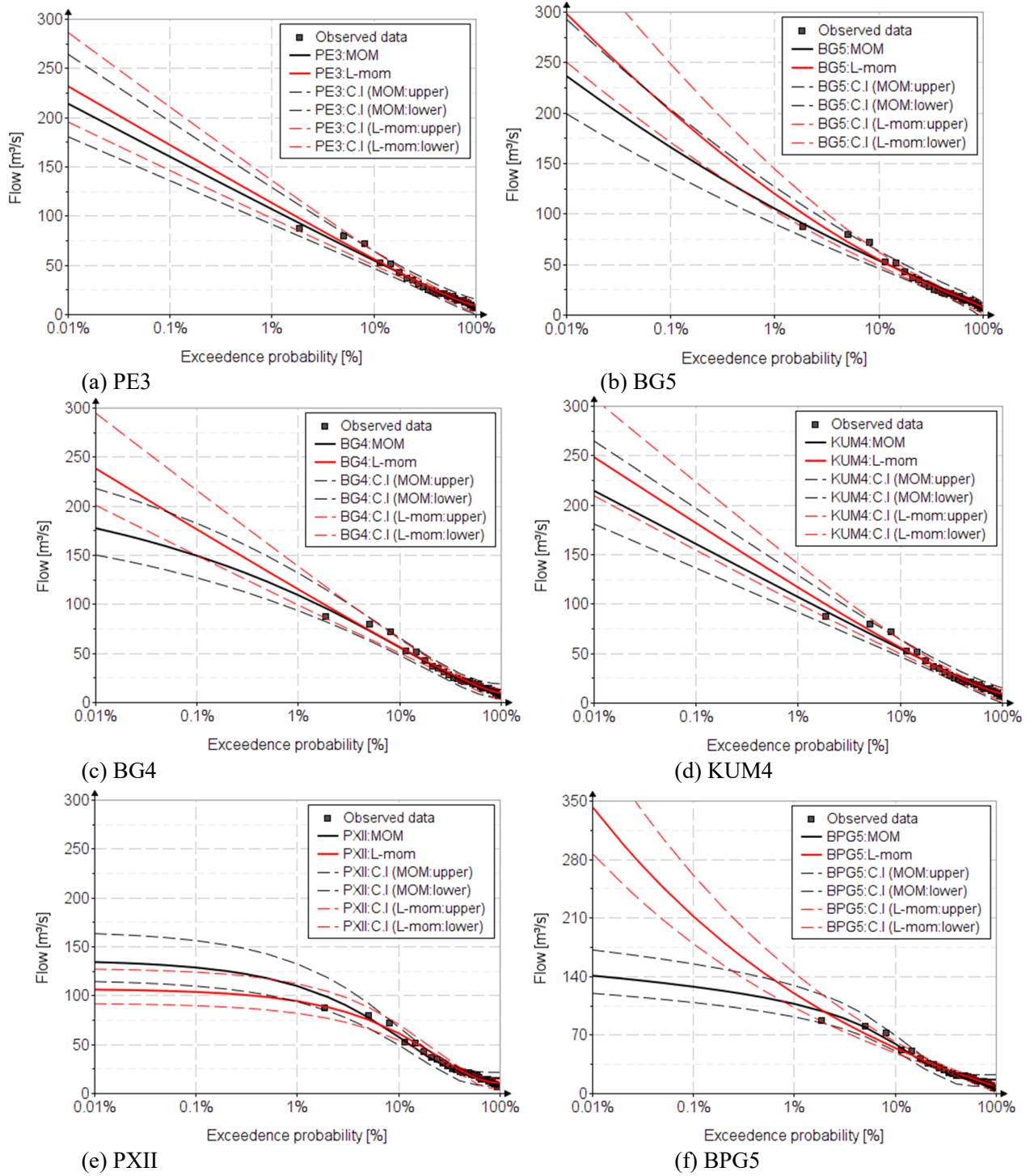
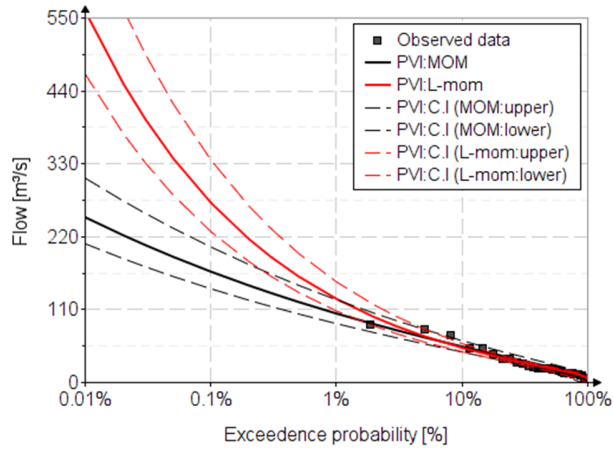


