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Recovery and Characterization of Orbital Angular Momentum Modes with Ghost Diffraction Holography

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Supplementary Materials

This document provides supplementary information to "Recovery and Characterization of Orbital Angular Momentum Modes with Ghost Diffraction Holography".

Principle of the Recovery Scheme

The proposed technique makes use of the holography scheme for the complex correlation function recovery and thereby the simultaneous recovery of amplitude and phase distributions of the OAM mode [1]. The independent addition of the holographic reference random field ($\psi_R(\boldsymbol{r})$) with the field contributions $\psi_m(\boldsymbol{r})$ and $\psi_0(\boldsymbol{r})$ modifies the resultant intensity distributions at the detector plane to

$$\begin{aligned} I_m(\boldsymbol{r}) &= \left(\psi_m(\boldsymbol{r})+\psi_R(\boldsymbol{r})\right)^*\left(\psi_m(\boldsymbol{r})+\psi_R(\boldsymbol{r})\right) \\ I_0(\boldsymbol{r}) &= \left(\psi_0(\boldsymbol{r})+\psi_R(\boldsymbol{r})\right)^*\left(\psi_0(\boldsymbol{r})+\psi_R(\boldsymbol{r})\right) \end{aligned} \tag{1}$$

where the respective field distributions at the detector plane is expressed as

$$\begin{aligned} \psi_m(\boldsymbol{r}) &= \frac{exp(ikz)}{i\lambda z}\int \psi_0(\hat{\boldsymbol{r}})\, exp\left(i\varphi_g(\hat{\boldsymbol{r}})\right) V_0^{\varepsilon l}(\hat{\boldsymbol{r}})\, exp\left[\frac{ik}{2z}\left(|\boldsymbol{r}|^2-2\boldsymbol{r}.\hat{\boldsymbol{r}}+|\hat{\boldsymbol{r}}|^2\right)\right]d\hat{\boldsymbol{r}} \\ \psi_0(\boldsymbol{r}) &= \frac{exp(ikz)}{i\lambda z}\int \psi_0(\hat{\boldsymbol{r}})\, exp\left(i\varphi_g(\hat{\boldsymbol{r}})\right)\, exp\left[\frac{ik}{2z}\left(|\boldsymbol{r}|^2-2\boldsymbol{r}.\hat{\boldsymbol{r}}+|\hat{\boldsymbol{r}}|^2\right)\right]d\hat{\boldsymbol{r}} \\ \psi_R(\boldsymbol{r}) &= \frac{exp(ikz)}{i\lambda z}\int \psi_R(\hat{\boldsymbol{r}})\, exp\left(i\varphi_r(\hat{\boldsymbol{r}})\right)\, exp\left[\frac{ik}{2z}\left(|\boldsymbol{r}|^2-2\boldsymbol{r}.\hat{\boldsymbol{r}}+|\hat{\boldsymbol{r}}|^2\right)\right]d\hat{\boldsymbol{r}} \end{aligned} \tag{2}$$

where $'\lambda'$ the wavelength of light source, $\varphi_g(\hat{\boldsymbol{r}})$ and $\varphi_r(\hat{\boldsymbol{r}})$ are the random phase introduced by the rotating ground glass diffuser and the reference ground glass diffuser, respectively, $'k = 2\pi/\lambda'$ the wave number, $'z'$ the propagation distances in the respective arms, $V_0^{\varepsilon l}(\hat{\boldsymbol{r}}) = exp(i\varepsilon l\phi)$ is the vortex information with zero radial index and l azimuthal index, where ε and l represent the sign and magnitude of the topological charge (TC) of the specific OAM mode, and $\hat{\boldsymbol{r}}$ the position coordinate at the diffuser plane.

