geometry

Article

Some Nice Configurations of Golden Triangles

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Abstract

It is well known among geometry scholars that the golden triangle, an isosceles triangle with sides and base in golden ratio, maintains a significant relationship with regular polygons, notably the regular pentagon, pentagram, and decagon. Extensive mathematical literature addresses this subject. Furthermore, its close association with the golden ratio—a mathematical concept describing a harmonious and proportionate relationship between segments—renders it a noteworthy element in the fields of geometry, art, and architecture. Nevertheless, the interrelationships among these mathematical constructs frequently reveal unexpected configurations, thereby accentuating intriguing patterns. The purpose of this investigation is to highlight these novel configurations, which indicate new connections between the golden triangle and regular polygons.

Keywords: golden triangle; euclidean geometry; geometry of triangles; regular polygons

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1. Introduction

The golden ratio of a segment is one of many mathematical concepts that permeate the history of mathematics. Numerous books have been written on this subject because, often in unexpected ways, the golden ratio appears in various mathematical and physical investigations, as noted, for instance, in [1–3]. Let us recall its definition and some of its properties. The golden ratio of a segment originates from the classical problem presented in Euclid's Elements Book II ([4], Prop. 11) : To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square of the remaining segment. This can be visualized with the so-called golden rectangle (see Figure 1), where the larger rectangle and the smaller rectangle have sides in the same ratio.

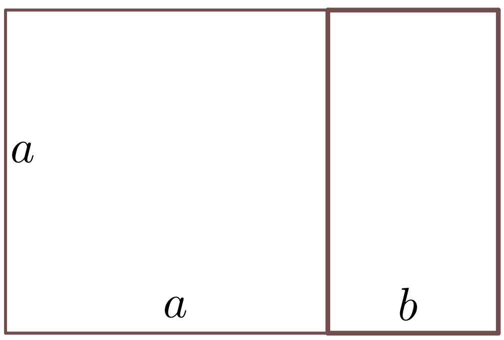

Figure 1. The Golden rectangle.

This problem reappears in Book VI ([4], Prop. 30): to cut a given straight line in extreme and mean ratio. If we denote the lengths of the two parts of a segment by a and b, with a representing the larger part, then we have

$$\varphi = \frac{a+b}{a} = \frac{a}{b}$$

where the Greek letter φ (phi) denotes the so-called golden ratio, since the above relation holds for any b, we can choose $b = 1$, obtaining $\varphi = a$, we derive the equation

$$\frac{1+\varphi}{\varphi} = \varphi,$$

or after usual simplifications

$$\varphi^2 - \varphi - 1 = 0.$$

Hence, since φ is the ratio of positive quantities, we select the positive root, namely

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.618033\ldots$$

According to ([5], p. 6)

> Some of the greatest mathematical minds of all ages, from Pythagoras and Euclid in ancient Greece, through the medieval Italian mathematician Leonardo of Pisa and the Renaissance astronomer Johannes Kepler, to present-day scientific figures such as Oxford physicist Roger Penrose, have spent endless hours over this simple ratio and its properties.

The golden ratio is closely related to the construction of the regular pentagon and the regular decagon inscribed in a circle. In the case of the decagon, it is divided into ten isosceles triangles, each with a vertex angle of 36° and base angles of 72°. In each of these triangles, the ratio between one of the oblique sides and the base is the golden number φ. Thus, each of the triangles is golden. The construction of the golden triangle is found in Euclid's Elements Book VI ([4], Prop. 10): To construct an isosceles triangle having each of the angles at the base double of the remaining one. If ABC is a Golden triangle (Figure 2), it has the property

$$\frac{AB}{BC} = \varphi$$

After recalling these basic notions, let's illustrate our geometric construction. Let $\triangle ABC$ be a triangle (see Figure 3), and on each of its sides we construct the isosceles triangles $\triangle AVB$, $\triangle BV'C$, and $\triangle CV''A$. Suppose $AV = BV = k\,AB$, $BV' = CV' = k\,BC$, $CV'' = AV'' = k\,CA$, for some $k > 1/2$, (large enough for $\triangle ABV$, $\triangle BCV'$, and $\triangle CAV''$ can be formed), and $k \neq 1$. Let A', B', and C' be the midpoints of BC, CA, and AB, respectively. Then the lines AA', BB', and CC' concur, say at G. This point G is the centroid of $\triangle ABC$. Let T be the intersection of the lines GA and VV''.

