```
> # NAME OF MAPLE PROGRAM: Gram Schmidt Conjugate Direction Method
  Mar 15 2023.mws  Programmed by Ivie Stein Jr. This method is also
  called CGS -- no derivatives by Magnus R. Hestenes.  No
  derivative evaluations and no line searches are used. THE
  QUOTIENT CONVERGENCE FACTOR FOR THIS PROBLEM IS COMPUTED AT THE
  END OF THIS COMPUTER PROGRAM SHOWING QUADRATIC CONVERGENCE.  When
  the output for the iterations is compared with that of Newton's
  method, the results agree to approximately 150 significant
  figures.  The function for minimization to which CGS is applied
  is Rosenbrock's banana valley function.  The main idea here to
  show how to implement CGS and to see quadratic convergence
  numerically, which is what Hestenes states on p. 202 that CGS has
  Newton's algorithm as its limit.
> #
> # References: CGS [CONJUGATE GRAM-SCHMIDT -- NO DERIVATIVES,
  FUNCTION EVALUATIONS WITH NO LINE SEARCHES] Algorithms for
  Unconstrained Minimization of Functions, R. F. Dennemeyer and E.
  H. Mookini, Communicated by M. R. Hestenes, Journal of
  Optimization Theory and Applications, Vol. 16, Nos. 1/2, 1975,
  pp.67-85.  Particular attention should be called to pp. 75-76,
  the CGS-F [CONJUGATE GRAM-SCHMIDT - FUNCTION EVALUATION ONLY]
  Algorithm.  Also see Conjugate Direction Methods in Optimization
  by Magnus R. Hestenes, Springer-Verlag, New York, 1980, pages 198
  -202.
> restart: Digits:=400;
```

$$Digits := 400 \tag{1}$$

```
> sigma:=.1e-120; rho:=2*sigma; epsilon:= .1e-60;
```

$$\sigma := 1.\ 10^{-121}$$
$$\rho := 2.\ 10^{-121} \tag{2}$$
$$\epsilon := 1.\ 10^{-61}$$

```
> u[1,1]:=1; u[1,2]:=0;u[2,1]:=0; u[2,2]:=1;
```

$$u_{1,1} := 1$$
$$u_{1,2} := 0 \tag{3}$$
$$u_{2,1} := 0$$
$$u_{2,2} := 1$$

```
> p[1,1]:=u[1,1];p[1,2]:=u[1,2];
```

$$p_{1,1} := 1 \tag{4}$$
$$p_{1,2} := 0$$

```
> f:=(x1,x2)->100*(x2-x1^2)^2+(x1-1)^2;
```

(5)

$$f := (x1, x2) \rightarrow 100\left(x2 - x1^2\right)^2 + (x1 - 1)^2 \qquad (5)$$

```
> x[1,1]:=-1.2;x[1,2]:=1;
```

