1 S1

This section outlines the steps of Shannon entropy and Rényi entropy, also checking that the steps are corrected.

Shannon entropy:

$$
\begin{aligned}
H &= -\int_0^\infty f(x)\left[\ln\left(\sqrt{\frac{\lambda}{2\pi x^3}}\,e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}\right)\right]dx \\
&= -\int_0^\infty f(x)\left[\ln\left(\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}}\cdot x^{-\frac{3}{2}}\cdot e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}\right)\right]dx \\
&= -\int_0^\infty f(x)\left[\frac{1}{2}\ln\left(\frac{\lambda}{2\pi}\right)-\frac{3}{2}\ln(x)-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right]dx \\
&= -\frac{1}{2}\ln\left(\frac{\lambda}{2\pi}\right)\underbrace{\int_0^\infty f(x)\,dx}_{\text{Integral of pdf}=1}+\frac{3}{2}\int_0^\infty f(x)\ln(x)\,dx+\frac{\lambda}{2\mu^2}\int_0^\infty f(x)\,\frac{(x-\mu)^2}{x}\,dx \\
&= -\frac{1}{2}\ln\left(\frac{\lambda}{2\pi}\right)+\frac{3}{2}\underbrace{\int_0^\infty \ln(x)\,f(x)\,dx}_{\text{Expectation}}+\frac{\lambda}{2\mu^2}\underbrace{\int_0^\infty \frac{(x-\mu)^2}{x}\,f(x)\,dx}_{\text{Expectation}} \\
&= -\frac{1}{2}\ln\left(\frac{\lambda}{2\pi}\right)+\frac{3}{2}\,\mathbb{E}\left[\ln(x)\right]+\frac{\lambda}{2\mu^2}\,\mathbb{E}\left[\frac{(x-\mu)^2}{x}\right] \qquad \to \textcircled{1}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[\ln(x)\right] &= \int_0^\infty \ln(x)\,f(x)\,dx \\
&= \int_0^\infty \sqrt{\frac{\lambda}{2\pi x^3}}\,e^{-\left(\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right)}\,\ln(x)\,dx && \because (x-\mu)=x^2-2x\mu+\mu^2 \\
&= \int_0^\infty \sqrt{\frac{\lambda}{2\pi(\mu\,y)^3}}\,e^{-\left(\frac{\lambda(\mu\,y-\mu)^2}{2\mu^2\,\mu\,y}\right)}\,\ln(\mu\,y)\,\mu\,dy && \begin{aligned}\text{let: } x &= \mu\,y \\ dx &= \mu\,dy \\ x=0 &\Rightarrow y=0 \\ x=\infty &\Rightarrow y=\infty\end{aligned} \\
&= \int_0^\infty \sqrt{\frac{\lambda}{2\pi(\mu\,y)^3}}\,e^{-\left(\frac{\lambda(y-1)^2}{2\mu\,y}\right)}\,\left(\ln(\mu)+\ln(y)\right)\,\mu\,dy \\
&= \int_0^\infty \sqrt{\frac{\lambda}{2\pi(\mu\,y)^3}}\,e^{-\left(\frac{\lambda(y-1)^2}{2\mu\,y}\right)}\,\ln(\mu)\,\mu\,dy+\int_0^\infty \sqrt{\frac{\lambda}{2\pi(\mu\,y)^3}}\,e^{-\left(\frac{\lambda(y-1)^2}{2\mu\,y}\right)}\,\ln(y)\,\mu\,dy \\
&= \ln(\mu)\cdot\underbrace{\int_0^\infty \sqrt{\frac{\lambda}{2\pi x^3}}\,e^{-\left(\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right)}\,dx}_{\text{Integral of pdf}=1}+\int_0^\infty \sqrt{\frac{\lambda}{2\pi(\mu\,y)^3}}\,e^{-\left(\frac{\lambda(y-1)^2}{2\mu\,y}\right)}\,\ln(y)\,\mu\,dy \\
&= \ln(\mu)+\int_0^\infty \sqrt{\frac{\lambda}{2\pi(\mu\,y)^3}}\,e^{-\left(\frac{\lambda(y-1)^2}{2\mu\,y}\right)}\,\ln(y)\,\mu\,dy \\
&= \ln(\mu)+\left(\frac{\lambda}{2\pi\,\mu^3}\right)^{1/2}\cdot\mu\int_0^\infty y^{-3/2}\,e^{-\left(\frac{\lambda(y-1)^2}{2\mu\,y}\right)}\,\ln(y)\,dy \\
&= \ln(\mu)+\left(\frac{\lambda}{2\pi\,\mu}\right)^{1/2}\int_0^\infty y^{-3/2}\,e^{-\left(\frac{\lambda(y^2-2y+1)}{2\mu\,y}\right)}\,\ln(y)\,dy \\
&= \ln(\mu)+\left(\frac{\lambda}{2\pi\,\mu}\right)^{1/2}\int_0^\infty y^{-3/2}\,e^{-\frac{\lambda}{2\mu}\left(y+\frac{1}{y}-2\right)}\,\ln(y)\,dy && \begin{aligned}\text{let: } y &= e^t \\ \ln(y) &= t \\ dy &= e^t\,dt \\ x=0 &\Rightarrow y=-\infty \\ x=\infty &\Rightarrow y=\infty\end{aligned} \\
&= \ln(\mu)+\left(\frac{\lambda}{2\pi\,\mu}\right)^{1/2}\int_0^\infty y^{-3/2}\,e^{-\frac{\lambda}{2\mu}\left(y+y^{-1}-2\right)}\,\ln(y)\,dy && \cosh(t)=y+\frac{1}{y} \\
&= \ln(\mu)+\left(\frac{\lambda}{2\pi\,\mu}\right)^{1/2}\int_{-\infty}^\infty e^{-3t/2}\,e^{-\frac{\lambda}{2\mu}\left(e^t+e^{-t}-2\right)}\,t\,e^t\,dt && =\frac{e^t+e^{-t}}{2}
\end{aligned}
$$

$$
\begin{aligned}
&= \ln(\mu) + \left(\frac{\lambda}{2\pi\,\mu}\right)^{1/2} \int_{-\infty}^{\infty} e^{-t/2}\, e^{-\frac{\lambda}{2\mu}(2\cosh(t)-2)}\, t\, dt \\
&= \ln(\mu) + \left(\frac{\lambda}{2\pi\,\mu}\right)^{1/2} \int_{-\infty}^{\infty} e^{-t/2}\, e^{-\frac{\lambda}{\mu}(\cosh(t)-1)}\, t\, dt \\
&= \ln(\mu) + \left(\frac{\lambda}{2\pi\,\mu}\right)^{1/2} \int_{-\infty}^{\infty} e^{-t/2}\, e^{-\frac{\lambda}{\mu}\cosh(t)+\frac{\lambda}{\mu}}\, t\, dt \\
&= \ln(\mu) + \left(\frac{\lambda}{2\pi\,\mu}\right)^{1/2} e^{\frac{\lambda}{\mu}} \int_{-\infty}^{\infty} t\, e^{-t/2}\, e^{-\frac{\lambda}{\mu}\cosh(t)}\, dt \\
&= \ln(\mu) + \left(\frac{\lambda}{2\pi\,\mu}\right)^{1/2} e^{\frac{\lambda}{\mu}} \left[\int_{-\infty}^{0} t\, e^{-t/2}\, e^{-\frac{\lambda}{\mu}\cosh t}\, dt + \int_{0}^{\infty} t\, e^{-t/2}\, e^{-\frac{\lambda}{\mu}\cosh t}\, dt\right] \\
&= \ln(\mu) + \left(\frac{\lambda}{2\pi\,\mu}\right)^{1/2} e^{\frac{\lambda}{\mu}} \left[\mathbb{V} + \int_{0}^{\infty} t\, e^{-t/2}\, e^{-\frac{\lambda}{\mu}\cosh t}\, dt\right] \qquad \rightarrow ②
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{V} &= \int_{-\infty}^{0} t\, e^{-t/2}\, e^{-\frac{\lambda}{\mu}\cosh t}\, dt \\
&= \int_{\infty}^{0} (-u)\, e^{u/2}\, e^{-\frac{\lambda}{\mu}\cosh u}\, (-du) \\
&= -\int_{0}^{\infty} u\, e^{u/2}\, e^{-\frac{\lambda}{\mu}\cosh u}\, du \\
&= -\int_{0}^{\infty} t\, e^{t/2}\, e^{-\frac{\lambda}{\mu}\cosh t}\, dt \qquad \rightarrow ③ \\
\mathbb{U} &= \int_{0}^{\infty} t\, e^{-t/2}\, e^{-\frac{\lambda}{\mu}\cosh t}\, dt \qquad \rightarrow ④
\end{aligned}
$$

$$
\begin{aligned}
\text{let: } -t &= u \\
e^{-t/2} &= e^{u/2} \\
dt &= -du \\
t = -\infty &\Rightarrow u = \infty \\
t = 0 &\Rightarrow u = 0
\end{aligned}
$$

From ③ and ④ shown above.

$$
\begin{aligned}
\therefore \mathbb{V} + \mathbb{U} &= \int_{0}^{\infty} t\, e^{-t/2}\, e^{-\frac{\lambda}{\mu}\cosh t}\, dt - \int_{0}^{\infty} t\, e^{t/2}\, e^{-\frac{\lambda}{\mu}\cosh t}\, dt \\
&= \int_{0}^{\infty} t e^{-\frac{\lambda}{\mu}\cosh t}(e^{-t/2} - e^{t/2}) dt \\
&= \int_{0}^{\infty} t e^{-\frac{\lambda}{\mu}\cosh t}\left[-2\sinh\left(\frac{t}{2}\right)\right] dt \\
&= -2\int_{0}^{\infty} t \sinh\left(\frac{t}{2}\right)\, e^{-\frac{\lambda}{\mu}\cosh t} dt
\end{aligned}
$$

$$
\because K_\nu(z) = \int_{0}^{\infty} e^{-z\cosh t} \cosh(\nu t)\, dt
$$

Differentiating two sides with respect to order ν:

$$
\begin{aligned}
\therefore \frac{\partial K_\nu(z)}{\partial \nu} &= \int_{0}^{\infty} t\, e^{-z\cosh t}\, \frac{\partial}{\partial \nu}[\cosh(\nu t)] dt \\
&= \int_{0}^{\infty} t\, e^{-z\cosh t} \sinh(\nu t)\, dt \\
\therefore \left.\frac{\partial K_\nu(z)}{\partial \nu}\right|_{\nu=-\frac{1}{2}} &= \int_{0}^{\infty} t \sinh\left(\frac{t}{2}\right)\, e^{-\frac{\lambda}{\mu}\cosh t} dt \\
\therefore \mathbb{V} + \mathbb{U} &= -2\left.\frac{\partial K_\nu(z)}{\partial \nu}\right|_{\nu=-\frac{1}{2}} \qquad \rightarrow ⑤
\end{aligned}
$$