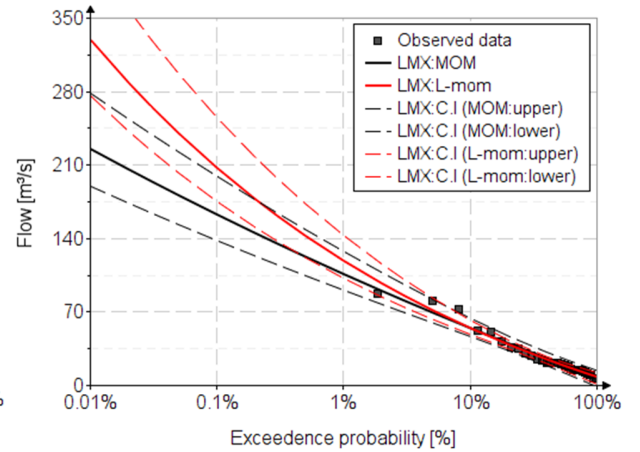
Supplementary Materials

Figure S1 presents the results of the analyzed distributions, highlighting the confidence interval (C.I) for each distribution, for both parameter estimation methods.

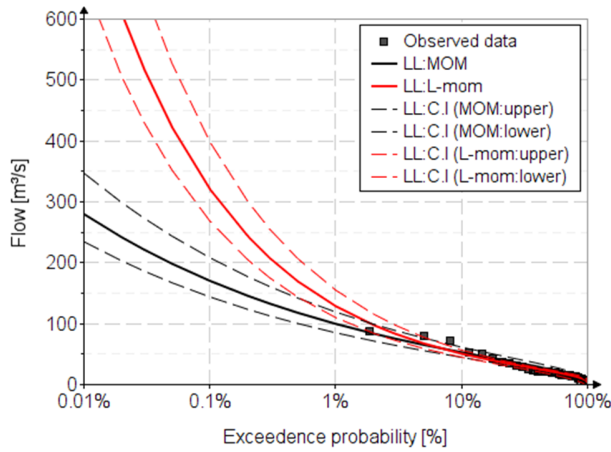




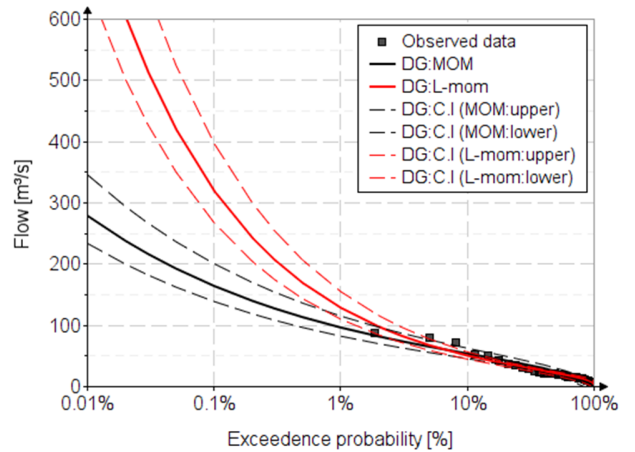
(g) PVI



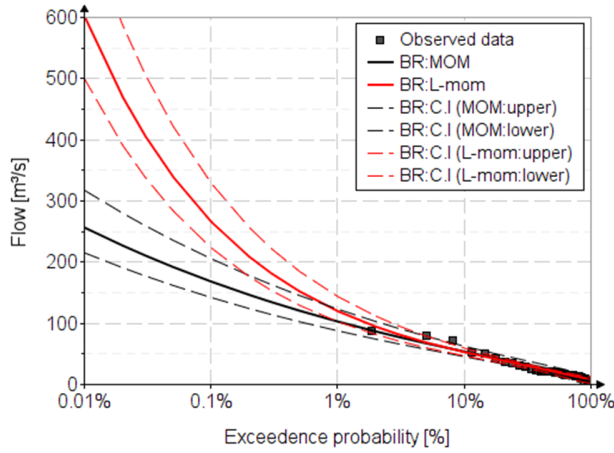
(h) LMX



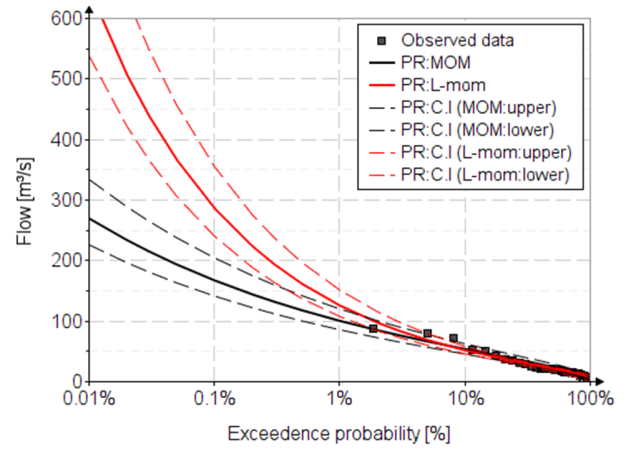
(i) LL



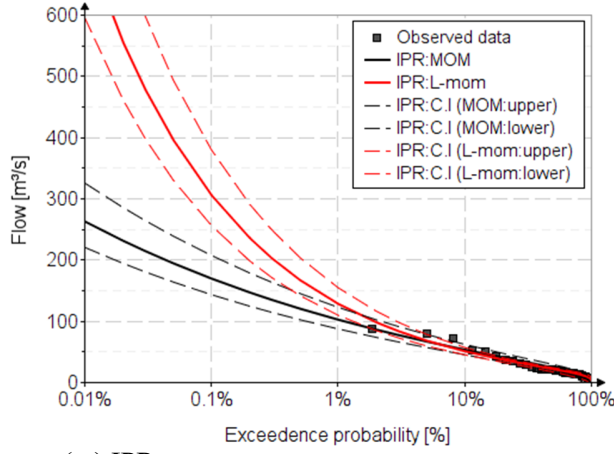
(j) DG



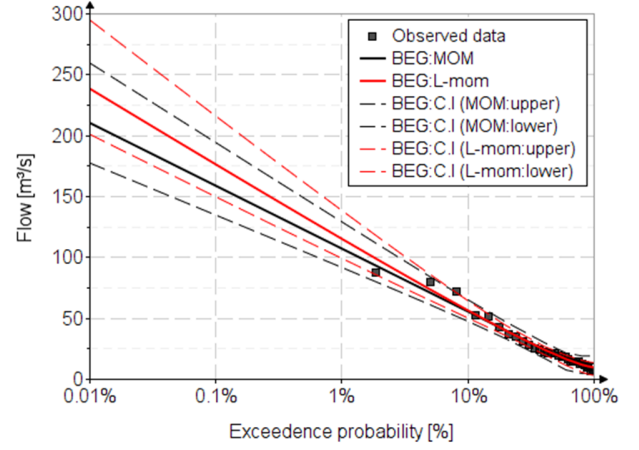
(k) BR



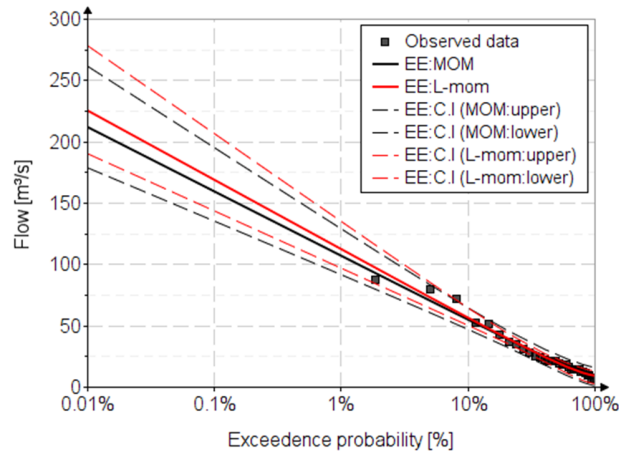
(l) PR



(m) IPR



(n) BEG



(o) EE

Figure S1. The fitting distributions with Confidence Intervals.

The raw and central moments of the analysed distributions were determined using the following methodology presented below, by substitution using the probability density function.

In the case of estimation with L-moments, the methodology by substitution was used on the expression of the inverse function.

Dagum distribution