$$x_{1,1} := -1.2 \qquad (6)$$

$$x_{1,2} := 1$$

```
> for j from 1 to 10 do
  c[1]:=(f(x[1,1]-sigma*p[1,1],x[1,2]-sigma*p[1,2])-f(x[1,1]+sigma*
  p[1,1],x[1,2]+sigma*p[1,2]))/(2*sigma):
  d[1]:=(f(x[1,1]-sigma*p[1,1],x[1,2]-sigma*p[1,2])-2*f(x[1,1],x[1,
  2])+f(x[1,1]+sigma*p[1,1],x[1,2]+sigma*p[1,2]))/(sigma)^2:
  a[1]:=c[1]/d[1]:
  x[2,1]:=x[1,1]+a[1]*p[1,1]:
  x[2,2]:=x[1,2]+a[1]*p[1,2]:
  for i from 1 to 2 do  x[2,i]:=x[1,i]+a[1]*p[1,i]  end do:
  c[2,1]:=(f(x[1,1]+rho*u[2,1]-sigma*p[1,1],x[1,2]+rho*u[2,2]-
  sigma*p[1,2])-f(x[1,1]+rho*u[2,1]+sigma*p[1,1],x[1,2]+rho*u[2,2]+
  sigma*p[1,2]))/(2*sigma):
  a[2,1]:=c[2,1]/d[1]:
  b[2,1]:=(a[2,1]-a[1])/rho:
  for i from 1 to 2 do  pbar[2,i]:=u[2,i]+b[2,1]*p[1,i] end do:
  for i from 1 to 2 do  p[2,i]:=pbar[2,i]/sqrt((pbar[2,1])^2+(pbar
  [2,2])^2) end do:
  for k from 2 to 2 do c[k]:=(f(x[1,1]-sigma*p[k,1],x[1,2]-sigma*p
  [k,2])-f(x[1,1]+sigma*p[k,1],x[1,2]+sigma*p[k,2]))/(2*sigma) end
  do:
  for k from 2 to 2 do c[k]:=(f(x[1,1]-sigma*p[k,1],x[1,2]-sigma*p
  [k,2])-f(x[1,1]+sigma*p[k,1],x[1,2]+sigma*p[k,2]))/(2*sigma) end
  do:
  for k from 2 to 2 do  d[k]:=(f(x[1,1]-sigma*p[k,1],x[1,2]-sigma*p
  [k,2])-2*f(x[1,1],x[1,2])+f(x[1,1]+sigma*p[k,1],x[1,2]+sigma*p[k,
  2]))/(sigma)^2 end do:
  for k from 2 to 2 do a[k]:=c[k]/d[k] end do;
  for i from 1 to 2 do x[3,i]:=x[2,i]+a[2]*p[2,i] end do;
  for i from 1 to 2 do x[1,i]:=x[3,i] end do;
  print(x[3,1],x[3,2]);
  y[j,1]:=x[3,1]; y[j,2]:=x[3,2];
  end do:
```

−1.17528089887640449438202247191011235955056179775280898876404494382022471910112\ **(7)**
35955056179775280898876404494382022471910112359550561797752808988764044943820\
22503613562241339799575017784196302955476385899599063106706626632095624545653\
06481440912509674639864703902575382355071454879441452007444547473807797479062\

356184419492629449758964584558511222162874755052322102412994529646406950670 83\
56491638996898,
1.380674157303370786516853932584269662921348314606741573033707865168539325842\
696629213483146067415730337078651685393258426966292134831460674157303370786 51\
685402042924537704569465577844371058943913248593013907069149961129309829301 19\
137004232586274555681462511706338595527551048956180036175377366770773944259 35\
536306542541044603925086291637216988147804186506447417430390019305769005325 58\
6801449328372258
0.763114871176469844688739143569694142868157968250115974673912081716956066139 0609\
021881608502365630541977775839599757812016113843520776215016413418427775270 82\
845762572586818531181957672473959104600606206553090724183657017121849636887 76\
359991776082248187200051775702816803767430973174770077126157839150551520768 15\
804417425946732758640040430829572364912046211601976154062518963595207559030 86\
352507102063,
−3.175033854748195523834199768500873792101206865200764967713500096840255220 8\
344839194118077563924403728692818179818201079455323877003487967699482155665 07\
539610029631811379592903003105884610774761749135654710926879553912032536534 70\
459250971562095841172618241802799879336895108246057106435211700671371886325 15\
391283948553069777087302863128910174489918089479518767121292762729997802009 76\
346502684615559282
0.763429678884074086744924104964792540475396607427795508587873921581855893289 8646\
482364674795954495588369412150193784423299961881843322375890429475621786112 47\
085636963261067743329781659316114602940965860314605479379040304417123242971 56\
469376379981964791159288659425212681659619161649004567562545764869859047085 20\
717215181475645822355421580007082962390792639606924575726684382926459903213 67\
3346318782139,
0.582824775497147708487268219815099320525790412676089108814007944328038145710\
583650867194200275681901712573435007118542437504036237799467175690109626533 51\
852829807383682842326522297107863266694760185506477583674878318164876638187 46\
028073566434877364754467756491291851900671728536898777354453185348877438930 28\
574261195774291826233119434729470944101829656673188566082190365487370676889 60\
9811901121737938
0.999995311084990798761506689816157322909488415309187087322826894829297948742 1585\
484610948839622924722798137959627191552038291118380882492396900585248918451 10\
298800248903135686472012222754518982146443494876298140010575669833123616876 64\
093037640099056649624644467654721692200471639160151875242998690002494595474 94\
513521359771514574880002536450739381340951613816731872402614670150774139474 71\
7166787959216,
0.944027323853348119531028797598028624487126114932192014496159164458585814202\
469387234648854261116208872571063865963349935045169772287889689731512144243 41\
441126696983898214728201379082196009238549067274424164432438720554689763725 44\
492185749973835067332734527719340320377983254177520593209327667280299323277 34\
450080274513756417323179440422277547253225755926836122437533322716563734756 19\
72815346677143635
0.999995695653668756174124455273130825642998193865079334428331265111569047347 9582\
859724787612803061441372669263962218478683776838828501134838190145247751145 29\
298592980939369475981830765347601257667038297619949181295651003544984150318 92\
951143786622741802177192799288885651705312853639715286094886870998336292766 89\