Let us consider the intensity distributions at the detector plane. On assuming the scattered field from the diffusers obeys the Gaussian statistics and by utilizing the fourth order moment of the field at the detector plane [2], the fourth order correlation is expressed in terms of the respective second order correlation function as

$$\begin{aligned} \left\langle \Delta I_m(\boldsymbol{r}_1)\Delta I_0(\boldsymbol{r}_2)\right\rangle &= \left|\left\langle \left(\psi_m(\boldsymbol{r}_1)+\psi_R(\boldsymbol{r}_1)\right)^*\left(\psi_0(\boldsymbol{r}_2)+\psi_R(\boldsymbol{r}_2)\right)\right\rangle\right|^2 \\ &= \left|\left\langle \psi_m^*(\boldsymbol{r}_1)\psi_0(\boldsymbol{r}_2)+\psi_m^*(\boldsymbol{r}_1)\psi_R(\boldsymbol{r}_2)+\psi_R^*(\boldsymbol{r}_1)\psi_0(\boldsymbol{r}_2)+\psi_R^*(\boldsymbol{r}_1)\psi_R(\boldsymbol{r}_2)\right\rangle\right|^2 \\ &= \left|\left\langle \psi_m^*(\boldsymbol{r}_1)\psi_0(\boldsymbol{r}_2)\right\rangle+\left\langle \psi_R^*(\boldsymbol{r}_1)\psi_R(\boldsymbol{r}_2)\right\rangle\right|^2 \\ &= \left|W(\boldsymbol{r}_1,\boldsymbol{r}_2)+W^R(\boldsymbol{r}_1,\boldsymbol{r}_2)\right|^2 \end{aligned} \tag{3}$$

where $\langle\ldots\rangle$ represents the ensemble averaging, $\Delta I_m(\boldsymbol{r}_1) = I_m(\boldsymbol{r}_1)-\langle I_m(\boldsymbol{r}_1)\rangle$ and $\Delta I_0(\boldsymbol{r}_2) = I_0(\boldsymbol{r}_2)-\langle I_0(\boldsymbol{r}_2)\rangle$ represents the fluctuation of the intensity value with respect to its average value for the respective intensity distributions at the detector plane, $W(\boldsymbol{r}_1,\boldsymbol{r}_2) = \left\langle \psi_m^*(\boldsymbol{r}_1)\psi_0(\boldsymbol{r}_2)\right\rangle$, and $W^R(\boldsymbol{r}_1,\boldsymbol{r}_2) = \left\langle \psi_R^*(\boldsymbol{r}_1)\psi_R(\boldsymbol{r}_2)\right\rangle$ are the second order correlation functions corresponding to vortex encoded field and a reference random field, respectively. Here, we are justified in taking the contribution from the cross terms zero. i.e., $\left\langle \psi_m^*(\boldsymbol{r}_1)\psi_R(\boldsymbol{r}_2)\right\rangle = \left\langle \psi_R^*(\boldsymbol{r}_1)\psi_0(\boldsymbol{r}_2)\right\rangle = 0$, since independent diffusers are used to generate the respective random fields. Equation (3) describes that the two-point intensity correlation at the detector plane is the result of the superposition of second order complex field correlation functions, like the superposition of the optical field in conventional holography. Therefore, the resulting intensity correlation function is described as 'correlation hologram' [3,4]. Here, the recording and reconstruction of information take place in terms of the coherence functions rather than in terms of the optical fields as usually in the conventional holography.

Now let us turn our attention to the ensemble averaging in the intensity correlation. The execution of ensemble averaging can be performed either with temporal averaging or with space averaging. Here, we rely on space averaging by considering the time frozen speckle fields at the detector plane under the assumption of spatial stationarity [1,5,6]. The spatial averaging is performed in the execution of the cross-correlation of intensity images by fixing the distance $\Delta\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ equivalent to the pixel size at the detector plane, and by moving a specific window size in the intensity distribution $I_m(\boldsymbol{r})$ over the entire intensity image $I_0(\boldsymbol{r})$. We apply a change of variables $\boldsymbol{r}_1 = \boldsymbol{r} + \Delta\boldsymbol{r}$ and $\boldsymbol{r}_2 = \boldsymbol{r}$ with $\Delta\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, the Equation (3) modifies to

$$\left\langle \Delta I_m(\boldsymbol{r}+\Delta\boldsymbol{r})\Delta I_0(\boldsymbol{r})\right\rangle = \left| W(\boldsymbol{r}+\Delta\boldsymbol{r},\boldsymbol{r}) + W^R(\boldsymbol{r}+\Delta\boldsymbol{r},\boldsymbol{r})\right|^2 \tag{4}$$