To determine the raw moments of the Dagum distribution, the following substitutions in the expression for the probability density function are required:

$$f(x) = \frac{\alpha \cdot \gamma \cdot \left(\frac{x}{\beta}\right)^{\alpha-1}}{\beta \cdot \left(1 + \left(\frac{x}{\beta}\right)^{\alpha}\right)^{\gamma+1}}$$

It is noted:

$$t = \left(\frac{x}{\beta}\right)^\alpha \quad x = \beta \cdot t^{\frac{1}{\alpha}} \quad dx = \frac{\beta \cdot t^{\frac{1}{\alpha}-1}}{\alpha} \cdot dt$$

The complementary cumulative distribution function is obtained as:

$$F(x) = 1 - \int_0^x f(x) dx = 1 - \int_0^x \frac{\alpha \cdot \gamma \cdot \left(\frac{x}{\beta}\right)^{\alpha-1}}{\beta \cdot \left(1 + \left(\frac{x}{\beta}\right)^\alpha\right)^{\gamma+1}} dx = 1 - \gamma \cdot \int_0^t \frac{t^{\gamma-1}}{(1+t)^{\gamma+1}} \cdot dt = 1 - t^\gamma \cdot (1+t)^{-\gamma} =$$

$$1 - \left(\frac{x}{\beta}\right)^{\alpha\gamma} \cdot \left(1 + \left(\frac{x}{\beta}\right)^\alpha\right)^{-\gamma} = 1 - \left(1 + \left(\frac{x}{\beta}\right)^\alpha\right)^{-\gamma}$$

