Supplementary Note S1. Numerical checks of the BLLU probability mass function

The explicit finite-sum representation in Equation (3) was checked against the defining mixture integral in Equation (2). For the integral calculation, the interval was split at $c = 2^{-1/\theta}$, and endpoint powers were handled by adaptive quadrature. For Equation (3), the incomplete-beta representation was evaluated using arbitrary-precision arithmetic (300 decimal digits). The full support was examined for the three parameter configurations used in the Monte Carlo study: Scenario I $(\lambda, \theta) = (1, 0.5)$, Scenario II (0.3, 1), and Scenario III (2, 2), for $n \in \{10, 20, 50, 100, 150\}$.

Table S1.1. Full-support comparison of Equation (3) with direct integration of Equation (2). Entries other than n are maxima across the three Monte Carlo parameter configurations, except for the minimum probability, which is the minimum over all tested support points and configurations.

n	Maximum absolute discrepancy	Maximum $\lvert \Sigma_x p_x^{(2)} - 1 \rvert$	Minimum computed probability
10	1.34×10^{-12}	1.25×10^{-12}	6.76×10^{-4}
20	6.66×10^{-13}	6.22×10^{-13}	5.20×10^{-5}
50	3.33×10^{-13}	3.06×10^{-13}	1.61×10^{-6}
100	1.67×10^{-13}	1.53×10^{-13}	1.09×10^{-7}
150	2.96×10^{-12}	1.34×10^{-12}	2.22×10^{-8}

For all full-support checks above, both the arbitrary-precision values from Equation (3) and the direct-integral values were nonnegative. The PMF from Equation (3) summed to one up to the final conversion to standard floating-point values; the directly integrated PMFs also normalized to one within a maximum absolute error of 1.34×10^{-12}.

An additional shape configuration, $(\lambda, \theta) = (0.3, 0.2)$, was checked for the same values of n by comparing the arbitrary-precision finite-sum formula with the equivalent quantile-integral representation, evaluated using 512-point Gauss–Legendre quadrature on each half of the unit interval. Across the tested values of n, all computed probabilities were nonnegative, the largest absolute discrepancy was below 6.1×10^{-15}, and the absolute normalization error was below 5.6×10^{-16}.

For a large value $n = 2000$ with $(\lambda, \theta) = (2, 2)$, seven support points $x \in \{0, 100, 500, 1000, 1500, 1900, 2000\}$ were checked by comparing Equation (3), evaluated with 1100-digit arithmetic, with direct integration of Equation (2), evaluated at 80-digit precision. The two calculations agreed to relative discrepancies below 10^{-70} at all seven points.

Finally, a diagnostic calculation showed why the alternating sum should not be the default numerical implementation for moderate or large n. Even when the H_r values were first computed at high precision and then rounded to standard double precision, direct evaluation of the alternating sum began to produce negative probabilities at $n = 50$ in all three Monte Carlo configurations, and the resulting probability sums were far from one. This supports using the equivalent integral or quantile-based representation for numerical evaluation, while retaining Equation (3) as an explicit analytical representation.