Using ⑤ to substitute in equation ② shown above.

$$
\therefore \mathbb{E}\left[\ln(x)\right] = \ln(\mu) + \left(\frac{\lambda}{2\pi\,\mu}\right)^{1/2} e^{\frac{\lambda}{\mu}} \left[-2\left.\frac{\partial K_\nu(z)}{\partial \nu}\right|_{\nu=-\frac{1}{2}}\right]
$$

dividing and multiplying by $K_{-\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}e^{-z} = \sqrt{\frac{\pi\mu}{2\lambda}}e^{-\lambda/\mu}$:

$$\therefore \mathbb{E}\left[\ln(x)\right] = \ln(\mu) + \frac{1}{K_{-\frac{1}{2}}(z)} \left.\frac{\partial K_\nu(z)}{\partial \nu}\right|_{\nu=-\frac{1}{2}}$$

$$= \ln(\mu) + \left.\frac{\partial \ln K\nu(z)}{\partial \nu}\right|_{\nu=-1/2,\, z=\lambda/\mu}$$

$$\because \left.\frac{\partial K_\nu(z)}{\partial \nu}\right|_{\nu=-\frac{1}{2},\, z=\lambda/\mu} = -\sqrt{\frac{\pi}{2x}}\, E_1(2z)\, e^z$$

$$\therefore \left.\frac{\partial \ln K_\nu(z)}{\partial \nu}\right|_{\nu=-1/2,\, z=\lambda/\mu} = -e^{\frac{2\lambda}{\mu}}\ E_1\left(\frac{2\lambda}{\mu}\right)$$

$$\therefore \mathbb{E}\left[\ln(x)\right] = \ln(\mu) - e^{\frac{2\lambda}{\mu}}\ E_1\left(\frac{2\lambda}{\mu}\right) \qquad \rightarrow ⑥$$

$$\mathbb{E}\left[\frac{(x-\mu)^2}{x}\right] = \int_0^\infty \frac{(x-\mu)^2}{x}\, f(x)\, dx$$

$$= \int_0^\infty f(x) \cdot \left(x^2 - 2\mu x + \mu^2\right)/x\, dx$$

$$= \int_0^\infty \sqrt{\frac{\lambda}{2\pi x^3}} \left[e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}\right] \left[\frac{x^2 - 2\mu x + \mu^2}{x}\right] dx$$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \int_0^\infty x^{-\frac{5}{2}}\, e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}} \left(x^2 - 2\mu x + \mu^2\right)\, dx$$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \left[\int_0^\infty x^{-\frac{1}{2}}\, e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}\, dx - 2\mu \int_0^\infty x^{-\frac{3}{2}}\, e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}\, dx \right.$$

$$\left. + \mu^2 \int_0^\infty x^{-\frac{5}{2}}\, e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}\, dx\right]$$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \left[\mathbb{W} - 2\mu\, \mathbb{H} + \mu^2\, \mathbb{G}\right] \qquad \rightarrow ⑦$$

let: $x = \mu\, y$
$dx = \mu\, dy$
$x = 0 \Rightarrow y = 0$
$x = \infty \Rightarrow y = \infty$

$$\mathbb{W} = \int_0^\infty x^{-\frac{1}{2}}\, e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}\, dx$$

$$= \int_0^\infty (\mu\, y)^{-\frac{1}{2}}\, e^{-\frac{\lambda(\mu\, y-\mu)^2}{2\mu^2 \mu\, y}}\, \mu\, dy$$

$$= \mu^{\frac{1}{2}} \int_0^\infty y^{-\frac{1}{2}}\, e^{-\frac{\lambda(y-1)^2}{2\mu\, y}}\, dy$$

$$= \mu^{\frac{1}{2}} \int_0^\infty y^{-\frac{1}{2}}\, e^{-\frac{\lambda}{2\mu}\left(y+\frac{1}{y}-2\right)}\, dy$$

$$= \mu^{\frac{1}{2}} \int_0^\infty y^{-\frac{1}{2}} \left[e^{-\frac{\lambda}{2\mu}y} \cdot e^{-\frac{\lambda}{2\mu\, y}} \cdot e^{\frac{\lambda}{\mu}}\right] dy$$

$$= \mu^{\frac{1}{2}}\, e^{\frac{\lambda}{\mu}} \int_0^\infty y^{-\frac{1}{2}} \left[e^{-\frac{\lambda\, y}{2\mu} - \frac{\lambda}{2\mu\, y}}\right] dy \ \Rightarrow \int_0^\infty x^{-\frac{1}{2}} \left[e^{-\gamma\, x - \frac{\beta}{x}}\right] dx = \sqrt{\frac{\pi}{\gamma}} e^{-2\sqrt{\beta\gamma}}$$

$$= \mu^{\frac{1}{2}}\, e^{\frac{\lambda}{\mu}} \sqrt{\frac{\pi}{\frac{\lambda}{2\mu}}}\, e^{-2\sqrt{\frac{\lambda}{2\mu} \cdot \frac{\lambda}{2\mu}}}$$

note: $x = y$

$$\gamma = \frac{\lambda}{2\mu},\ \beta = \frac{\lambda}{2\mu}$$

$$= \mu \sqrt{\frac{2\pi}{\lambda}} \qquad \rightarrow ⑧$$

$$
\begin{aligned}
\mathbb{H} &= \int_0^\infty x^{-\frac{3}{2}}\, e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}\, dx \\
&= \int_0^\infty (\mu\, y)^{-\frac{3}{2}}\, e^{-\frac{\lambda(\mu\, y-\mu)^2}{2\mu^2 \mu\, y}}\, \mu\, dy \\
&= \mu^{-\frac{1}{2}} \int_0^\infty y^{-\frac{3}{2}}\, e^{-\frac{\lambda(y-1)^2}{2\mu\, y}}\, dy \\
&= \mu^{-\frac{1}{2}}\, e^{\frac{\lambda}{\mu}} \int_0^\infty y^{-\frac{3}{2}} \left[e^{-\frac{\lambda\, y}{2\mu}-\frac{\lambda}{2\mu\, y}}\right] dy \;\Rightarrow \int_0^\infty x^{\nu-1}\left[e^{-\gamma\, x-\frac{\beta}{x}}\right] dx = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu\left(2\sqrt{\gamma\,\beta}\right) \\
&= \mu^{-\frac{1}{2}} \cdot e^{\frac{\lambda}{\mu}} \cdot 2\left(\frac{\lambda}{2\mu}\Big/\frac{\lambda}{2\mu}\right)^{-\frac{1}{2}/2} \cdot K_{-\frac{1}{2}}\left(2\sqrt{\frac{\lambda}{2\mu}\cdot\frac{\lambda}{2\mu}}\right) \\
&= 2\,\mu^{-\frac{1}{2}} \cdot e^{\frac{\lambda}{\mu}} \cdot K_{-\frac{1}{2}}\left(\frac{\lambda}{\mu}\right) \\
&= \sqrt{\frac{2\pi}{\lambda}} \qquad \to ⑨
\end{aligned}
$$

note: $\nu = -1/2$

$$K_{\pm\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2\,z}}\, e^{-z}$$

$$
\begin{aligned}
\mathbb{G} &= \int_0^\infty x^{-\frac{5}{2}}\, e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}\, dx \\
&= \int_0^\infty (\mu\, y)^{-\frac{5}{2}}\, e^{-\frac{\lambda(\mu\, y-\mu)^2}{2\mu^2 \mu\, y}}\, \mu\, dy \\
&= \mu^{-\frac{3}{2}} \int_0^\infty y^{-\frac{5}{2}}\, e^{-\frac{\lambda(y-1)^2}{2\mu\, y}}\, dy \\
&= \mu^{-\frac{3}{2}}\, e^{\frac{\lambda}{\mu}} \int_0^\infty y^{-\frac{5}{2}} \left[e^{-\frac{\lambda\, y}{2\mu}-\frac{\lambda}{2\mu\, y}}\right] dy \;\Rightarrow \int_0^\infty x^{\nu-1}\left[e^{-\gamma\, x-\frac{\beta}{x}}\right] dx = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu\left(2\sqrt{\gamma\,\beta}\right) \\
&= \mu^{-\frac{3}{2}} \cdot e^{\frac{\lambda}{\mu}} \cdot 2\left(\frac{\lambda}{2\mu}\Big/\frac{\lambda}{2\mu}\right)^{-\frac{3}{2}/2} \cdot K_{-\frac{3}{2}}\left(2\sqrt{\frac{\lambda}{2\mu}\cdot\frac{\lambda}{2\mu}}\right) \\
&= 2\,\mu^{-\frac{3}{2}} \cdot e^{\frac{\lambda}{\mu}} \cdot K_{-\frac{3}{2}}\left(\frac{\lambda}{\mu}\right) \\
&= 2\,\mu^{-\frac{3}{2}} \cdot e^{\frac{\lambda}{\mu}} \cdot \sqrt{\frac{\pi}{2\left(\frac{\lambda}{\mu}\right)}}\left(1+\frac{1}{\left(\frac{\lambda}{\mu}\right)}\right) e^{-\left(\frac{\lambda}{\mu}\right)} \\
&= 2\,\mu^{-\frac{3}{2}} \sqrt{\frac{\pi\,\mu}{2\lambda}}\left(1+\frac{\mu}{\lambda}\right) \\
&= \mu^{-1}\sqrt{\frac{2\pi}{\lambda}}\left(1+\frac{\mu}{\lambda}\right) \qquad \to ⑩
\end{aligned}
$$

note: $\nu = -\frac{3}{2}$

$$
\begin{aligned}
&\because z\,K_{\nu-1}(z) - z\,K_{\nu+1}(z) = -2\nu K_\nu(z) \\
&\therefore K_{\nu+1}(z) = K_{\nu-1}(z) + \frac{-2\nu}{z} K_\nu(z) \\
&\therefore K_{\frac{3}{2}}(z) = K_{-\frac{1}{2}}(z) + \frac{1}{z} K_{\frac{1}{2}}(z) \\
&\therefore K_{\frac{3}{2}}(z) = \sqrt{\frac{\pi}{2\,z}}\, e^{-z} + \frac{1}{z}\sqrt{\frac{\pi}{2\,z}}\, e^{-z} \\
&\therefore K_{\frac{3}{2}}(z) = \sqrt{\frac{\pi}{2z}}\left(1+\frac{1}{z}\right) e^{-z} \\
&\because K_{-\nu}(z) = K_\nu(z) \\
&\therefore K_{-\frac{3}{2}}(z) = \sqrt{\frac{\pi}{2z}}\left(1+\frac{1}{z}\right) e^{-z}
\end{aligned}
$$