Checking the condition that the integral of $\int_0^\infty f(x)$ is 1:

$$\int_0^\infty f(x) dx = \int_0^\infty \frac{\alpha \cdot \gamma \cdot \left(\frac{x}{\beta}\right)^{\alpha-1}}{\beta \cdot \left(1 + \left(\frac{x}{\beta}\right)^\alpha\right)^{\gamma+1}} dx = \gamma \cdot \int_0^\infty \frac{t^{\gamma-1}}{(1+t)^{\gamma+1}} \cdot dt = \gamma \cdot \frac{\Gamma(\gamma) \cdot \Gamma(1)}{\Gamma(\gamma+1)} = \gamma \cdot \frac{1}{\gamma} = 1$$

Thus, the first four raw moments are:

$$m_1' = \int_0^\infty x \cdot f(x) dx = \int_0^\infty x \cdot \frac{\alpha \cdot \gamma \cdot \left(\frac{x}{\beta}\right)^{\alpha-1}}{\beta \cdot \left(1 + \left(\frac{x}{\beta}\right)^\alpha\right)^{\gamma+1}} dx = \gamma \cdot \beta \cdot \int_0^\infty \frac{t^{\gamma+\frac{1}{\alpha}-1}}{(1+t)^{\gamma+1}} \cdot dt = \beta \cdot \frac{\Gamma\left(\gamma + \frac{1}{\alpha}\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right)}{\Gamma(\gamma)}$$