Since Equation (4) utilizes the correlations of intensity fluctuations corresponding to the complex field given in Equation (2), the contribution of phase factor $exp\left(ik\frac{|\boldsymbol{r}|^2}{2z}\right)$ or $exp\left(ik\frac{|(\boldsymbol{r}+\Delta\boldsymbol{r})|^2}{2z}\right)$ outside the propagation integral is canceled out while estimating the fourth order correlation as in Equation (4). Removal of the common phase curvature outside the integration helps to achieve spatial stationarity and replace the ensemble averaging $\langle\ldots\rangle$ by the spatial averaging $\langle\ldots\rangle_S$ at any arbitrary plane along z [5].

On substitution of complex field distributions given in Equation (2) into Equation (4) and applying the spatial averaging, the first term in the RHS of Equation (4) modifies to

$$\begin{aligned}
W(\boldsymbol{r}+\Delta\boldsymbol{r},\boldsymbol{r}) &= \left\langle \psi_m^*(\boldsymbol{r}+\Delta\boldsymbol{r})\psi_0(\boldsymbol{r})\right\rangle \\
&= \iint\left[\int \frac{1}{\lambda^2 z^2} exp\left(-i\left(\varphi_g(\hat{\boldsymbol{r}}_1)-\varphi_g(\hat{\boldsymbol{r}}_2)\right)\right) exp\left(-i\frac{k}{2z}\left(|\hat{\boldsymbol{r}}_1|^2-|\hat{\boldsymbol{r}}_2|^2\right)\right) exp\left(-i\frac{2\pi}{\lambda z}\left((\boldsymbol{r}+\Delta\boldsymbol{r}).\hat{\boldsymbol{r}}_1-\boldsymbol{r}.\hat{\boldsymbol{r}}_2\right)\right) d\boldsymbol{r}\right] \\
&\quad \times \psi_0^*(\hat{\boldsymbol{r}}_1)\mathrm{V}_0^{\varepsilon 1}(\hat{\boldsymbol{r}}_1)\psi_0(\hat{\boldsymbol{r}}_2)d\hat{\boldsymbol{r}}_1 d\hat{\boldsymbol{r}}_2 \\
&= \frac{1}{\lambda^2 z^2}\iint\left[\int \exp\left(-i\frac{2\pi}{\lambda z}(\hat{\boldsymbol{r}}_1-\hat{\boldsymbol{r}}_2).\boldsymbol{r}\right)d\boldsymbol{r}\right] exp\left(-i\left(\varphi_g(\hat{\boldsymbol{r}}_1)-\varphi_g(\hat{\boldsymbol{r}}_2)\right)\right) exp\left(-i\frac{k}{2z}\left(|\hat{\boldsymbol{r}}_1|^2-|\hat{\boldsymbol{r}}_2|^2\right)\right) \\
&\quad \times exp\left(-i\frac{2\pi}{\lambda z}(\Delta\boldsymbol{r}.\hat{\boldsymbol{r}}_1)\right)\psi_0^*(\hat{\boldsymbol{r}}_1)\mathrm{V}_0^{\varepsilon 1}(\hat{\boldsymbol{r}}_1)\psi_0(\hat{\boldsymbol{r}}_2)d\hat{\boldsymbol{r}}_1 d\hat{\boldsymbol{r}}_2 \\
&= \frac{1}{\lambda^2 z^2}\iint \psi_0^*(\hat{\boldsymbol{r}}_1)\mathrm{V}_0^{\varepsilon 1}(\hat{\boldsymbol{r}}_1)\psi_0(\hat{\boldsymbol{r}}_2)\delta(\hat{\boldsymbol{r}}_1-\hat{\boldsymbol{r}}_2) exp\left(-i\left(\varphi_g(\hat{\boldsymbol{r}}_1)-\varphi_g(\hat{\boldsymbol{r}}_2)\right)\right) exp\left(-i\frac{k}{2z}\left(|\hat{\boldsymbol{r}}_1|^2-|\hat{\boldsymbol{r}}_2|^2\right)\right) \\
&\quad \times exp\left(-i\frac{2\pi}{\lambda z}\Delta\boldsymbol{r}.\hat{\boldsymbol{r}}_1\right)d\hat{\boldsymbol{r}}_1 d\hat{\boldsymbol{r}}_2
\end{aligned} \tag{5}$$