Using ⑧, ⑨ and ⑩ to substitute in equation ⑦ shown above.

$$
\begin{aligned}
\therefore \mathbb{E}\left[\frac{(x-\mu)^2}{x}\right] &= \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} \left[\mathbb{W} - 2\mu\,\mathbb{H} + \mu^2\,\mathbb{G}\right] \\
&= \sqrt{\frac{\lambda}{2\pi}}\left[\mu\sqrt{\frac{2\pi}{\lambda}} - 2\mu\sqrt{\frac{2\pi}{\lambda}} + \mu^2\left(\mu^{-1}\sqrt{\frac{2\pi}{\lambda}}\left(1+\frac{\mu}{\lambda}\right)\right)\right] \\
&= \frac{\mu^2}{\lambda} \qquad \to ⑪
\end{aligned}
$$

Then using ⑥ and ⑪ to substitute in equation ① shown above to obtain the Shannon entropy.

$$\therefore H = -\frac{1}{2}\ln\left(\frac{\lambda}{2\pi}\right) + \frac{3}{2}\left[\ln(\mu) - e^{\frac{2\lambda}{\mu}}\ E_1\left(\frac{2\lambda}{\mu}\right)\right] + \frac{\lambda}{2\mu^2}\left(\frac{\mu^2}{\lambda}\right)$$
$$= \frac{1}{2}\ln\frac{2\pi e}{\lambda} + \frac{3}{2}\ln\mu - \frac{3}{2}e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)$$

note:

$$\frac{1}{2}\ln\frac{2\pi}{\lambda} + \frac{1}{2} = \frac{1}{2}\ln\frac{2\pi}{\lambda} + \frac{1}{2}\ln e$$
$$= \frac{1}{2}\left(\ln\frac{2\pi}{\lambda} + \ln e\right)$$
$$= \frac{1}{2}\ln\left(\frac{2\pi}{\lambda}\cdot e\right)$$
$$= \frac{1}{2}\ln\frac{2\pi e}{\lambda}$$

Rényi entropy:

$$H_\alpha = \frac{1}{1-\alpha}\ln\left(\int_0^\infty \left[\sqrt{\frac{\lambda}{2\pi x^3}}\,e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}\right]^\alpha dx\right)$$
$$= \frac{1}{1-\alpha}\ln\left(\int_0^\infty \left[\left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}} x^{-\frac{3}{2}} e^{-\frac{\lambda(x-\mu)^2}{2\mu^2 x}}\right]^\alpha dx\right)$$
$$= \frac{1}{1-\alpha}\ln\left(\int_0^\infty \left(\frac{\lambda}{2\pi}\right)^{\frac{\alpha}{2}} x^{-\frac{3\alpha}{2}} e^{-\frac{\alpha\lambda(x-\mu)^2}{2\mu^2 x}}\,dx\right)$$
$$= \frac{1}{1-\alpha}\ln\left(\left(\frac{\lambda}{2\pi}\right)^{\frac{\alpha}{2}}\int_0^\infty x^{-\frac{3\alpha}{2}} e^{-\frac{\alpha\lambda x}{2\mu^2}+\frac{\alpha\lambda}{\mu}-\frac{\alpha\lambda}{2x}}\,dx\right)$$
$$= \frac{1}{1-\alpha}\left(\frac{\alpha}{2}\ln\left(\frac{\lambda}{2\pi}\right) + \ln\left[e^{\frac{\alpha\lambda}{\mu}}\int_0^\infty x^{-\frac{3\alpha}{2}} e^{-\frac{\alpha\lambda x}{2\mu^2}-\frac{\alpha\lambda}{2x}}\,dx\right]\right)$$
$$= \frac{1}{1-\alpha}\left(\frac{\alpha}{2}\ln\left(\frac{\lambda}{2\pi}\right) + \frac{\alpha\lambda}{\mu} + \ln\left[\int_0^\infty x^{-\frac{3\alpha}{2}} e^{-\frac{\alpha\lambda x}{2\mu^2}-\frac{\alpha\lambda}{2x}}\,dx\right]\right)$$
$$= \frac{1}{1-\alpha}\left(\frac{\alpha}{2}\ln\left(\frac{\lambda}{2\pi}\right) + \frac{\alpha\lambda}{\mu} + \ln\left[2\mu^{1-\frac{3\alpha}{2}} K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)\right]\right)$$
$$= \frac{1}{1-\alpha}\left[\frac{\alpha}{2}\ln\left(\frac{\lambda}{2\pi}\right) + \left(1-\frac{3\alpha}{2}\right)\ln(\mu) + \ln(2) + \frac{\alpha\lambda}{\mu} + \ln K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)\right]$$

Proof:

To prove that $\lim_{\alpha\to 1} H_\alpha = H$, we follow the following steps:

$$\lim_{\alpha\to 1} f(\alpha) = \lim_{\alpha\to 1}\left[\ \frac{1}{1-\alpha}\left[\frac{\alpha}{2}\ln\left(\frac{\lambda}{2\pi}\right) + \left(1-\frac{3\alpha}{2}\right)\ln(\mu) + \ln(2) + \frac{\alpha\lambda}{\mu} + \ln K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)\right]\ \right]$$

First, using direct substitution: $(\alpha \to 1)$

$$\text{denominator} = 1-\alpha = 1-1 = 0$$

$$\text{numerator} = \frac{1}{2}\ln\left(\frac{\lambda}{2\pi}\right) + \left(1-\frac{3}{2}\right)\ln(\mu) + \ln(2) + \frac{\lambda}{\mu} + \ln\left(K_{1-\frac{3}{2}}\left(\frac{\lambda}{\mu}\right)\right)$$
$$= \frac{1}{2}\ln\left(\frac{\lambda}{2\pi}\right) + \left(1-\frac{3}{2}\right)\ln(\mu) + \ln(2) + \frac{\lambda}{\mu} + \ln\left(\sqrt{\frac{\pi\mu}{2\lambda}}\,e^{-\frac{\lambda}{\mu}}\right)$$
$$= 0$$

$$\therefore \frac{\text{numerator}}{\text{denominator}} = \frac{0}{0}$$

Second, using L'Hôpital's rule:

$$\lim_{\alpha\to 1} f(\alpha) = \lim_{\alpha\to 1} \frac{N(\alpha)}{1-\alpha} = \lim_{\alpha\to 1} \frac{N'(\alpha)}{-1} = N'(1)$$

Differentiate both the numerator and the denominator with respect to α;

$$N(\alpha) = \frac{\alpha}{2}\ln\left(\frac{\lambda}{2\pi}\right) + \left(1-\frac{3\alpha}{2}\right)\ln(\mu) + \ln(2) + \frac{\alpha\lambda}{\mu} + \ln\left(K_{\nu(\alpha)}(z(\alpha))\right)$$
$$N'(\alpha) = \frac{1}{2}\ln\left(\frac{\lambda}{2\pi}\right) + \frac{\lambda}{\mu} - \frac{3}{2}\ln\mu + \frac{d}{d\alpha}\ln\left(K_{\nu(\alpha)}(z(\alpha))\right)$$

Using the chain rule:

$$\frac{d}{d\alpha}\ln K_\nu(z) = \frac{1}{K_\nu(z)}\left(\frac{\partial K_\nu}{\partial \nu}\cdot\frac{d\nu}{d\alpha} + \frac{\partial K_\nu}{\partial z}\cdot\frac{dz}{d\alpha}\right)$$