$$m_2' = \int_0^\infty x^2 \cdot f(x) dx = \int_0^\infty x^2 \cdot \frac{\alpha \cdot \gamma \cdot \left(\frac{x}{\beta}\right)^{\alpha-1}}{\beta \cdot \left(1 + \left(\frac{x}{\beta}\right)^\alpha\right)^{\gamma+1}} dx = \gamma \cdot \beta^2 \cdot \int_0^\infty \frac{t^{\gamma+\frac{2}{\alpha}-1}}{(1+t)^{\gamma+1}} \cdot dt = \beta^2 \cdot \frac{\Gamma\left(\gamma + \frac{2}{\alpha}\right) \cdot \Gamma\left(1 - \frac{2}{\alpha}\right)}{\Gamma(\gamma)}$$

$$m_3' = \int_0^\infty x^3 \cdot f(x) dx = \int_0^\infty x^3 \cdot \frac{\alpha \cdot \gamma \cdot \left(\frac{x}{\beta}\right)^{\alpha-1}}{\beta \cdot \left(1 + \left(\frac{x}{\beta}\right)^\alpha\right)^{\gamma+1}} dx = \gamma \cdot \beta^3 \cdot \int_0^\infty \frac{t^{\gamma+\frac{3}{\alpha}-1}}{(1+t)^{\gamma+1}} \cdot dt = \beta^3 \cdot \frac{\Gamma\left(\gamma + \frac{3}{\alpha}\right) \cdot \Gamma\left(1 - \frac{3}{\alpha}\right)}{\Gamma(\gamma)}$$

$$m_4' = \int_0^\infty x^4 \cdot f(x) dx = \int_0^\infty x^4 \cdot \frac{\alpha \cdot \gamma \cdot \left(\frac{x}{\beta}\right)^{\alpha-1}}{\beta \cdot \left(1 + \left(\frac{x}{\beta}\right)^\alpha\right)^{\gamma+1}} dx = \gamma \cdot \beta^4 \cdot \int_0^\infty \frac{t^{\gamma+\frac{4}{\alpha}-1}}{(1+t)^{\gamma+1}} \cdot dt = \beta^4 \cdot \frac{\Gamma\left(\gamma + \frac{4}{\alpha}\right) \cdot \Gamma\left(1 - \frac{4}{\alpha}\right)}{\Gamma(\gamma)}$$

The characteristic moment of order r of the Dagum distribution has the following expression:

$$g_r = \beta^r \cdot \frac{\Gamma\left(\gamma + \frac{r}{\alpha}\right) \cdot \Gamma\left(1 - \frac{r}{\alpha}\right)}{\Gamma(\gamma)}$$

The first six central moments (except the arithmetic mean) can be obtained as:

$$\begin{aligned}
m_1 &= m_1' = \mu = g_1 \\
m_2 &= \sigma^2 = m_2' - m_1'^2 = g_2 - g_1^2 \\
m_3 &= m_3' - 3 \cdot m_2' \cdot m_1' + 2 \cdot m_1'^3 = g_3 - 3 \cdot g_2 \cdot g_1 + 2 \cdot g_1^3 \\
m_4 &= m_4' - 4 \cdot m_3' \cdot m_1' + 6 \cdot m_2' \cdot m_1'^2 - 3 \cdot m_1'^4 = g_4 - 4 \cdot g_3 \cdot g_1 + 6 \cdot g_2 \cdot g_1^2 - 3 \cdot g_1^4 \\
m_5 &= m_5' - 5 \cdot m_4' \cdot m_1' + 10 \cdot m_3' \cdot m_1'^2 - 10 \cdot m_2' \cdot m_1'^3 + 4 \cdot m_1'^5 = \\
&g_5 - 5 \cdot g_4 \cdot g_1 + 10 \cdot g_3 \cdot g_1^2 - 10 \cdot g_2 \cdot g_1^3 + 4 \cdot g_1^5 \\
m_6 &= m_6' - 6 \cdot m_5' \cdot m_1' + 15 \cdot m_4' \cdot m_1'^2 - 20 \cdot m_3' \cdot m_1'^3 + 15 \cdot m_2' \cdot m_1'^4 - 5 \cdot m_1'^6 = \\
&g_6 - 6 \cdot g_5 \cdot g_1 + 15 \cdot g_4 \cdot g_1^2 - 20 \cdot g_3 \cdot g_1^3 + 15 \cdot g_2 \cdot g_1^4 - 5 \cdot g_1^6
\end{aligned}$$

