Supplementary Note S2. Numerical assessment of the two-moment mapping for the LLU mixing distribution

Purpose

This note documents the numerical checks used to assess the local identifiability of the method-of-moments mapping $g(\lambda,\theta) = (m_1(\lambda,\theta), m_2(\lambda,\theta))$. The results are numerical diagnostics; they are not presented as a proof of global identifiability.

Moment representation

For $r = 1, 2$, the moments were calculated from the quantile representation of the LLU distribution using the equivalent integral:

$$m_r(\lambda,\theta) = (\lambda/2)\int_0^1 z^{\lambda-1}\left[1+z^{r/\theta}\right]/[1+z]^{r/\theta}dz\,, \qquad r = 1{,}2.$$

The numerical Jacobian was formed with respect to the log-parameters, $J = \partial(m_1, m_2)/\partial(\log\lambda, \log\theta)$, using centered finite differences with step $h = 10^{-5}$. Since λ and θ are positive, changing from the original parameters to their logarithms preserves the sign of the Jacobian determinant.

Grid assessment

We evaluated $\det(J)$ on a 101×101 logarithmically spaced grid over $0.1 \le \lambda \le 5$ and $0.1 \le \theta \le 5$, using 400 −point Gauss–Legendre quadrature for the moment integrals.

Diagnostic	Result
Grid points	10,201
Nonpositive determinants	0
Minimum det(J)	1.58656×10^{-6} at $(\lambda,\theta) = (5, 0.1)$
Maximum det(J)	0.0195398 at $(\lambda,\theta) = (0.8269, 0.7353)$

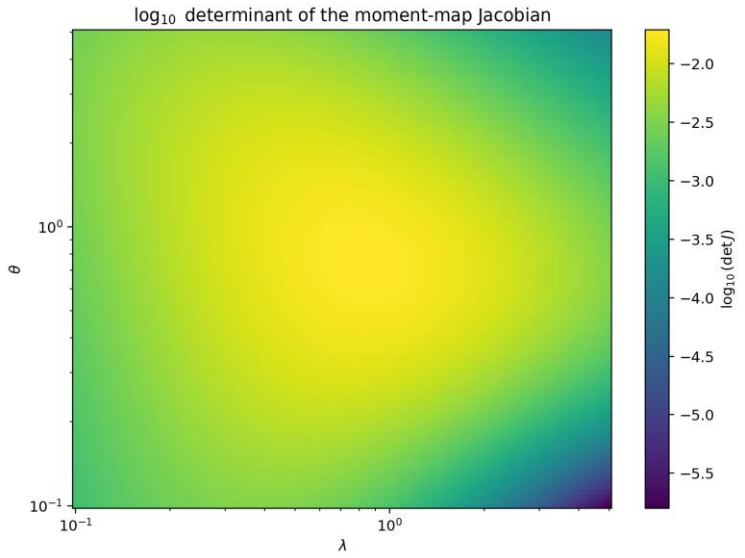


Figure S2.1. Logarithm of the numerically evaluated Jacobian determinant. All 10,201 grid values were positive.

Additional numerical checks

Quadrature sensitivity

The positivity result was unchanged when the quadrature order was varied from 80 to 400 nodes. The table reports the minimum and maximum determinants over the same 101 × 101 grid for each order.

Nodes	Minimum det(J)	Maximum det(J)	Nonpositive points
80	1.50811×10^{-6}	0.01953982	0
100	1.53495×10^{-6}	0.01953982	0
150	1.56461×10^{-6}	0.01953982	0
200	1.57581×10^{-6}	0.01953982	0
400	1.58656×10^{-6}	0.01953982	0

Determinants in the three Monte Carlo scenarios

Scenario	λ	θ	det(J)
I	1.0	0.5	0.0175570
II	0.3	1.0	0.0114899
III	2.0	2.0	0.00498403

Multiple-start inversion check

We constructed 300 deterministic target parameter pairs over the interior region $0.15 \leq \lambda, \theta \leq 4.5$. For each target pair, its two moments were matched by bounded nonlinear least-squares optimization from five distinct starting points: (0.2, 0.2), (0.5, 1), (1, 1), (2, 3), and (4, 4.8), with parameter bounds 0.1 to 5. Across the 300 targets, the maximum best-fit Euclidean norm of the two moment residuals was 5.12×10^{-15}. Among fits with residual norm below 10^{-8}, the largest pairwise distance between log-parameter estimates for a given target was 4.38×10^{-9}.

Interpretation and limitation

The positive finite-difference determinants, their stability across quadrature orders, and the multi-start inversion check provide consistent numerical evidence for local identifiability in the investigated region and no indication of distinct moment-matching parameter pairs in the numerical checks. These calculations do not prove that the moment mapping is globally one-to-one, nor do they replace an analytic identifiability proof.

Reproducibility

The accompanying Python script, reproduce_identifiability.py, implements the numerical procedures described in this note. The CSV files contain the full determinant grid, inversion-start diagnostics, and numerical summaries. The heatmap is also supplied to facilitate independent inspection of the Jacobian determinant over the investigated parameter region.