So:

$$\frac{d}{d\alpha}\ln K_\nu(z) = -\frac{3}{2}\cdot\frac{1}{K_\nu(z)}\frac{\partial K_\nu}{\partial \nu} + \frac{\lambda}{\mu}\cdot\frac{1}{K_\nu(z)}\frac{\partial K_\nu}{\partial z}$$

$$\because z\frac{d}{dz}K_\nu(z) + \nu K_\nu(z) = -zK_{\nu-1}(z)$$

$$\frac{d}{dz}K_\nu(z) = -K_{\nu-1}(z) - \frac{\nu}{z}K_\nu(z)$$

$$\therefore \frac{1}{K_\nu(z)}\frac{\partial K_\nu}{\partial z} = -\frac{K_{\nu-1}(z)}{K_\nu(z)} - \frac{\nu}{z},$$

$$= \left[-\frac{K-3/2}{K_{-1/2}} - \frac{-1/2}{\lambda/\mu}\right]$$

Let: $\nu(\alpha) = 1 - \frac{3\alpha}{2}$

$\therefore \nu'(\alpha) = -\frac{3}{2}$

, $z(\alpha) = \frac{\alpha\lambda}{\mu}$

$\therefore z'(\alpha) = \frac{\lambda}{\mu}$

when: $\alpha = 1$,

$\therefore \nu = -\frac{1}{2}$, $z = \frac{\lambda}{\mu}$

$$\because \left.\frac{1}{K_\nu(z)}\frac{\partial K_\nu}{\partial \nu}\right|_{\alpha=1} = \left.\frac{\partial \ln K\nu(z)}{\partial \nu}\right|_{\nu=-1/2,\, z=\lambda/\mu}$$

$$\therefore \left.\frac{d}{d\alpha}\ln K_\nu(z)\right|_{\alpha=1} = -\frac{3}{2}\left[\left.\frac{\partial \ln K\nu(z)}{\partial \nu}\right|_{\nu=-1/2,\, z=\lambda/\mu}\right] + \frac{\lambda}{\mu}\left[-\frac{K-3/2}{K_{-1/2}} - \frac{-1/2}{\lambda/\mu}\right]$$

$$= -\frac{3}{2}\left.\frac{\partial \ln K\nu(z)}{\partial \nu}\right|_{\nu=-1/2,\, z=\lambda/\mu} - \frac{\lambda}{\mu} - \frac{1}{2}$$

$$\therefore N'(1) = \frac{1}{2}\ln\left(\frac{\lambda}{2\pi}\right) + \frac{\lambda}{\mu} - \frac{3}{2}\ln\mu - \frac{3}{2}\left.\frac{\partial \ln K\nu(z)}{\partial \nu}\right|_{\nu=-1/2,\, z=\lambda/\mu} - \frac{\lambda}{\mu} - \frac{1}{2}$$

$$= \frac{1}{2}\ln\left(\frac{\lambda}{2\pi}\right) - \frac{3}{2}\ln\mu - \frac{3}{2}\left.\frac{\partial \ln K\nu(z)}{\partial \nu}\right|_{\nu=-1/2,\, z=\lambda/\mu} - \frac{1}{2}$$

$$\therefore \lim_{\alpha\to 1} f(\alpha) = \frac{N'(1)}{-1} = -\left[\frac{1}{2}\ln\left(\frac{\lambda}{2\pi}\right) - \frac{3}{2}\ln\mu - \frac{3}{2}\left.\frac{\partial \ln K\nu(z)}{\partial \nu}\right|_{\nu=-1/2,\, z=\lambda/\mu} - \frac{1}{2}\right]$$

$$= \frac{1}{2}\ln\frac{2\pi}{\lambda} + \frac{3}{2}\ln\mu + \frac{1}{2} + \frac{3}{2}\left.\frac{\partial \ln K_\nu(z)}{\partial \nu}\right|_{\nu=-1/2,\, z=\lambda/\mu}$$

$$= \frac{1}{2}\ln\frac{2\pi e}{\lambda} + \frac{3}{2}\ln\mu - \frac{3}{2}e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)$$

$$\therefore \lim_{\alpha\to 1} H_\alpha = H$$

The integral $K_\nu(z) = \int_0^\infty e^{-z\cosh t}\cosh(\nu t)\,dt$ is expressed in terms of the modified Bessel function of the second kind via its integral representation (Edition (2007), Eq. 8.432-1, p. 917). The evaluation at half-integer order employs the closed form $K_{\pm\frac{1}{2}}(z)$ (Edition, 2007, Eq. 8.469-3, p. 925), we also used the following forms $z\,K_{\nu-1}(z) - z\,K_{\nu+1}(z) = -2\nu K_\nu(z)$ (Edition, 2007, Eq. 8.486-10, p. 929), $z\frac{d}{dz}K_\nu(z) + \nu K_\nu(z) = -zK_{\nu-1}(z)$ (Edition, 2007, Eq. 8.486-12, p. 929) and $K_{-\nu}(z) = K_\nu(z)$ (Edition, 2007, Eq. 8.486-16, p. 929), we also used the following equations $\int_0^\infty x^{\nu-1}\left[e^{-\gamma\,x-\frac{\beta}{x}}\right]dx = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2}K_\nu\left(2\sqrt{\gamma\,\beta}\right)$ (Edition, 2007, Eq. 3.471-9, p. 368) and $\int_0^\infty x^{-\frac{1}{2}}\left[e^{-\gamma\,x-\frac{\beta}{x}}\right]dx = \sqrt{\frac{\pi}{\gamma}}e^{-2\sqrt{\beta\gamma}}$ (Edition, 2007, Eq. 3.471-15, p. 369).

2 S2

This section outlines the steps involved in solving Lindley derivatives. Bayesian estimators of the entropies (Shannon and Rényi) under the squared error loss function (SE), the general entropy loss function (GE) and the LINEX loss function, respectively:

♦ Bayesian estimators of the Shannon entropy under the SE loss function. In that case, $u(\mu,\lambda) = H$, thus;

$$\hat{u}_{S.\mu} = \left(\frac{3\lambda}{\mu^2}\right)e^{\frac{2\lambda}{\mu}}\,E_1\left(\frac{2\lambda}{\mu}\right),\qquad \hat{u}_{S.\lambda} = \frac{1}{\lambda} - \left(\frac{3}{\mu}\right)e^{\frac{2\lambda}{\mu}}\,E_1\left(\frac{2\lambda}{\mu}\right),$$

$$\hat{u}_{S.\mu\mu} = \frac{3}{2\mu^2}\left[\frac{2\lambda}{\mu} - \left(\frac{4\lambda}{\mu} + \left(\frac{2\lambda}{\mu}\right)^2\right)e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)\right], \hat{u}_{S.\lambda\lambda} = \frac{-1}{\lambda^2}\left[1 - \frac{3\lambda}{\mu} + 6\left(\frac{\lambda}{\mu}\right)^2 e^{\frac{2\lambda}{\mu}}\,E_1\left(\frac{2\lambda}{\mu}\right)\right],$$

$$\hat{u}_{S.\mu\lambda} = \hat{u}_{S.\lambda\mu} = \frac{3}{2\lambda\mu}\left[\left(\frac{2\lambda}{\mu} + \left(\frac{2\lambda}{\mu}\right)^2\right)e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right) - \frac{2\lambda}{\mu}\right].$$

♦ Bayesian estimators of the Rényi entropy under the SE loss function. In that case, $u(\mu,\lambda) = H_\alpha$, thus;

$$\hat{u}_{R.\mu} = \frac{1}{1-\alpha}\left[\left(1-\frac{3\alpha}{2}\right)\frac{1}{\mu} - \frac{\alpha\lambda}{\mu^2} - \frac{\alpha\lambda}{\mu^2}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)\right],$$

$$\hat{u}_{R.\mu\mu} = \frac{1}{1-\alpha}\left[\left(\frac{3\alpha}{2}-1\right)\frac{1}{\mu^2} + \frac{\alpha\lambda}{\mu^3} - \frac{2\alpha\lambda}{\mu^3}\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)} + \frac{\alpha^2\lambda^2}{\mu^4}\left[\frac{K''_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)} - \left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)^2\right]\right],$$

$$\hat{u}_{R.\lambda} = \frac{1}{1-\alpha}\left[\frac{\alpha}{2\lambda} + \frac{\alpha}{\mu} - \frac{\alpha}{\mu}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)\right],$$

$$\hat{u}_{R.\lambda\lambda} = \frac{1}{1-\alpha}\left[-\frac{\alpha}{2\lambda^2} - \left(\frac{\alpha}{\mu}\right)^2\left[\frac{K''_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)} - \left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)^2\right]\right],$$

$$\hat{u}_{R.\mu\lambda} = \hat{u}_{R.\lambda\mu} = \frac{1}{1-\alpha}\left[-\frac{\alpha}{\mu^2} - \frac{\alpha}{\mu^2}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right) - \frac{\alpha^2\lambda}{\mu^3}\left[\frac{K''_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)} - \left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)^2\right]\right].$$

• Bayesian estimators of the Shannon entropy under the GE loss function. In that case, $u(\mu,\lambda) = H^{-q}$, thus;

$$\hat{u}_{S.\mu} = -qH^{-q-1}\left[\frac{3\lambda}{\mu^2}\,e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)\right],\qquad \hat{u}_{S.\lambda} = -qH^{-q-1}\left[\frac{1}{\lambda} - \frac{3}{\mu}\,e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)\right],$$

$$\hat{u}_{S.\mu\mu} = qH^{-q-2}\left[(q+1)\left\{\frac{3\lambda}{\mu^2}\,e^{\frac{2\lambda}{\mu}}\,E_1\left(\frac{2\lambda}{\mu}\right)\right\}^2 - \frac{3H}{2\mu^2}\left\{\frac{2\lambda}{\mu} - \left(\frac{4\lambda}{\mu} + \left(\frac{2\lambda}{\mu}\right)^2\right)e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)\right\}\right],$$

$$\hat{u}_{S.\lambda\lambda} = qH^{-q-2}\left[\frac{(q+1)}{\lambda^2}\left\{1 - \frac{3\lambda}{\mu}e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)\right\}^2 - \frac{H}{\lambda^2}\left\{-1 + \frac{3\lambda}{\mu} - 6\left(\frac{\lambda}{\mu}\right)^2 e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)\right\}\right],$$

$$\hat{u}_{S.\mu\lambda} = qH^{-q-2}\left[\frac{3\,(q+1)\,\lambda}{\mu^2}\,e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)\left\{1 - \frac{3\lambda}{\mu}\,e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)\right\}\right.$$

$$- \frac{3H}{2\lambda\mu}\left\{\left(\frac{2\lambda}{\mu}+\left(\frac{2\lambda}{\mu}\right)^2\right)e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)-\frac{2\lambda}{\mu}\right\}\Bigg].$$