To proceed with our geometric construction, we need to identify the conditions under which $\triangle B'TC'$ is isosceles on the base $B'C'$. If that occurs, then the median TH must also serve as the height and bisector of the angle $\angle B'TC'$, for the well-known properties of isosceles triangles. This means that HT must coincide with the axis $V'A'$ of the base BC of $\triangle ABC$, which must therefore be isosceles on the base BC. In the same way, for $\triangle C'T'A'$ and $\triangle A'T'B'$ to be isosceles, then $\triangle ABC$ must be isosceles both on the base AC and on the base AB. It follows that $\triangle ABC$ must be equilateral. Consequently, we will start from an equilateral triangle in the next section, choosing $k = \varphi$ since this case yields several interesting configurations.

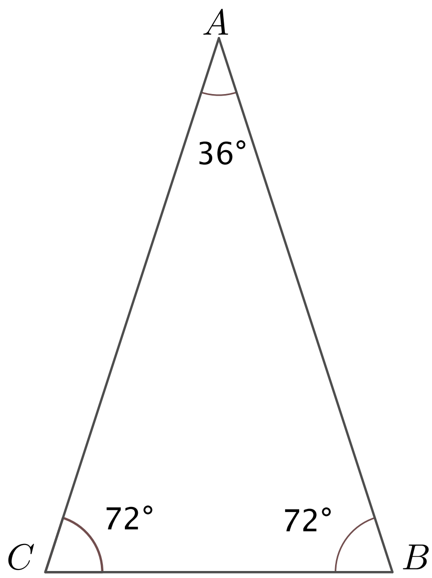

Figure 2. The Golden triangle.

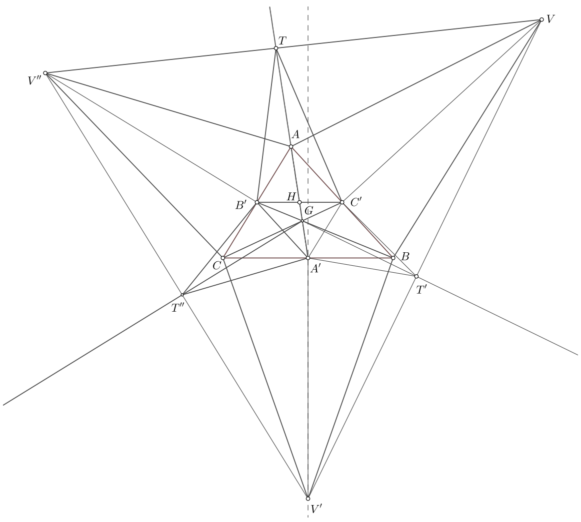

Figure 3. The triangle ABC.

2. The Equilateral Triangle

Let $\triangle ABC$ be an equilateral triangle (see Figure 4) with sides $AB = BC = CA = a$. Let $\triangle AVB$, $\triangle BV'C$, and $\triangle CV''A$ be three golden triangles, for which $AV = BV' = CV'' = a\varphi$, where φ is the golden number.

Let $\triangle A'B'C'$ be the equilateral triangle (see Figure 5) whose vertices are the midpoints of the sides of $\triangle ABC$, such that $B'C' = \frac{a}{2}$. Let r, s, and t denote the axes of $\triangle ABC$ (and of $\triangle A'B'C'$), and draw line v that connects the vertices V and V''. The lines r and v will be perpendicular to one another. In fact, the triangle $\triangle VGV''$ is isosceles, since $GV = GC' + C'V$ is congruent to $GV'' = GB' + B'V''$, furthermore $\angle C'GA \cong \angle GB'A = 60°$, so that GT is the bisector of $\angle VGV''$ and perpendicular to side VV''.

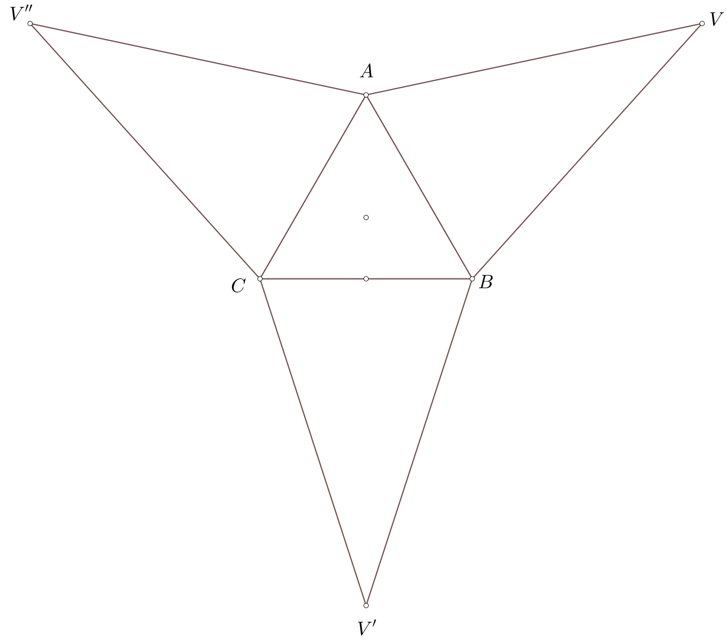

Figure 4. The equilateral triangle $\triangle ABC$ with its golden triangles.