where $\int \exp\left(-i\frac{2\pi}{\lambda z}(\hat{\boldsymbol{r}}_1-\hat{\boldsymbol{r}}_2).\boldsymbol{r}\right)d\boldsymbol{r} = \delta(\hat{\boldsymbol{r}}_1-\hat{\boldsymbol{r}}_2)$. On considering $\hat{\boldsymbol{r}}_2 = \hat{\boldsymbol{r}}_1 = \hat{\boldsymbol{r}}$ and $\psi_0^*(\hat{\boldsymbol{r}})\psi_0(\hat{\boldsymbol{r}}) = I_0(\hat{\boldsymbol{r}})$, the Equation (5) modifies to

$$W(\Delta\boldsymbol{r}) = \frac{1}{\lambda^2 z^2}\int I_0(\hat{\boldsymbol{r}})\mathrm{V}_0^{\varepsilon 1}(\hat{\boldsymbol{r}}) exp\left(-i\frac{2\pi}{\lambda z}\Delta\boldsymbol{r}.\hat{\boldsymbol{r}}\right)d\hat{\boldsymbol{r}} \tag{6}$$

Similarly, the second term in the RHS of the Equation (4) results into

$$
\begin{aligned}
W^{R}(\boldsymbol{r}+\Delta\boldsymbol{r},\boldsymbol{r}) &= \left\langle \psi_R^*(\boldsymbol{r}+\Delta\boldsymbol{r})\psi_R(\boldsymbol{r})\right\rangle \\
&= \iint\left[\int \frac{1}{\lambda^2 z^2} exp\left(-i\left(\varphi_r(\hat{\boldsymbol{r}}_1)-\varphi_r(\hat{\boldsymbol{r}}_2)\right)\right) exp\left(-i\frac{k}{2z}\left(\left|\hat{\boldsymbol{r}}_1\right|^2-\left|\hat{\boldsymbol{r}}_2\right|^2\right)\right) exp\left(-i\frac{2\pi}{\lambda z}\left((\boldsymbol{r}+\Delta\boldsymbol{r}).\hat{\boldsymbol{r}}_1-\boldsymbol{r}.\hat{\boldsymbol{r}}_2\right)\right) d\boldsymbol{r}\right] \\
&\quad \times \psi_R^*(\hat{\boldsymbol{r}}_1)\psi_R(\hat{\boldsymbol{r}}_2) d\hat{\boldsymbol{r}}_1 d\hat{\boldsymbol{r}}_2 \\
&= \frac{1}{\lambda^2 z^2}\iint\left[\int exp\left(-i\frac{2\pi}{\lambda z}(\hat{\boldsymbol{r}}_1-\hat{\boldsymbol{r}}_2).\boldsymbol{r}\right) d\boldsymbol{r}\right] exp\left(-i\left(\varphi_r(\hat{\boldsymbol{r}}_1)-\varphi_r(\hat{\boldsymbol{r}}_2)\right)\right) exp\left(-i\frac{k}{2z}\left(\left|\hat{\boldsymbol{r}}_1\right|^2-\left|\hat{\boldsymbol{r}}_2\right|^2\right)\right) \\
&\quad \times exp\left(-i\frac{2\pi}{\lambda z}\Delta\boldsymbol{r}.\hat{\boldsymbol{r}}_1\right)\psi_R^*(\hat{\boldsymbol{r}}_1)\psi_R(\hat{\boldsymbol{r}}_2) d\hat{\boldsymbol{r}}_1 d\hat{\boldsymbol{r}}_2 \\
&= \frac{1}{\lambda^2 z^2}\iint \psi_R^*(\hat{\boldsymbol{r}}_1)\psi_R(\hat{\boldsymbol{r}}_2)\delta(\hat{\boldsymbol{r}}_1-\hat{\boldsymbol{r}}_2) exp\left(-i\left(\varphi_r(\hat{\boldsymbol{r}}_1)-\varphi_r(\hat{\boldsymbol{r}}_2)\right)\right) exp\left(-i\frac{k}{2z}\left(\left|\hat{\boldsymbol{r}}_1\right|^2-\left|\hat{\boldsymbol{r}}_2\right|^2\right)\right) \\
&\quad \times exp\left(-i\frac{2\pi}{\lambda z}\Delta\boldsymbol{r}.\hat{\boldsymbol{r}}_1\right) d\hat{\boldsymbol{r}}_1 d\hat{\boldsymbol{r}}_2
\end{aligned}
\tag{7}
$$