- Bayesian estimators of the Rényi entropy under the GE loss function. In that case, $u(\mu,\lambda)=H_\alpha^{-q}$, thus;

$$\hat{u}_{R.\mu}=-q\,H_\alpha^{-q-1}\left[\frac{1}{1-\alpha}\left[\left(1-\frac{3\alpha}{2}\right)\frac{1}{\mu}-\frac{\alpha\lambda}{\mu^2}-\frac{\alpha\lambda}{\mu^2}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)\right]\right],$$

$$\hat{u}_{R.\mu\mu}=\frac{(q^2+q)\,H_\alpha^{-q-2}}{[1-\alpha]^2}\left[\left(1-\frac{3\alpha}{2}\right)\frac{1}{\mu}-\frac{\alpha\lambda}{\mu^2}-\frac{\alpha\lambda}{\mu^2}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)\right]^2$$
$$-\frac{q\,H_\alpha^{-q-1}}{1-\alpha}\left[\left(\frac{3\alpha}{2}-1\right)\frac{1}{\mu^2}+\frac{\alpha\lambda}{\mu^3}-\frac{2\alpha\lambda}{\mu^3}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)\right.$$
$$\left.+\frac{\alpha^2\lambda^2}{\mu^4}\left[\frac{K''_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}-\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)^2\right]\right],$$

$$\hat{u}_{R.\lambda}=-q\,H_\alpha^{-q-1}\left[\frac{1}{1-\alpha}\left[\frac{\alpha}{2\lambda}+\frac{\alpha}{\mu}-\frac{\alpha}{\mu}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)\right]\right],$$

$$\hat{u}_{R.\lambda\lambda}=\frac{(q^2+q)\,H_\alpha^{-q-2}}{[1-\alpha]^2}\left[\frac{\alpha}{2\lambda}+\frac{\alpha}{\mu}-\frac{\alpha}{\mu}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)\right]^2$$
$$-\frac{q\,H_\alpha^{-q-1}}{1-\alpha}\left[-\frac{\alpha}{2\lambda^2}-\left(\frac{\alpha}{\mu}\right)^2\left[\frac{K''_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}-\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)^2\right]\right],$$

$$\hat{u}_{R.\mu\lambda}=\hat{u}_{R.\lambda\mu}=\frac{(q^2+q)\,H_\alpha^{-q-2}}{(1-\alpha)^2}\left[\left(1-\frac{3\alpha}{2}\right)\frac{1}{\mu}-\frac{\alpha\lambda}{\mu^2}-\frac{\alpha\lambda}{\mu^2}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)\right]$$
$$\left[\frac{\alpha}{2\lambda}+\frac{\alpha}{\mu}-\frac{\alpha}{\mu}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)\right]-\frac{q\,H_\alpha^{-q-1}}{1-\alpha}\left[-\frac{\alpha}{\mu^2}\right.$$
$$\left.-\frac{\alpha}{\mu^2}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)-\frac{\alpha^2\lambda}{\mu^3}\left[\frac{K''_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}-\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)^2\right]\right].$$

▲ Bayesian estimators of the Shannon entropy under the LINEX loss function. In that case, $u(\mu,\lambda)=e^{-H\,p}$, thus;

$$\hat{u}_{S.\mu}=-pe^{-pH}\left[\frac{3\lambda}{\mu^2}\,e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)\right],\qquad \hat{u}_{S.\lambda}=-pe^{-pH}\left[\frac{1}{\lambda}-\frac{3}{\mu}\,e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)\right],$$

$$\hat{u}_{S.\mu\mu}=pe^{-pH}\left[p\left\{\frac{3\lambda}{\mu^2}\,e^{\frac{2\lambda}{\mu}}\,E_1\left(\frac{2\lambda}{\mu}\right)\right\}^2-\frac{3}{2\mu^2}\left\{\frac{2\lambda}{\mu}-\left[\frac{4\lambda}{\mu}+\left(\frac{2\lambda}{\mu}\right)^2\right]e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)\right\}\right],$$

$$\hat{u}_{S.\lambda\lambda}=pe^{-pH}\left[\frac{p}{\lambda^2}\left\{1-\frac{3\lambda}{\mu}e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)\right\}^2-\frac{1}{\lambda^2}\left\{-1+\frac{3\lambda}{\mu}-6\left(\frac{\lambda}{\mu}\right)^2e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)\right\}\right],$$

$$\hat{u}_{S.\mu\lambda}=pe^{-pH}\left[\frac{3\,p\,\lambda}{\mu^2}e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)\left\{1-\frac{3\lambda}{\mu}e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)\right\}\right.$$
$$\left.-\frac{3}{2\lambda\mu}\left\{\left(\frac{2\lambda}{\mu}+\left(\frac{2\lambda}{\mu}\right)^2\right)e^{\frac{2\lambda}{\mu}}E_1\left(\frac{2\lambda}{\mu}\right)-\frac{2\lambda}{\mu}\right\}\right].$$

▲ Bayesian estimators of the Rényi entropy under the LINEX loss function. In that case, $u(\mu,\lambda)=e^{-(H_\alpha)\,p}$, thus;

$$\hat{u}_{R.\mu}=-p\,e^{-p\,H_\alpha}\left[\frac{1}{1-\alpha}\left[\left(1-\frac{3\alpha}{2}\right)\frac{1}{\mu}-\frac{\alpha\lambda}{\mu^2}-\frac{\alpha\lambda}{\mu^2}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)\right]\right],$$

$$\hat{u}_{R.\mu\mu}=\frac{p^2\,e^{-p\,H_\alpha}}{[\,1-\alpha\,]^2}\left[\left(1-\frac{3\alpha}{2}\right)\frac{1}{\mu}-\frac{\alpha\lambda}{\mu^2}-\frac{\alpha\lambda}{\mu^2}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)\right]^2-\frac{p\,e^{-p\,H_\alpha}}{1-\alpha}\left[\left(\frac{3\alpha}{2}-1\right)\frac{1}{\mu^2}+\frac{\alpha\lambda}{\mu^3}\right.$$
$$\left.-\frac{2\alpha\lambda}{\mu^3}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)+\frac{\alpha^2\lambda^2}{\mu^4}\left[\frac{K''_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}-\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)^2\right]\right],$$

$$\hat{u}_{R.\lambda}=-p\,e^{-p\,H_\alpha}\left[\frac{1}{1-\alpha}\left[\frac{\alpha}{2\lambda}+\frac{\alpha}{\mu}-\frac{\alpha}{\mu}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)\right]\right],$$

$$\hat{u}_{R.\lambda\lambda}=\frac{p^2\,e^{-p\,H_\alpha}}{[\,1-\alpha\,]^2}\left[\frac{\alpha}{2\lambda}+\frac{\alpha}{\mu}-\frac{\alpha}{\mu}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)\right]^2$$
$$-\frac{p\,e^{-p\,H_\alpha}}{1-\alpha}\left[-\frac{\alpha}{2\lambda^2}-\left(\frac{\alpha}{\mu}\right)^2\left[\frac{K''_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}-\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)^2\right]\right],$$

$$\hat{u}_{R.\mu\lambda}=\hat{u}_{R.\lambda\mu}=\frac{p^2\,e^{-p\,H_\alpha}}{(1-\alpha)^2}\left[\left(1-\frac{3\alpha}{2}\right)\frac{1}{\mu}-\frac{\alpha\lambda}{\mu^2}-\frac{\alpha\lambda}{\mu^2}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)\right]\left[\frac{\alpha}{2\lambda}+\frac{\alpha}{\mu}-\frac{\alpha}{\mu}\,\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right]$$
$$-\frac{p\,e^{-p\,H_\alpha}}{1-\alpha}\left[-\frac{\alpha}{\mu^2}-\frac{\alpha}{\mu^2}\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)-\frac{\alpha^2\lambda}{\mu^3}\left[\frac{K''_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}-\left(\frac{K'_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}{K_{1-\frac{3\alpha}{2}}\left(\frac{\alpha\lambda}{\mu}\right)}\right)^2\right]\right].$$

The derivatives of the maximum likelihood estimators equations. The log-likelihood function is expressed as:

$$\begin{aligned}\ell(\mu,\lambda|x) &\propto \frac{m}{2}\ln\left(\frac{\lambda}{2\pi}\right)+\sum_{i=1}^{m}\ln\left[\frac{1}{x_i}\right]-\sum_{i=1}^{m}\frac{\lambda(x_i-\mu)^2}{2\mu^2x_i}+\sum_{i=1}^{m}R_i\\ &\times \ln\left(1-\left(\Phi\left[\sqrt{\frac{\lambda}{x_i}}\left(\frac{x_i}{\mu}-1\right)\right]+e^{\frac{2\lambda}{\mu}}\,\Phi\left[-\sqrt{\frac{\lambda}{x_i}}\left(\frac{x_i}{\mu}+1\right)\right]\right)\right)\end{aligned}$$

For μ, which is given by:

$$\ell_\mu=-\sum_{i=1}^{m}\frac{\lambda(\mu-x_i)}{\mu^3}+\sum_{i=1}^{m}R_i\frac{\Phi\sqrt{\lambda}\left[\frac{\sqrt{x_i}}{\mu^2}\left(e^{\frac{2\lambda}{\mu}}-1\right)+e^{\frac{2\lambda}{\mu}}\left(\frac{2\lambda\sqrt{x_i}}{\mu^3}-\frac{2\lambda}{\mu^2\sqrt{x_i}}\right)\right]}{\left(1-\left(\Phi\left[\sqrt{\frac{\lambda}{x_i}}\left(\frac{x_i}{\mu}-1\right)\right]+e^{\frac{2\lambda}{\mu}}\,\Phi\left[-\sqrt{\frac{\lambda}{x_i}}\left(\frac{x_i}{\mu}+1\right)\right]\right)\right)} \tag{1}$$

$$B_i=\tfrac{\lambda(x_i-\mu)^2}{2\mu^2x_i} \qquad B_{i,\mu\mu\lambda}=3\mu^{-4}x_i-2\mu^{-3}$$
$$B_{i,\mu}=\tfrac{\lambda(\mu-x_i)}{\mu^3} \qquad B_{i,\mu\lambda}=-\mu^{-3}x_i+\mu^{-2}$$
$$B_{i,\mu\mu}=3\mu^{-4}\lambda x_i-2\mu^{-3}\lambda \qquad B_{i,\mu\lambda\lambda}=0$$
$$B_{i,\mu\mu\mu}=-12\mu^{-5}\lambda x_i+6\mu^{-4}\lambda \qquad B_{i,\mu\lambda\mu}=3\mu^{-4}x_i-2\mu^{-3}$$

$$A_i=1-\left(\Phi\left[\sqrt{\frac{\lambda}{x_i}}\left(\frac{x_i}{\mu}-1\right)\right]+e^{\frac{2\lambda}{\mu}}\,\Phi\left[-\sqrt{\frac{\lambda}{x_i}}\left(\frac{x_i}{\mu}+1\right)\right]\right)$$