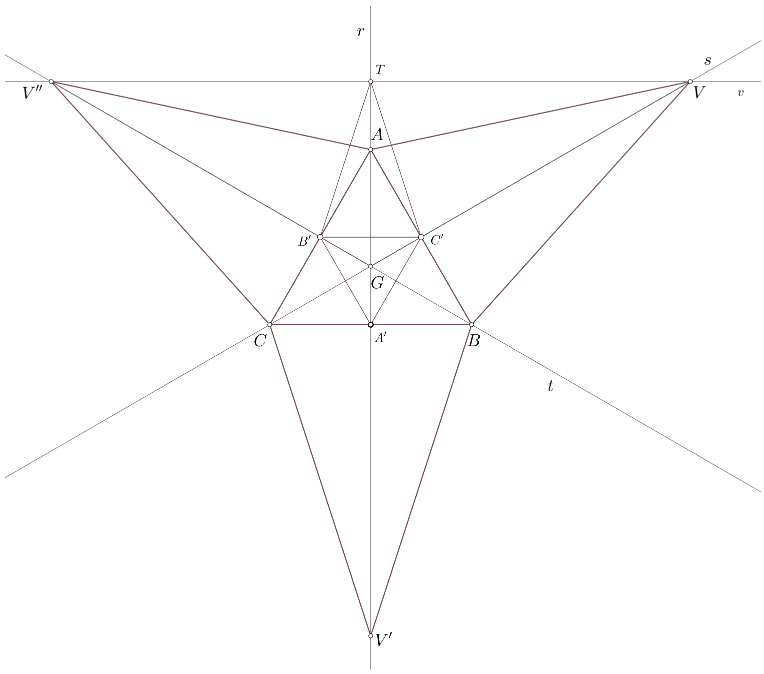

Figure 5. The triangle $\triangle B'TC'$.

We denote with T the intersection of lines r and v.

Theorem 1. *The triangle $\triangle B'TC'$ is isosceles and golden.*

Proof. Let us consider, for simplicity, Figure 6.

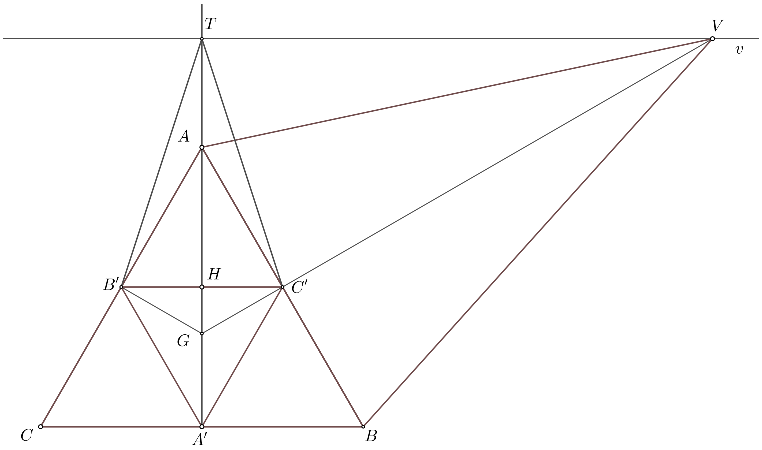

Figure 6. The triangle $\triangle B'TC'$.

Triangles $\triangle TC'G$ and $\triangle TB'G$ are congruent, sharing side CT, with $GC' \cong GB'$, and $\angle C'GT \cong \angle TGB' = 60°$, so $C'T \cong B'T$ and the triangle $\triangle B'TC'$ is isosceles. Now, we must demonstrate that it is also golden:

$$\frac{C'T}{B'C'} = \varphi.$$

First of all, in the right triangle $\triangle GHC'$, the angle $\angle C'GH = 60°$, so in the right triangle $\triangle GTV$ the angle $\angle GVT = 30°$, hence