On considering $\hat{\boldsymbol{r}}_2=\hat{\boldsymbol{r}}_1=\hat{\boldsymbol{r}}$ and $\psi_R^*(\hat{\boldsymbol{r}})\psi_R(\hat{\boldsymbol{r}})=I_R(\hat{\boldsymbol{r}})$ at the reference diffuser plane, Equation (7) modifies to

$$
W^{R}(\Delta\boldsymbol{r}) = \frac{1}{\lambda^2 z^2}\int I_R(\hat{\boldsymbol{r}})\ exp\left(-i\frac{2\pi}{\lambda z}\Delta\boldsymbol{r}.\hat{\boldsymbol{r}}\right) d\hat{\boldsymbol{r}} \tag{8}
$$

By substituting Equation (6) and (8) in Equation (4), the intensity correlation function in terms of the position vector $\Delta\boldsymbol{r}$ is expressed as

$$
\begin{aligned}
\left\langle \Delta I_m(\boldsymbol{r}+\Delta\boldsymbol{r})\Delta I_0(\boldsymbol{r})\right\rangle &= \left|W(\Delta\boldsymbol{r})+W^{R}(\Delta\boldsymbol{r})\right|^2 \\
&= \left|\frac{1}{\lambda^2 z^2}\left\{\int I_0(\hat{\boldsymbol{r}})\mathrm{V}_0^{\varepsilon 1}(\hat{\boldsymbol{r}}) exp\left(-i\frac{2\pi}{\lambda z}\Delta\boldsymbol{r}.\hat{\boldsymbol{r}}\right) d\hat{\boldsymbol{r}} + \int I_R(\hat{\boldsymbol{r}}) exp\left(-i\frac{2\pi}{\lambda z}\Delta\boldsymbol{r}.\hat{\boldsymbol{r}}\right) d\hat{\boldsymbol{r}}\right\}\right|^2
\end{aligned}
\tag{9}
$$

Equation (9) describes the cross-correlation of intensity fluctuations in the proposed scheme, which results into the generation of the correlation hologram, where the respective second order correlation functions are expressed as

$$
\begin{aligned}
W(\Delta\boldsymbol{r}) &\propto \int I_0(\hat{\boldsymbol{r}})\mathrm{V}_0^{\varepsilon 1}(\hat{\boldsymbol{r}}) exp\left(-i\frac{2\pi}{\lambda z}\Delta\boldsymbol{r}.\hat{\boldsymbol{r}}\right) d\hat{\boldsymbol{r}} \\
W^{R}(\Delta\boldsymbol{r}) &\propto \int circ\left(\frac{\hat{\boldsymbol{r}}-\boldsymbol{r}_s}{a}\right) exp\left[-i\frac{2\pi}{\lambda z}\Delta\boldsymbol{r}.\hat{\boldsymbol{r}}\right] d\hat{\boldsymbol{r}}
\end{aligned}
\tag{10}
$$

where $I_R(\hat{\boldsymbol{r}}) = circ\left(\frac{\hat{\boldsymbol{r}}-\boldsymbol{r}_s}{a}\right)$ is utilized in the proposed approach for the generation of $W^{R}(\Delta\boldsymbol{r})$.

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