$$A_{i,\mu} = \Phi\sqrt{\lambda}\left[\frac{\sqrt{x_i}}{\mu^2}\left(e^{\frac{2\lambda}{\mu}}-1\right)+e^{\frac{2\lambda}{\mu}}\left(\frac{2\lambda\sqrt{x_i}}{\mu^3}-\frac{2\lambda}{\mu^2\sqrt{x_i}}\right)\right]$$

$$= \frac{\Phi\sqrt{\lambda}\,\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^2} + \frac{2\Phi\lambda^{3/2}\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^3} - \frac{2\Phi\lambda^{3/2}\,e^{\frac{2\lambda}{\mu}}}{\mu^2\sqrt{x_i}} - \frac{\Phi\sqrt{\lambda}\,\sqrt{x_i}}{\mu^2}$$

$$A_{i,\mu\mu} = -\frac{4\Phi\,\lambda^{5/2}\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^5} - \frac{8\Phi\,\lambda^{3/2}\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^4} - \frac{2\Phi\,\sqrt{\lambda}\,\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^3} + \frac{4\Phi\,\lambda^{5/2}\,e^{\frac{2\lambda}{\mu}}}{\sqrt{x_i}\,\mu^4} + \frac{4\Phi\,\lambda^{3/2}\,e^{\frac{2\lambda}{\mu}}}{\sqrt{x_i}\,\mu^3} + \frac{2\Phi\,\sqrt{\lambda}\,\sqrt{x_i}}{\mu^3}$$

$$A_{i,\mu\mu\mu} = \frac{4\Phi\,\lambda^{3/2}\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^5} + \frac{6\Phi\,\sqrt{\lambda}\,\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^4} - \frac{6\Phi\,\sqrt{\lambda}\,\sqrt{x_i}}{\mu^4} - \frac{12\Phi\,\lambda^{3/2}\,e^{\frac{2\lambda}{\mu}}}{\sqrt{x_i}\,\mu^4} + \frac{32\Phi\,\lambda^{3/2}\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^5} - \frac{24\Phi\,\lambda^{5/2}\,e^{\frac{2\lambda}{\mu}}}{\sqrt{x_i}\,\mu^5} + \frac{36\Phi\,\lambda^{5/2}\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^6} - \frac{8\Phi\,\lambda^{7/2}\,e^{\frac{2\lambda}{\mu}}}{\sqrt{x_i}\,\mu^6} + \frac{8\Phi\,\lambda^{7/2}\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^7}$$

$$A_{i,\mu\mu\lambda} = -\frac{10\Phi\,\sqrt{x_i}\,\lambda^{3/2}\,e^{\frac{2\lambda}{\mu}}}{\mu^5} - \frac{8\Phi\,\sqrt{x_i}\,\lambda^{5/2}\,e^{\frac{2\lambda}{\mu}}}{\mu^6} - \frac{12\Phi\,\sqrt{x_i}\,\lambda^{1/2}\,e^{\frac{2\lambda}{\mu}}}{\mu^4} - \frac{16\Phi\,\sqrt{x_i}\,\lambda^{3/2}\,e^{\frac{2\lambda}{\mu}}}{\mu^5} - \frac{\Phi\,\sqrt{x_i}\,\lambda^{-1/2}\,e^{\frac{2\lambda}{\mu}}}{\mu^3} - \frac{4\Phi\,\sqrt{x_i}\,\lambda^{1/2}\,e^{\frac{2\lambda}{\mu}}}{\mu^4} + \frac{10\Phi\,\lambda^{3/2}\,e^{\frac{2\lambda}{\mu}}}{\sqrt{x_i}\,\mu^4} + \frac{8\Phi\,\lambda^{5/2}\,e^{\frac{2\lambda}{\mu}}}{\sqrt{x_i}\,\mu^5} + \frac{6\Phi\,\lambda^{1/2}\,e^{\frac{2\lambda}{\mu}}}{\sqrt{x_i}\,\mu^3} + \frac{8\Phi\,\lambda^{3/2}\,e^{\frac{2\lambda}{\mu}}}{\sqrt{x_i}\,\mu^4} + \frac{\Phi\,\sqrt{x_i}\,\lambda^{-1/2}}{\mu^3}$$

$$A_{i,\mu\lambda} = \frac{\Phi\,\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{2\,\mu^2\sqrt{\lambda}} + \frac{5\Phi\,\sqrt{x_i}\,\sqrt{\lambda}\,e^{\frac{2\lambda}{\mu}}}{\mu^3} + \frac{4\Phi\,\lambda^{3/2}\,\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^4} - \frac{3\Phi\,\sqrt{\lambda}\,e^{\frac{2\lambda}{\mu}}}{\mu^2\sqrt{x_i}} - \frac{4\Phi\,\lambda^{3/2}\,e^{\frac{2\lambda}{\mu}}}{\mu^3\sqrt{x_i}} - \frac{\Phi\,\sqrt{x_i}}{2\,\mu^2\sqrt{\lambda}}$$

$$A_{i,\mu\lambda\lambda} = \frac{16\Phi\,\lambda^{5/2}e^{\frac{2\lambda}{\mu}}}{\mu^6\sqrt{x_i}} - \frac{16\Phi\,\lambda^{5/2}\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^7} + \frac{56\Phi\,\lambda^{3/2}e^{\frac{2\lambda}{\mu}}}{\mu^5\sqrt{x_i}} - \frac{72\Phi\,\lambda^{3/2}\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^6} + \frac{39\Phi\,\sqrt{\lambda}\,e^{\frac{2\lambda}{\mu}}}{\mu^4\sqrt{x_i}} - \frac{71\Phi\,\sqrt{\lambda}\,\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^5} + \frac{3\Phi\,e^{\frac{2\lambda}{\mu}}}{\sqrt{\lambda}\,\mu^3\sqrt{x_i}} - \frac{10\Phi\,\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\sqrt{\lambda}\,\mu^4} + \frac{\Phi\,\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{2\lambda^{3/2}\,\mu^3} - \frac{\Phi\,\sqrt{x_i}}{2\lambda^{3/2}\,\mu^3}$$

$$A_{i,\mu\lambda\mu} = \frac{8\Phi\,\lambda^{5/2}\,e^{\frac{2\lambda}{\mu}}}{\mu^5\sqrt{x_i}} - \frac{8\Phi\,\lambda^{5/2}\,\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^6} + \frac{18\Phi\,\lambda^{3/2}\,e^{\frac{2\lambda}{\mu}}}{\mu^4\sqrt{x_i}} - \frac{26\Phi\,\lambda^{3/2}\,\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^5} + \frac{6\Phi\,\sqrt{\lambda}\,e^{\frac{2\lambda}{\mu}}}{\mu^3\sqrt{x_i}} - \frac{16\Phi\,\sqrt{\lambda}\,\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\mu^4} - \frac{\Phi\,\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{\sqrt{\lambda}\,\mu^3} + \frac{\Phi\,\sqrt{x_i}}{\sqrt{\lambda}\,\mu^3}$$

For λ, which is given by:

$$\ell_\lambda = \frac{m}{2\lambda} - \sum_{i=1}^{m}\frac{(x_i-\mu)^2}{2\mu^2 x_i} + \sum_{i=1}^{m} R_i \times \left(\frac{\Phi\left(\frac{1}{2\sqrt{\lambda}}\left(\frac{\sqrt{x_i}}{\mu}-\frac{1}{\sqrt{x_i}}\right)+e^{\frac{2\lambda}{\mu}}\left(\frac{\sqrt{x_i}}{\mu}\left(\frac{-2\sqrt{\lambda}}{\mu}-\frac{1}{2\sqrt{\lambda}}\right)+\frac{1}{\sqrt{x_i}}\left(\frac{2\sqrt{\lambda}}{\mu}+\frac{1}{2\sqrt{\lambda}}\right)\right)\right)}{\left(1-\left(\Phi\left[\sqrt{\frac{\lambda}{x_i}}\left(\frac{x_i}{\mu}-1\right)\right]+e^{\frac{2\lambda}{\mu}}\,\Phi\left[-\sqrt{\frac{\lambda}{x_i}}\left(\frac{x_i}{\mu}+1\right)\right]\right)\right)}\right) \quad (2)$$

$$C = \tfrac{m}{2}\ln\left(\tfrac{\lambda}{2\pi}\right)$$

$$C_\lambda = \tfrac{m}{2\lambda} \qquad B_{i,\lambda} = \tfrac{(x_i-\mu)^2}{2\mu^2 x_i}$$

$$C_{\lambda\lambda} = -\tfrac{m}{2\lambda^2} \qquad B_{i,\lambda\lambda} = B_{i,\lambda\lambda\lambda} = B_{i,\lambda\lambda\mu} = B_{i,\lambda\mu\lambda} = 0$$

$$C_{\lambda\lambda\lambda} = \tfrac{m}{\lambda^3} \qquad B_{i,\lambda\mu} = -x_i\mu^{-3} + \mu^{-2}$$

$$C_{\lambda\mu} = C_{\lambda\lambda\mu} = C_{\lambda\mu\mu} = C_{\lambda\mu\lambda} = 0 \qquad B_{i,\lambda\mu\mu} = 3x_i\mu^{-4} - 2\mu^{-3}$$

$$\begin{aligned} A_{i,\lambda} &= \Phi\left(\frac{1}{2\sqrt{\lambda}}\left[\frac{\sqrt{x_i}}{\mu} - \frac{1}{\sqrt{x_i}}\right] + e^{\frac{2\lambda}{\mu}}\left[\frac{\sqrt{x_i}}{\mu}\left(\frac{-2\sqrt{\lambda}}{\mu} - \frac{1}{2\sqrt{\lambda}}\right) + \frac{1}{\sqrt{x_i}}\left(\frac{2\sqrt{\lambda}}{\mu} + \frac{1}{2\sqrt{\lambda}}\right)\right]\right) \\ &= \frac{\Phi\sqrt{x_i}}{2\mu\sqrt{\lambda}} - \frac{\Phi}{2\sqrt{x_i}\sqrt{\lambda}} - \frac{2\Phi\sqrt{x_i}\sqrt{\lambda}\,e^{\frac{2\lambda}{\mu}}}{\mu^2} - \frac{\Phi\sqrt{x_i}\,e^{\frac{2\lambda}{\mu}}}{2\mu\sqrt{\lambda}} + \frac{2\Phi\sqrt{\lambda}\,e^{\frac{2\lambda}{\mu}}}{\mu\sqrt{x_i}} + \frac{\Phi\,e^{\frac{2\lambda}{\mu}}}{2\sqrt{\lambda}\sqrt{x_i}} \end{aligned}$$