$$GT = \frac{1}{2}GV, \quad GH = \frac{1}{2}GC'.$$

One has

$$GC' = A'G = \frac{2}{3}A'H = \frac{\sqrt{3}}{6}a$$

$$GV = GC' + C'V = \frac{\sqrt{3}}{6}a + \frac{a}{2}\sqrt{4\varphi + 3}$$

so

$$GT = \frac{1}{2}GV = \frac{\sqrt{3}}{12}a + \frac{a}{4}\sqrt{4\varphi + 3}$$

$$HG = \frac{1}{3}A'H = \frac{\sqrt{3}}{12}a$$

thus

$$TH = GT - HG = \frac{a}{4}\sqrt{4\varphi + 3}.$$

With

$$TC' = \sqrt{HC'^2 + HT^2} = \frac{a\varphi}{2}$$

one gets

$$\frac{C'T}{B'C'} = \varphi.$$

□

By adding more golden triangles to the other two sides of triangle $\triangle A'B'C'$, we create the first configuration of Figure 7:

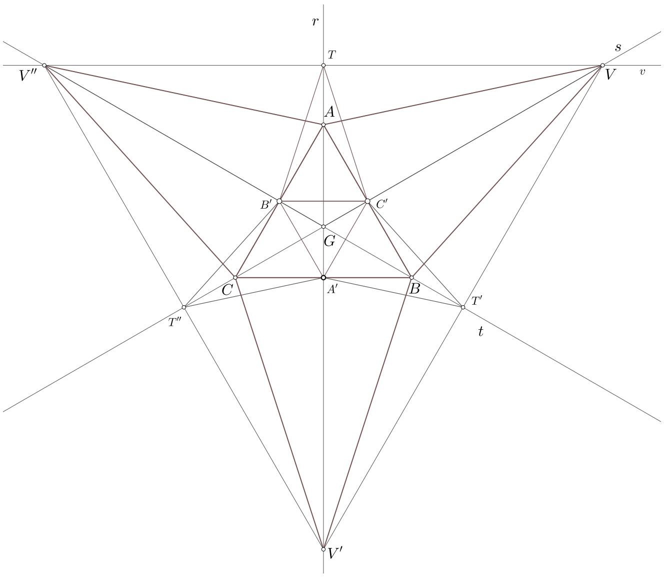

Figure 7. The first configuration of golden triangles.

If we apply this construction to the sides of the medial triangle of triangle $\triangle A'B'C'$, we create a second configuration of golden triangles, as illustrated in Figure 8. By using a similar approach, we can generate an infinite number of configurations.

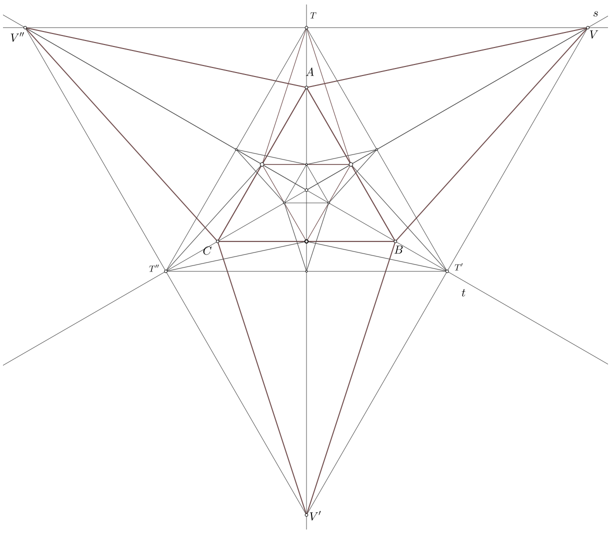

Figure 8. A second configuration of golden triangles.

3. A Hidden Harmony

Consider Figure 9, where we reference a Cartesian coordinate system. Let Ω be the center of circle Γ, which passes through V, A, and V''.

Theorem 2. $\Omega A / AV' = \varphi$.

Proof. We have to demonstrate that

$$\frac{\Omega A}{AV'} = \varphi.$$

To achieve this goal, we need to determine the coordinates of Ω. First, we have

$$A(0, \frac{a}{2}\sqrt{3}), \quad B\left(\frac{a}{2}, 0\right), \quad M\left(\frac{a}{4}, \frac{a\sqrt{3}}{4}\right), \quad V(x_0, y_0)$$

Since

$$BV = \sqrt{BK^2 + VK^2} \quad (\text{with } BV = a\varphi)$$

and

$$MV = \frac{a}{2}\sqrt{4\varphi + 3},$$

we have

$$(x_0 - \frac{a}{2})^2 + y_0^2 = a^2\varphi^2$$

$$(x_0 - \frac{a}{4})^2 + (y_0 - \frac{a\sqrt{3}}{4})^2 = \frac{a^2(4\varphi + 3)}{4}$$

from which we get the coordinates of $V(x_0, y_0)$:

$$V\left(\frac{a}{4} + \frac{a\sqrt{12\varphi + 9}}{4}, \quad \frac{a\sqrt{3}}{4} + \frac{a\sqrt{4\varphi + 3}}{4}\right).$$