885891428019680276223578549534138637884084584238913107934576442978358286025994\
4227199396050,
0.99999139132571701661947518199549774190517981590502864742814482422016587105 4\
096376355700816030461107979943954223099673401965269556580526937985645714465 26\
844267457842967495939538879964440536027238384023060657545497843328196718212 47\
707441202758665432114223481412935697722401723296141821013034230871609332184 52\
448217398256526181471189409438824938775950265268440332109543373888513592572 96\
61805784343022148
0.999999999999999872683403014585382083265695670928589357929203580566702339904031 8\
31249895065307315038424815851729603973839861090368769626987827630247445158577\
59876794923740557834304752288475454986724363375867557526757324058558948687511\
33049328200476984562375738103487300285520799853946958634974014577529925560642\
20493107475755844857232611911512959204796007446148441273061885449014663956097\
9264941558741,
0.999999999981472348028609875038378336326100680138963923102325279809701030007\
27132696619779677893176988575691573434138036857939680945461713986870554507588\
11406392400161310017770421872889039029944786513751089123887130757003465088804\
29306803998770646686458085473307445547986357251653023994529050266995393463271\
75138563901054473857424975858506852874838039794661058618262409691709531500078\
77429961718251350
0.99999999999999999999999995282309657289778112714924619461794960394467804810564309\
10235633827216215980456030901929247917074556716128506161196340114868025562287\
03290643033889891909162190265636668060253316550112077514435247822767584433466\
70762352084729486289898239952396124544059743961144023582759167296609999741024\
40592409034548561672002452501454537050606616823324073189102647649268568915316\
0044084231433,
0.999999999999999999999999905646191524843987472455436221592198190298454284085 7\
79799630203379259736667144277479943797663758911987877751897368128588660915906\
39016820014234303016751159067590750675456991500468183160604838053785692560834\
52597822943537779979262442905280699720468638852389845947296486505623891834443\
26617530782941334708380456329726692820023140721850046897032057291345055463109\
29306029725568088
0.9999999999999999999999999999999999999999999999999999984705704819301545730638222\
17117929757602966785499974646703991841998730010167226903852021124564235622522\
03380920437496748083103344131976303010312422794677975431925020256265080753320\
55546675770834849681823893644684750580178861844348676821429437216115944190440\
81622244399853210074004479342069104201924646924379147140378393336622207918492\
2611379933266,
0.999999999999999999999999999999999999999999999999777430919443950951897665721\
88394827410083666220512131550813239012669460671905714653190027207759211794319\
11709103694840364503755915147439310075737617310630202114453113534376561667647\
29383938823724463063898035236896847229160929720017913243431803446142395796680\
43761233702337382324323029245863047433814413681253704096867099743203966684020\
95162103140753247
0.9999999999999999999999999999999999999999999999999999999999999999999999999999999\
99999999999999999999931920191339442957328307380619720882900201489432751073091\
99602314690672276971544618112613864008735934092012670995609348020852336933541\
58655594688044103568315433943745293366273668893618327730888819026522234995721\