$$\begin{aligned} A_{i,\lambda\lambda} &= -\frac{1}{4}\lambda^{-\frac{3}{2}}\Phi\,x_i^{\frac{1}{2}}\mu^{-1} + \frac{1}{4}\lambda^{-\frac{3}{2}}\Phi\,x_i^{-\frac{1}{2}} - \lambda^{-\frac{1}{2}}\Phi\,x_i^{\frac{1}{2}}\mu^{-2}e^{\frac{2\lambda}{\mu}} - 4\lambda^{\frac{1}{2}}\Phi\,x_i^{\frac{1}{2}}\mu^{-3}e^{\frac{2\lambda}{\mu}} \\ &- \frac{1}{4}\lambda^{-\frac{3}{2}}\Phi\,x_i\mu^{-1}e^{\frac{2\lambda}{\mu}} - \lambda^{-\frac{1}{2}}\Phi\,x_i\mu^{-2}e^{\frac{2\lambda}{\mu}} + 4\lambda^{\frac{1}{2}}\Phi\,x_i\mu^{-2}e^{\frac{2\lambda}{\mu}} + 2\lambda^{-\frac{1}{2}}\Phi\,x_i^{-\frac{1}{2}}\mu^{-1}e^{\frac{2\lambda}{\mu}} \\ &- \frac{1}{4}\lambda^{-\frac{3}{2}}\Phi\,x_i^{-\frac{1}{2}}e^{\frac{2\lambda}{\mu}} \end{aligned}$$

$$\begin{aligned} A_{i,\lambda\lambda\lambda} &= -\frac{3}{8}\lambda^{-\frac{5}{2}}\Phi\,x_i^{-\frac{1}{2}} - \Phi\,x_i^{\frac{1}{2}}\mu^{-2}\left(-\frac{1}{2}\lambda^{-\frac{3}{2}}e^{\frac{2\lambda}{\mu}} + \frac{2\lambda^{-\frac{1}{2}}}{\mu}e^{\frac{2\lambda}{\mu}}\right) - 4\Phi\,x_i^{\frac{1}{2}}\mu^{-3}\left(-\frac{1}{2}\lambda^{-\frac{1}{2}}e^{\frac{2\lambda}{\mu}} + \frac{2\lambda^{\frac{1}{2}}}{\mu}e^{\frac{2\lambda}{\mu}}\right) \\ &- \frac{1}{4}\Phi\,x_i^{\frac{1}{2}}\mu^{-1}\left(-\frac{3}{2}\lambda^{-\frac{5}{2}}e^{\frac{2\lambda}{\mu}} + \frac{2\lambda^{-\frac{3}{2}}}{\mu}e^{\frac{2\lambda}{\mu}}\right) - \Phi\,x_i\mu^{-2}\left(-\frac{1}{2}\lambda^{-\frac{3}{2}}e^{\frac{2\lambda}{\mu}} + \frac{2\lambda^{-\frac{1}{2}}}{\mu}e^{\frac{2\lambda}{\mu}}\right) \\ &+ 4\Phi\,x_i\mu^{-2}\left(\frac{1}{2}\lambda^{-\frac{1}{2}}e^{\frac{2\lambda}{\mu}} + \frac{2\lambda^{\frac{1}{2}}}{\mu}e^{\frac{2\lambda}{\mu}}\right) + 2\Phi\,x_i^{-\frac{1}{2}}\mu^{-1}\left(-\frac{1}{2}\lambda^{-\frac{3}{2}}e^{\frac{2\lambda}{\mu}} + \frac{2\lambda^{-\frac{3}{2}}}{\mu}e^{\frac{2\lambda}{\mu}}\right) \\ &- \frac{1}{4}\Phi\,x_i^{-\frac{1}{2}}\left(-\frac{3}{2}\lambda^{-\frac{5}{2}}e^{\frac{2\lambda}{\mu}} + \frac{2\lambda^{-\frac{3}{2}}}{\mu}e^{\frac{2\lambda}{\mu}}\right) \end{aligned}$$

$$\begin{aligned} A_{i,\lambda\lambda\mu} &= \frac{1}{4}\lambda^{-\frac{3}{2}}\,\Phi\,x_i^{\frac{1}{2}}\mu^{-2} - \lambda^{-\frac{1}{2}}\Phi\,x_i^{\frac{1}{2}}\left(-2\mu^{-3}e^{\frac{2\lambda}{\mu}} - 2\lambda\mu^{-4}e^{\frac{2\lambda}{\mu}}\right) \\ &- 4\lambda^{\frac{1}{2}}\Phi\,x_i^{\frac{1}{2}}\left(-3\mu^{-4}e^{\frac{2\lambda}{\mu}} - 2\lambda\mu^{-5}e^{\frac{2\lambda}{\mu}}\right) - \frac{1}{4}\lambda^{-\frac{3}{2}}\Phi\,x_i\left(\mu^{-2}e^{\frac{2\lambda}{\mu}} - 2\lambda\mu^{-3}\right) \\ &- \lambda^{-\frac{1}{2}}\Phi\,x_i\left(-2\mu^{-3}e^{\frac{2\lambda}{\mu}} - 2\lambda\mu^{-4}e^{\frac{2\lambda}{\mu}}\right) + 4\lambda^{\frac{1}{2}}\Phi\,x_i\left(-2\mu^{-3}e^{\frac{2\lambda}{\mu}} - 2\lambda\mu^{-4}e^{\frac{2\lambda}{\mu}}\right) \\ &+ 2\lambda^{-\frac{1}{2}}\Phi\,x_i\left(\mu^{-2}e^{\frac{2\lambda}{\mu}} - 2\lambda\mu^{-3}e^{\frac{2\lambda}{\mu}}\right) + \frac{1}{2}\lambda^{-\frac{3}{2}}\Phi\,x^{-\frac{1}{2}}\mu^{-2}e^{\frac{2\lambda}{\mu}} \end{aligned}$$

$$\begin{aligned} A_{i,\lambda\mu} &= \frac{-1}{2}\lambda^{-\frac{1}{2}}\Phi\,x_i^{\frac{1}{2}}\mu^{-2} + 4\Phi\,x_i^{\frac{1}{2}}\lambda^{\frac{3}{2}}\mu^{-4}e^{\frac{2\lambda}{\mu}} + 4\Phi\,x_i^{\frac{1}{2}}\lambda^{\frac{1}{2}}\mu^{-3}e^{\frac{2\lambda}{\mu}} + \Phi\,x_i\lambda^{\frac{1}{2}}\mu^{-3}e^{\frac{2\lambda}{\mu}} \\ &+ \frac{1}{2}\Phi\,x_i\lambda^{-\frac{1}{2}}\mu^{-2}e^{\frac{2\lambda}{\mu}} - 4\Phi\,x_i^{-\frac{1}{2}}\lambda^{\frac{3}{2}}\mu^{-3}e^{\frac{2\lambda}{\mu}} - 3\Phi\,x_i^{-\frac{1}{2}}\lambda^{\frac{1}{2}}\mu^{-2}e^{\frac{2\lambda}{\mu}} \end{aligned}$$

$$\begin{aligned} A_{i,\lambda\mu\mu} &= \mu^{-3}\lambda^{-\frac{1}{2}}\Phi\,x_i^{\frac{1}{2}} + 4\Phi\,x_i^{\frac{1}{2}}\lambda^{\frac{3}{2}}\left(-4\mu^{-5}e^{\frac{2\lambda}{\mu}} - 2\lambda\mu^{-6}e^{\frac{2\lambda}{\mu}}\right) + 4\Phi\,x_i^{\frac{1}{2}}\lambda^{\frac{1}{2}}\left(-3\mu^{-4}e^{\frac{2\lambda}{\mu}} - 2\lambda\mu^{-5}e^{\frac{2\lambda}{\mu}}\right) \\ &+ \Phi\,x_i\lambda^{\frac{1}{2}}\left(-3\mu^{-4}e^{\frac{2\lambda}{\mu}} - 2\lambda\mu^{-5}e^{-\frac{2\lambda}{\mu}}\right) + \frac{1}{2}\Phi\,x_i\lambda^{-\frac{1}{2}}\left(-2\mu^{-3}e^{\frac{2\lambda}{\mu}} - 2\lambda\mu^{-4}e^{\frac{2\lambda}{\mu}}\right) \\ &- 4\Phi\,x_i^{-\frac{1}{2}}\lambda^{\frac{3}{2}}\left(-3\mu^{-4}e^{\frac{2\lambda}{\mu}} - 2\lambda\mu^{-5}e^{\frac{2\lambda}{\mu}}\right) - 3\Phi\,x_i^{-\frac{1}{2}}\lambda^{\frac{1}{2}}\left(-2\mu^{-3}e^{\frac{2\lambda}{\mu}} - 2\lambda\mu^{-4}e^{\frac{2\lambda}{\mu}}\right) \end{aligned}$$

$$A_{i,\lambda\mu\lambda} = \frac{1}{4}\lambda^{-\frac{3}{2}}\Phi\,x_i^{\frac{1}{2}}\mu^{-2} + 4\Phi\,x_i^{\frac{1}{2}}\mu^{-4}\left(\frac{3}{2}\lambda^{\frac{1}{2}}e^{\frac{2\lambda}{\mu}} + \frac{2\lambda^{\frac{3}{2}}}{\mu}e^{\frac{2\lambda}{\mu}}\right)$$

$$
\begin{aligned}
&+ \quad 4\Phi\, x_i^{\frac{1}{2}}\mu^{-3}\left(\frac{1}{2}\lambda^{-\frac{1}{2}}e^{\frac{2\lambda}{\mu}} + \frac{2\lambda^{\frac{1}{2}}}{\mu}e^{\frac{2\lambda}{\mu}}\right) + \Phi\, x_i\mu^{-3}\left(\frac{1}{2}\lambda^{-\frac{1}{2}}e^{\frac{2\lambda}{\mu}} + \frac{2\lambda^{\frac{1}{2}}}{\mu}e^{\frac{2\lambda}{\mu}}\right)\\
&+ \quad \frac{1}{2}\Phi\, x_i\mu^{-2}\left(\frac{-1}{2}\lambda^{-\frac{3}{2}}e^{\frac{2\lambda}{\mu}} + \frac{2\lambda^{-\frac{1}{2}}}{\mu}e^{\frac{2\lambda}{\mu}}\right) - 4\Phi\, x_i^{-\frac{1}{2}}\mu^{-3}\left(\frac{3}{2}\lambda^{\frac{1}{2}}e^{\frac{2\lambda}{\mu}} + \frac{2\lambda^{\frac{3}{2}}}{\mu}e^{\frac{2\lambda}{\mu}}\right)\\
&- \quad 3\Phi\, x_i^{-\frac{1}{2}}\mu^{-2}\left(\frac{1}{2}\lambda^{-\frac{1}{2}}e^{\frac{2\lambda}{\mu}} + \frac{2\lambda^{\frac{1}{2}}}{\mu}e^{\frac{2\lambda}{\mu}}\right)
\end{aligned}
$$