Skewness (γ_1), kurtosis (γ_2), γ_3 and γ_4 are determined with the following expressions:

$$\gamma_1 = C_s = \frac{m_3}{m_2^{3/2}}; \gamma_2 = C_k = \frac{m_4}{m_2^2}; \gamma_3 = \frac{m_5}{m_2^3}; \gamma_4 = \frac{m_6}{m_2^4}$$

Log-Logistic (Generalized Logistic) distribution

To determine the raw moments of the Log-Logistic distribution, the following substitutions in the expression for the probability density function are required:

$$f(x) = \frac{\alpha \cdot \left(\frac{x-\gamma}{\beta} \right)^{\alpha-1} \cdot \left(\left(\frac{x-\gamma}{\beta} \right)^{\alpha} + 1 \right)^{-2}}{\beta}$$

It is noted:

$$t = \left(\frac{x-\gamma}{\beta} \right)^{\alpha} \quad x = \beta \cdot t^{\frac{1}{\alpha}} + \gamma \quad dx = \frac{\beta \cdot t^{\frac{1}{\alpha}-1}}{\alpha} \cdot dt$$

The complementary cumulative distribution function is obtained as:

$$\begin{aligned}
F(x) &= 1 - \int_{\gamma + \frac{\alpha}{\beta}}^x f(x) dx = 1 - \int_{\gamma + \frac{\alpha}{\beta}}^x \frac{\alpha \cdot \left(\frac{x-\gamma}{\beta} \right)^{\alpha-1} \cdot \left(\left(\frac{x-\gamma}{\beta} \right)^{\alpha} + 1 \right)^{-2}}{\beta} dx = 1 - \int_0^t (t+1)^{-2} \cdot dt = \\
1 - \frac{t}{t+1} &= \frac{1}{\left(\frac{x-\gamma}{\beta} \right)^{\alpha} + 1} = \left(1 + \left(\frac{x-\gamma}{\beta} \right)^{\alpha} \right)^{-1}
\end{aligned}$$

Checking the condition that the integral of $\int_{\gamma + \frac{\alpha}{\beta}}^{\infty} f(x) dx$ is 1:

$$\int_{\gamma + \frac{\alpha}{\beta}}^{\infty} f(x) dx = \int_0^{\infty} \frac{\alpha \cdot \left(\frac{x-\gamma}{\beta} \right)^{\alpha-1} \cdot \left(\left(\frac{x-\gamma}{\beta} \right)^{\alpha} + 1 \right)^{-2}}{\beta} dx = \int_0^{\infty} (t+1)^{-2} \cdot dt = 1$$

Thus, the first six raw moments are:

$$m_1' = \int_{\gamma + \frac{\alpha}{\beta}}^{\infty} x \cdot f(x) = \int_{\gamma + \frac{\alpha}{\beta}}^{\infty} x \cdot \frac{\alpha \cdot \left(\frac{x-\gamma}{\beta}\right)^{\alpha-1} \cdot \left(\left(\frac{x-\gamma}{\beta}\right)^{\alpha} + 1\right)^{-2}}{\beta} dx = \int_0^{\infty} \left(\beta \cdot t^{\frac{1}{\alpha}} + \gamma\right) \cdot (t+1)^{-2} dt =$$

$$\int_0^{\infty} \beta \cdot t^{\frac{1}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} \gamma \cdot \frac{1}{(t+1)^2} dt = \beta \cdot \Gamma\left(\frac{1}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) + \gamma$$

$$m_2' = \int_{\gamma + \frac{\alpha}{\beta}}^{\infty} x^2 \cdot f(x) = \int_{\gamma + \frac{\alpha}{\beta}}^{\infty} x^2 \cdot \frac{\alpha \cdot \left(\frac{x-\gamma}{\beta}\right)^{\alpha-1} \cdot \left(\left(\frac{x-\gamma}{\beta}\right)^{\alpha} + 1\right)^{-2}}{\beta} dx = \int_0^{\infty} \left(\beta \cdot t^{\frac{1}{\alpha}} + \gamma\right)^2 \cdot (t+1)^{-2} dt =$$