47536272712618272128375048526773797310984875331963212068970454558434523054396\
0314223542872,
0.999999999999999999999999999999999999999999999999999999999999999999999999999\
9999999999999999999999998638403803397312639132551219275581756652393465319305 4\
97804330670604635576727255679149019180802935513956882876458161905672829515895\
54217965964228476671232755369852384423460473843540301807625002262637286856234\
78890310138931311859304358628996744014479120032170532161656333875701431959289\
88991483844850280
0.9999999999999999999999999999999999999999999999999999999999999999999999999999999\
99999999999999999999999999999999999999999999999999999999999999999999999999999\
99999999999999999999999999999999999999999999999996815015978998798800678561081\
71666855522573941340237165147442517019232354929004769556945208368413962473375\
17356861074435359782572117864347890624091281770267710569642520308499037574550\
5604644539594,
0.999999999999999999999999999999999999999999999999999999999999999999999999999\
99999999999999999999999999999999999999999999999999999999999999999999999999999\
99999999999999999999999999999999999999999999999953650759530615222633671258231\
96327742934447237040132664835688203768403619309111261936221962285230816141735\
05514831053647406996701905442584839378483879399741487324419866098536012537775\
64063159849505775

```
> for j from 1 to 9 do
  print(y[j,1],y[j,2])
  end do;
```

−1.175280898876404494382022471910112359550561797752808988764044943820224719101 12\ (8)
3595505617977528089887640449438202247191011235955056179775280898876404494382 0\
22503613562241339799575017784196302955476385899599063106706626632095624545653\
06481440912509674639864703902575382355071454879441452007444547473807797479062\
35618441949262944975896458455851122216287475505232210241299452964640695067083\
56491638996898,
1.380674157303370786516853932584269662921348314606741573033707865168539325842\
69662921348314606741573033707865168539325842696629213483146067415730337078651\
68540204292453770456946557784437105894391324859301390706914996112930982930119\
13700423258627455568146251170633859552755104895618003617537736677077394425935\
53630654254104460392508629163721698814780418650644741743039001930576900532558\
6801449328372258
0.763114871176469844688739143569694142868157968250115974673912081716956066139060 9\
02188160850236563054197777583959975781201611384352077621501641341842777527082\
84576257258681853118195767247395910460060620655309072418365701712184963688776\
35999177608224818720005177570281680376743097317477007712615783915055152076815\
80441742594673275864004043082957236491204621160197615406251896359520755903086\
352507102063,
−3.175033854748195523834199768500873792101206865200764967713500096840255220 8\
34483919411807756392440372869281817981820107945532387700348796769948215566507\
53961002963181137959290300310588461077476174913565471092687955391203253653470\
45925097156209584117261824180279987933689510824605710643521170067137188632515\
39128394855306977708730286312891017448991808947951876712129276272999780200976\
346502684615559282