Based on the above,

$$
\begin{aligned}
\ell_{\mu} &= \textstyle\sum_{i=1}^m B_{i,\mu} + \sum_{i=1}^m R_i \frac{A_{i,\mu}}{A_i}\\
\ell_{\mu\mu} &= \textstyle\sum_{i=1}^m B_{i,\mu\mu} + \sum_{i=1}^m R_i \left(\frac{A_{i,\mu\mu}A_i - A_{i,\mu}^2}{A_i^2}\right)\\
\ell_{\mu\mu\mu} &= \textstyle\sum_{i=1}^m B_{i,\mu\mu\mu} + \sum_{i=1}^m R_i \left[\frac{A_{i\mu\mu\mu}A_i^2 - 3A_{i,\mu}A_{i,\mu\mu}A_i + 2A_{i\mu}^3}{A_i^3}\right]\\
\ell_{\mu\mu\lambda} &= \textstyle\sum_{i=1}^m B_{i,\mu\mu\lambda} + \sum_{i=1}^m R_i \left[\frac{A_{i,\mu\mu\lambda}A_i^2 - A_iA_{i,\mu\mu}A_{i,\lambda} - 2A_iA_{i,\mu}A_{i,\mu\lambda} + 2A_{i,\mu}^2A_{i,\lambda}}{A_i^3}\right]\\
\ell_{\mu\lambda} &= \textstyle\sum_{i=1}^m B_{i,\mu\lambda} + \sum_{i=1}^m R_i \left(\frac{A_{i,\mu\lambda}A_i - A_{i,\mu}A_{i,\lambda}}{A_i^2}\right)\\
\ell_{\mu\lambda\lambda} &= \textstyle\sum_{i=1}^m B_{i,\mu\lambda\lambda} + \sum_{i=1}^m R_i \left[\frac{A_{i,\mu\lambda\lambda}A_i^2 - 2A_iA_{i,\mu\lambda}A_{i,\lambda} - A_iA_{i,\mu}A_{i,\lambda\lambda} + 2A_{i,\mu}A_{i,\lambda}^2}{A_i^3}\right]\\
\ell_{\mu\lambda\mu} &= \textstyle\sum_{i=1}^m B_{i,\mu\lambda\mu} + \sum_{i=1}^m R_i \left[\frac{A_{i,\mu\lambda\mu}A_i^2 - 2A_iA_{i,\mu}A_{i,\mu\lambda} - A_iA_{i,\mu\mu}A_{i,\lambda} + 2A_{i,\mu}^2A_{i,\lambda}}{A_i^3}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\ell_{\lambda} &= C_{\lambda} - \textstyle\sum_{i=1}^m B_{i,\lambda} + \sum_{i=1}^m R_i \frac{A_{i,\lambda}}{A_i}\\
\ell_{\lambda\lambda} &= C_{\lambda\lambda} - \textstyle\sum_{i=1}^m B_{i,\lambda\lambda} + \sum_{i=1}^m R_i \left(\frac{A_{i,\lambda\lambda}A_i - A_{i,\lambda}^2}{A_i^2}\right)\\
\ell_{\lambda\lambda\lambda} &= C_{\lambda\lambda\lambda} - \textstyle\sum_{i=1}^m B_{i,\lambda\lambda\lambda} + \sum_{i=1}^m R_i \left[\frac{A_{i,\lambda\lambda\lambda}A_i^2 - 3A_{i,\lambda}A_{i,\lambda\lambda}A_i + 2A_{i,\lambda}^3}{A_i^3}\right]\\
\ell_{\lambda\lambda\mu} &= C_{\lambda\lambda\mu} - \textstyle\sum_{i=1}^m B_{i,\lambda\lambda\mu} + \sum_{i=1}^m R_i \left[\frac{A_{i,\lambda\lambda\mu}A_i^2 - A_{i,\lambda\lambda}A_{i,\mu}A_i - 2A_{i,\lambda}A_{i,\lambda\mu}A_i + 2A_{i,\mu}A_{i,\lambda}^2}{A_i^3}\right]\\
\ell_{\lambda\mu} &= C_{\lambda\mu} - \textstyle\sum_{i=1}^m B_{i,\lambda\mu} + \sum_{i=1}^m R_i \left(\frac{A_{i,\lambda\mu}A_i - A_{i,\lambda}A_{i,\mu}}{A_i^2}\right)\\
\ell_{\lambda\mu\mu} &= C_{\lambda\mu\mu} - \textstyle\sum_{i=1}^m B_{i,\lambda\mu\mu} + \sum_{i=1}^m R_i \left[\frac{A_{i,\lambda\mu\mu}A_i^2 - A_{i,\lambda}A_{i,\mu\mu}A_i - 2A_{i,\lambda\mu}A_{i,\mu}A_i + 2A_{i,\lambda}A_{i,\mu}^2}{A_i^3}\right]\\
\ell_{\lambda\mu\lambda} &= C_{\lambda\mu\lambda} - \textstyle\sum_{i=1}^m B_{i,\lambda\mu\lambda} + \sum_{i=1}^m R_i \left[\frac{A_{i,\lambda\lambda\mu}A_i^2 - A_{i,\lambda\lambda}A_{i,\mu}A_i - 2A_{i,\lambda\mu}A_{i,\lambda}A_i + 2A_{i,\lambda}^2A_{i,\mu}}{A_i^3}\right]
\end{aligned}
$$

Then

$$
\begin{aligned}
\hat{\ell}_{\mu\mu\mu} &= \sum_{i=1}^m B_{i,\mu\mu\mu} + \sum_{i=1}^m R_i \left[\frac{A_{i,\mu\mu\mu}A_i^2 - 3A_{i,\mu}A_{i,\mu\mu}A_i + 2A_{i,\mu}^3}{A_i^3}\right]\\
\hat{\ell}_{\mu\mu\lambda} &= \sum_{i=1}^m B_{i,\mu\mu\lambda} + \sum_{i=1}^m R_i \left[\frac{A_{i,\mu\mu\lambda}A_i^2 - A_iA_{i,\mu\mu}A_{i,\lambda} - 2A_iA_{i,\mu}A_{i,\mu\lambda} + 2A_{i,\mu}^2A_{i,\lambda}}{A_i^3}\right]\\
\hat{\ell}_{\mu\lambda\mu} &= \sum_{i=1}^m B_{i,\mu\lambda\mu} + \sum_{i=1}^m R_i \left[\frac{A_{i,\mu\lambda\mu}A_i^2 - 2A_iA_{i,\mu}A_{i,\mu\lambda} - A_iA_{i,\mu\mu}A_{i,\lambda} + 2A_{i,\mu}^2A_{i,\lambda}}{A_i^3}\right]\\
\hat{\ell}_{\mu\lambda\lambda} &= \sum_{i=1}^m R_i \left[\frac{A_{i,\mu\lambda\lambda}A_i^2 - 2A_iA_{i,\mu\lambda}A_{i,\lambda} - A_iA_{i,\mu}A_{i,\lambda\lambda} + 2A_{i,\mu}A_{i,\lambda}^2}{A_i^3}\right]\\
\hat{\ell}_{\lambda\lambda\lambda} &= C_{\lambda\lambda\lambda} + \sum_{i=1}^m R_i \left[\frac{A_{i,\lambda\lambda\lambda}A_i^2 - 3A_{i,\lambda}A_{i,\lambda\lambda}A_i + 2A_{i,\lambda}^3}{A_i^3}\right]\\
\hat{\ell}_{\lambda\lambda\mu} &= \sum_{i=1}^m R_i \left[\frac{A_{i,\lambda\lambda\mu}A_i^2 - A_{i,\lambda\lambda}A_{i,\mu}A_i - 2A_{i,\lambda\mu}A_{i,\lambda}A_i + 2A_{i,\lambda}^2A_{i,\mu}}{A_i^3}\right]\\
\hat{\ell}_{\lambda\mu\lambda} &= \sum_{i=1}^m R_i \left[\frac{A_{i,\lambda\lambda\mu}A_i^2 - A_{i,\lambda\lambda}A_{i,\mu}A_i - 2A_{i,\lambda\mu}A_{i,\lambda}A_i + 2A_{i,\lambda}^2A_{i,\mu}}{A_i^3}\right]\\
\hat{\ell}_{\lambda\mu\mu} &= -\sum_{i=1}^m B_{i,\lambda\mu\mu} + \sum_{i=1}^m R_i \left[\frac{A_{i,\lambda\mu\mu}A_i^2 - A_{i,\lambda}A_{i,\mu\mu}A_i - 2A_{i,\lambda\mu}A_{i,\mu}A_i + 2A_{i,\lambda}A_{i,\mu}^2}{A_i^3}\right]
\end{aligned}
$$