$$\int_0^{\infty} \beta^2 \cdot t^{\frac{2}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} 2 \cdot \gamma \cdot \beta \cdot t^{\frac{1}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} \gamma^2 \cdot \frac{1}{(t+1)^2} dt =$$

$$\beta^2 \cdot \Gamma\left(\frac{2}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{2}{\alpha}\right) + 2 \cdot \gamma \cdot \beta \cdot \Gamma\left(\frac{1}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) + \gamma^2$$

$$m_3' = \int_{\gamma + \frac{\alpha}{\beta}}^{\infty} x^3 \cdot f(x) = \int_{\gamma + \frac{\alpha}{\beta}}^{\infty} x^3 \cdot \frac{\alpha \cdot \left(\frac{x-\gamma}{\beta}\right)^{\alpha-1} \cdot \left(\left(\frac{x-\gamma}{\beta}\right)^{\alpha} + 1\right)^{-2}}{\beta} dx = \int_0^{\infty} \left(\beta \cdot t^{\frac{1}{\alpha}} + \gamma\right)^3 \cdot (t+1)^{-2} dt =$$

$$\int_0^{\infty} \beta^3 \cdot t^{\frac{3}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} 3 \cdot \gamma \cdot \beta^2 \cdot t^{\frac{2}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} 3 \cdot \gamma^2 \cdot \beta \cdot t^{\frac{1}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} \gamma^3 \cdot \frac{1}{(t+1)^2} dt =$$

$$\beta^3 \cdot \Gamma\left(\frac{3}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{3}{\alpha}\right) + 3 \cdot \gamma \cdot \beta^2 \cdot \Gamma\left(\frac{2}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{2}{\alpha}\right) + 3 \cdot \gamma^2 \cdot \beta \cdot \Gamma\left(\frac{1}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) + \gamma^3$$

$$m_4' = \int_{\gamma + \frac{\alpha}{\beta}}^{\infty} x^4 \cdot f(x) = \int_{\gamma + \frac{\alpha}{\beta}}^{\infty} x^4 \cdot \frac{\alpha \cdot \left(\frac{x-\gamma}{\beta}\right)^{\alpha-1} \cdot \left(\left(\frac{x-\gamma}{\beta}\right)^{\alpha} + 1\right)^{-2}}{\beta} dx = \int_0^{\infty} \left(\beta \cdot t^{\frac{1}{\alpha}} + \gamma\right)^4 \cdot (t+1)^{-2} dt =$$

$$\int_0^{\infty} \beta^4 \cdot t^{\frac{4}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} 4 \cdot \gamma \cdot \beta^3 \cdot t^{\frac{3}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} 6 \cdot \gamma^2 \cdot \beta^2 \cdot t^{\frac{2}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} 4 \cdot \gamma^3 \cdot \beta \cdot t^{\frac{1}{\alpha}} \cdot \frac{1}{(t+1)^2} dt +$$

$$\int_0^{\infty} \gamma^4 \cdot \frac{1}{(t+1)^2} dt = \beta^4 \cdot \Gamma\left(\frac{4}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{4}{\alpha}\right) + 4 \cdot \gamma \cdot \beta^3 \cdot \Gamma\left(\frac{3}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{3}{\alpha}\right) + 6 \cdot \gamma^2 \cdot \beta^2 \cdot \Gamma\left(\frac{2}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{2}{\alpha}\right) +$$

$$4 \cdot \gamma^3 \cdot \beta \cdot \Gamma\left(\frac{1}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) + \gamma^4$$

