}{Q_1} - \frac{1}{Q_2}\right)} \cdot (S_2 - S_1)}}$$

Where:

L	total length of the transport pathway, [L]
L_1	length of the first (cascade) “Pipe” component, [L]
L_2	length of the second (cascade) “Pipe” component, [L]
L_3	length of the third (cascade) “Pipe” component, [L]
Q_1	flow through the first (cascade) “Pipe” component, [$L^3 \cdot T$]
Q_2	flow through the second (cascade) “Pipe” component, [$L^3 \cdot T$]
Q_3	flow of the third (cascade) “Pipe” component, [$L^3 \cdot T$]
S_1	cross section first (cascade) “Pipe” component, [L^2]
S_2	cross section of second (cascade) “Pipe” component, [L^2]
S_3	cross section third (cascade) “Pipe” component, [L^2]
T	total flow time by transport pathway, [T]
V	total volume of the transport pathway, [L^3]
η	total porosity of the transport path, [-]
η_1	porosity of the first (cascade) “Pipe” component, [-]
η_3	porosity of the third (cascade) “Pipe” component, [-]