0.76342967888407408674492410496479254047539660742779550858787392158185589328986 46\
48236467479595449558836941215019378442329996188184332237589042947562178611247\
08563696326106774332978165931611460294096586031460547937904030441712324297156\
46937637998196479115928865942521268165961916164900456756254576486985904708520\
71721518147564582235542158000708296239079263960692457572668438292645990321367\
3346318782139,
0.58282477549714770848726821981509932052579041267608910881400794432803814571 0\
58365086719420027568190171257343500711854243750403623779946717569010962653351\
85282980738368284232652229710786326669476018550647758367487831816487663818746\
02807356643487736475446775649129185190067172853689877735445318534887743893028\
57426119577429182623311943472947094410182965667318856608219036548737067688960\
9811901121737938
0.99999531108499079876150668981615732290948841530918708732282689482929794874215 85\
48461094883962292472279813795962719155203829111838088249239690058524891845110\
29880024890313568647201222275451898214644349487629814001057566983312361687664\
09303764009905664962464446765472169220047163916015187524299869000249459547494\
51352135977151457488000253645073938134095161381673187240261467015077413947471\
7166787959216,
0.94402732385334811953102879759802862448712611493219201449615916445858581420 2\
46938723464885426111620887257106386596334993504516977228788968973151214424341\
44112669698389821472820137908219600923854906727442416443243872055468976372544\
49218574997383506733273452771934032037798325417752059320932766728029932327734\
45008027451375641732317944042227754725322575592683612243753332271656373475619\
72815346677143635
0.99999569565366875617412445527313082564299819386507933442833126511156904734795 82\
85972478761280306144137266926396221847868377683882850113483819014524775114529\
29859298093936947598183076534760125766703829761994918129565100354498415031892\
95114378662274180217719279928888565170531285363971528609488687099833629276689\
88589142801968027622357854953413863788408458423891310793457644297835828602599\
4227199396050,
0.99999139132571701661947518199549774190517981590502864742814482422016587105 4\
09637635570081603046110797994395422309967340196526955658052693798564571446526\
84426745784296749593953887996444053602723838402306065754549784332819671821247\
70744120275866543211422348141293569772240172329614182101303423087160933218452\
44821739825652618147118940943882493877595026526844033210954337388851359257296\
61805784343022148
0.99999999999999987268340301458538208326569567092858935792920358056670233990403 18\
31249895065307315038424815851729603973839861090368769626987827630247445158577\
59876794923740557834304752288475454986724363375867557526757324058558948687511\
33049328200476984562375738103487300285520799853946958634974014577529925560642\
20493107475755844857232611911512959204796007446148441273061885449014663956097\
9264941558741,
0.99999999998147234802860987503837833632610068013896392310232527980970103000 7\
27132696619779677893176988575691573434138036857939680945461713986870554507588\
11406392400161310017770421872889039029944786513751089123887130757003465088804\
29306803998770646686458085473307445547986357251653023994529050266995393463271\
75138563901054473857424975858506852874838039794661058618262409691709531500078\
77429961718251350

0.9999999999999999999999995282309657289778112714924619461794960394467804810564309\
10235633827216215980456030901929247917074556716128506161196340114868025562287\
03290643033889891909162190265636668060253316550112077514435247822767584433466\
70762352084729486289898239952396124544059743961144023582759167296609999741024\
40592409034548561672002452501454537050606616823324073189102647649268568915316\
0044084231433,
0.999999999999999999999999056461915248439874724554362215921981902984542840857\
79799630203379259736667144277479943797663758911987877751897368128588660915906\
39016820014234303016751159067590750675456991500468183160604838053785692560834\
52597822943537779979262442905280699720468638852389845947296486505623891834443\
26617530782941334708380456329726692820023140721850046897032057291345055463109\
29306029725568088
0.9999999999999999999999999999999999999999999999999999984705704819301545730638222\
17111792975760296678549997464670399184199873001016722690385202112456423562252\
03380920437496748083103344131976303010312422794677975431925020256265080753320\
55546675770834849681823893644684750580178861844348676821429437216115944190440\
81622244399853210074004479342069104201924646924379147140378393336622207918492\
2611379933266,
0.999999999999999999999999999999999999999999999999777430919443950951897665721\
88394827410083666220512131550813239012669460671905714653190027207759211794319\
11709103694840364503755915147439310075737617310630202114453113534376561667647\
29383938823724463063898035236896847229160929720017913243431803446142395796680\
43761233702337382324323029245863047433814413681253704096867099743203966684020\
95162103140753247
0.9999999999999999999999999999999999999999999999999999999999999999999999999999999\
99999999999999999999319201913394429573283073806197208829002014894327510730 91\
99602314690672276971544618112613864008735934092012670995609348020852336933541\
58655594688044103568315433943745293366273668893618327730888819026522234995721\
47536272712618272128375048526773797310984875331963212068970454558434523054396\
0314223542872,
0.999999999999999999999999999999999999999999999999999999999999999999999999999\
99999999999999999999999986384038033973126391325512192755817566523934653193054\
97804330670604635576727255679149019180802935513956882876458161905672829515895\
54217965964228476671232755369852384423460473843540301807625002262637286856234\
78890310138931311859304358628996744014479120032170532161656333875701431959289\
88991483844850280

```
> for i from 1 to 9 do
  C[i]:=sqrt((y[i+1,1]-1)^2+(y[i+1,2]-1)^2)/(sqrt((y[i,1]-1)^2+(y
  [i,2]-1)^2)^2);
  od;
```