$$\begin{aligned}
m'_5 &= \int_{\gamma+\frac{\alpha}{\beta}}^{\infty} x^5 \cdot f(x) = \int_{\gamma+\frac{\alpha}{\beta}}^{\infty} x^5 \cdot \frac{\alpha \cdot \left(\frac{x-\gamma}{\beta}\right)^{\alpha-1} \cdot \left(\left(\frac{x-\gamma}{\beta}\right)^{\alpha} + 1\right)^{-2}}{\beta} dx = \int_0^{\infty} \left(\beta \cdot t^{\frac{1}{\alpha}} + \gamma\right)^5 \cdot (t+1)^{-2} dt = \\
&\int_0^{\infty} \beta^5 \cdot t^{\frac{5}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} 5 \cdot \gamma \cdot \beta^4 \cdot t^{\frac{4}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} 10 \cdot \gamma^2 \cdot \beta^3 \cdot t^{\frac{3}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \\
&\int_0^{\infty} 10 \cdot \gamma^3 \cdot \beta^2 \cdot t^{\frac{2}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} 5 \cdot \gamma^4 \cdot \beta \cdot t^{\frac{1}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} \gamma^5 \cdot \frac{1}{(t+1)^2} dt = \\
&\beta^5 \cdot \Gamma\left(\frac{5}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{5}{\alpha}\right) + 5 \cdot \gamma \cdot \beta^4 \cdot \Gamma\left(\frac{4}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{4}{\alpha}\right) + 10 \cdot \gamma^2 \cdot \beta^3 \cdot \Gamma\left(\frac{3}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{3}{\alpha}\right) + \\
&+ 10 \cdot \gamma^3 \cdot \beta^2 \cdot \Gamma\left(\frac{2}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{2}{\alpha}\right) + 5 \cdot \gamma^4 \cdot \beta \cdot \Gamma\left(\frac{1}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) + \gamma^5
\end{aligned}$$

$$\begin{aligned}
m'_6 &= \int_{\gamma+\frac{\alpha}{\beta}}^{\infty} x^6 \cdot f(x) = \int_{\gamma+\frac{\alpha}{\beta}}^{\infty} x^6 \cdot \frac{\alpha \cdot \left(\frac{x-\gamma}{\beta}\right)^{\alpha-1} \cdot \left(\left(\frac{x-\gamma}{\beta}\right)^{\alpha} + 1\right)^{-2}}{\beta} dx = \int_0^{\infty} \left(\beta \cdot t^{\frac{1}{\alpha}} + \gamma\right)^6 \cdot (t+1)^{-2} dt = \\
&\int_0^{\infty} \beta^6 \cdot t^{\frac{6}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} 6 \cdot \gamma \cdot \beta^5 \cdot t^{\frac{5}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} 15 \cdot \gamma^2 \cdot \beta^4 \cdot t^{\frac{4}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \\
&\int_0^{\infty} 20 \cdot \gamma^3 \cdot \beta^3 \cdot t^{\frac{3}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} 15 \cdot \gamma^4 \cdot \beta^2 \cdot t^{\frac{2}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} 6 \cdot \gamma^5 \cdot \beta \cdot t^{\frac{1}{\alpha}} \cdot \frac{1}{(t+1)^2} dt + \int_0^{\infty} \gamma^6 \cdot \frac{1}{(t+1)^2} dt = \\
&\beta^6 \cdot \Gamma\left(\frac{6}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{6}{\alpha}\right) + 6 \cdot \gamma \cdot \beta^5 \cdot \Gamma\left(\frac{5}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{5}{\alpha}\right) + 15 \cdot \gamma^2 \cdot \beta^4 \cdot \Gamma\left(\frac{4}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{4}{\alpha}\right) + \\
&+ 20 \cdot \gamma^3 \cdot \beta^3 \cdot \Gamma\left(\frac{3}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{3}{\alpha}\right) + 15 \cdot \gamma^4 \cdot \beta^2 \cdot \Gamma\left(\frac{2}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{2}{\alpha}\right) + 6 \cdot \gamma^5 \cdot \beta \cdot \Gamma\left(\frac{1}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{1}{\alpha}\right) + \gamma^6
\end{aligned}$$

The characteristic moment of order r of the Log-Logistic distribution has the following expression:

$$g_r = \beta^r \cdot \Gamma\left(\frac{r}{\alpha} + 1\right) \cdot \Gamma\left(1 - \frac{r}{\alpha}\right)$$

The arithmetic mean (expected value) results:

$$m_1 = m'_1 = \mu = \gamma + g_1$$

For the calculation of other central moments, the same equivalence relations between central and raw moments are applied, following Dagum distribution.