$C_1 :=$ **(9)**

0.85748506720898984877583489067318065281924119136215069533554664522207638304 4\
16965949350528861919297215624474021827876713243400519816171450379702419044348\
48340595980557589820918362813729474288169479585799164366476824995326799244419\
38921372418333501248029997031581155243272406598445063838553061960389463816771\
95146205394837048351572450678479481307330792354845435568154103210276877091840\

93111647745448873

$C_2 :=$

0.02742513043836409762027027421747699476251896358266367298525578321520099250 7\
19884464293524593479024407381310820803401760938509862650665811324134056298470\
47313476456156937315679558786904877105416286775081213644518654625726580741017\
28886075589853832001074612067868128814587935978298755666846430444119344799806\
98416384974341032353194261891417421391298291640353445083896721606033116051759\
463356366318470857

$C_3 :=$

0.24335873781663447663321733172166189645022417919690425199935273042453064618 0\
31209366938666888880546068726628944158798115770435015107404261837077681740520\
45429477944800281603629271196798498255346125144501232623650571267233330277882\
06271601458711076013231459046153818055786555937815591421555060833715876516522\
08441343894441770115760878373346810013672008713231619555599060938894429720207\
22950957776429992

$C_4 :=$

0.00307212810630861588840571134427927543481475437959316970486775351474609562 7\
98846240992211480165226640288552827299393910864964814530967100693027333038672\
07345615088552450551008205596404264969965760805827846764810583149350659995172\
41102704965799920339594916641746283342909940420746794181901141117320411372133\
62755854371314033405797552580901763100891309986532510522973555603260456991179\
6255884066720425407

$C_5 :=$

0.20000343192245995025647178570635381076922019256018841233485795206506806759 3\
22807640285512126724041821853950969628127936307613622462784302278218789818939\
10593261096956135330955780471003188300634552966385745558435031061708573718226\
19845795670551371039962793505544678715492783644187511421816391846910957749950\
24873156499153523932697552739745688678888902006180962985916592438497652243887 3\
55663505505196182

$C_6 :=$

0.00307307862015473402489287370616607759587259233434857851333375666022591417 2\
84808086671688166183589202179737835168731275897016114386900850712873241117706\
75127330662072275333800569352616326531059049439056549248224515302572183435239\
11371758913784083963831260476016994066616079413473236139081683737389463740726\
11358831797918831084820807892626845972562376789228153217639536671615098696831\
3538097816203323889

$C_7 :=$

0.20000274322779885511638236207839425420265244805469308739798444951438290075 0\
89145509919809959230317291822138097655134257807508027233627718494472294846840\
26567655850692489191770848255933023594876391696451343564374057072179265682266\
68036902221108441932191528338487290954477833751770339217467379549511201482277\
46651243631772036480091428824323023266769700268163489338356394497616531343316\
39721406905156624

$C_8 :=$

0.00307307862015504702478975842811575435701694963122006433059445893220928841 3\
99028747312704929157030570036482504607544652537080925753069133944106462206597\

1580527788403276329301846585496462116220142912443985479441996865990407680984231281495602590949178330733239813033789280707946848577277513246541229946918588340831330804866809553373730033084039749568245022330775246099991696538686425316376695332088012783

$C_9 :=$

0.2000027432277988551163822865956109919915261312711602376214016420805604135822555504033062463250995047830720401809748980943681177027182974226561594495921894730752929485771974946276308901292227253515285109866313592125428522736176976315374606258875416987491126924903841078438824685874245219328992249939604541455554939995179053041982677757968777930094462915403652342580748003985274638539384401008455818819

\> # *THE QUOTIENT CONVERGENCE FACTOR OF ORTEGA AND RHEINBOLDT, P. 281, IS* $Q_2\{y_n\} = 0.200002743227798855116382$ ..;   *REFER TO* C_9 *ABOVE*;
THIS INDICATES QUADRATIC CONVERGENCE OF CGS FOR THIS PROBLEM;
THE RESULTS AGREE WITH THE NEWTON METHOD APPLIED TO THE GRAD$(F) = 0$,
WHERE F IS THE FUNCTION TO BE MINIMIZED.

\>