Thus, we acquire the coordinates of the midpoint H:

$$H\left(\frac{a}{8} + \frac{a\sqrt{12\varphi + 9}}{8}, \quad \frac{3\sqrt{3}a}{8} + \frac{a\sqrt{4\varphi + 3}}{8}\right).$$

The axis equation of segment AV is

$$y = -\frac{\varphi^2\sqrt{3} + \sqrt{4\varphi + 3}}{\varphi}x + \frac{a\varphi^2\sqrt{4\varphi + 3} + a\sqrt{3}(2\varphi + 1)}{2\varphi},$$

where $-\frac{\varphi^2\sqrt{3} + \sqrt{4\varphi + 3}}{\varphi}$ is its slope. It intersects the y-axis at point Ω, so we have

$$\Omega\left(0, \frac{a\varphi^2\sqrt{4\varphi + 3} + a\sqrt{3}(2\varphi + 1)}{2\varphi}\right).$$

Summing up, we have

$$\Omega A = \Omega O - AO,$$

so:

$$\Omega A = \frac{a\varphi(\sqrt{4\varphi + 3} + \sqrt{3})}{2}.$$

And finally, with

$$AV' = AO + OV' = \frac{a(\sqrt{4\varphi + 3} + \sqrt{3})}{2},$$

we find

$$\frac{\Omega A}{AV'} = \varphi.$$

□

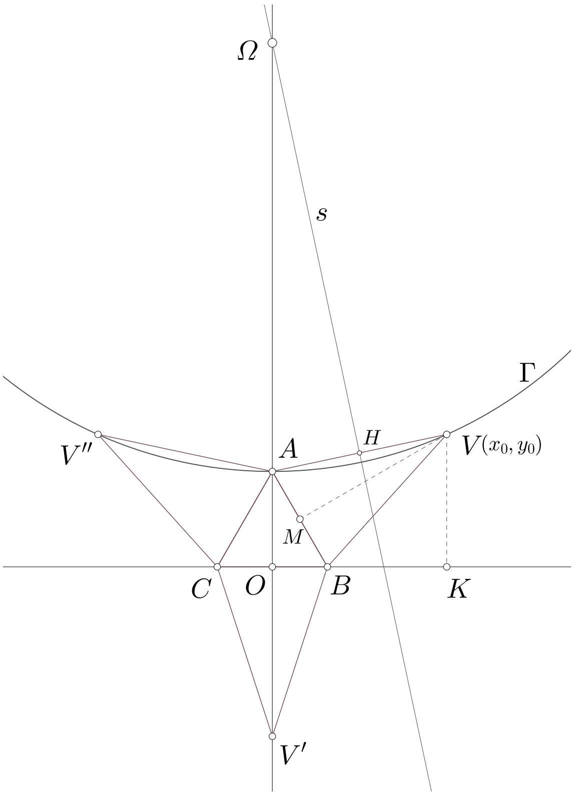

Figure 9. The segments ΩA and AV'.

This indicates that if we report the segment AV' (see Figure 10) on Γ, we obtain the golden triangle $\triangle A\Omega M$.

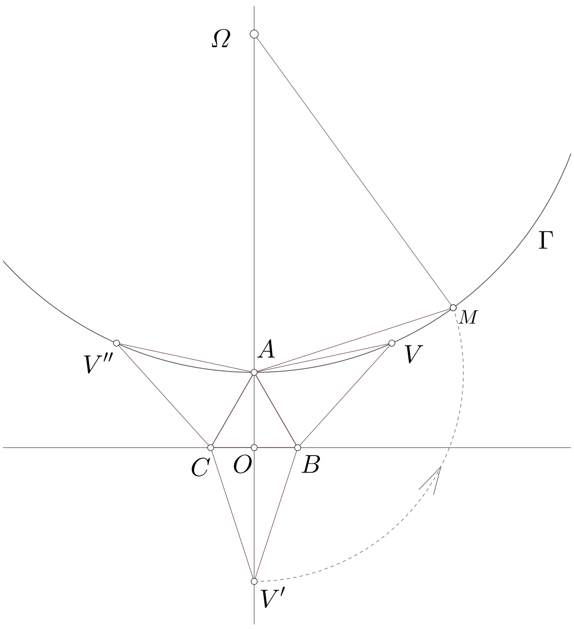

Figure 10. The golden triangle $\triangle A\Omega M$.

Thus, AM denotes a side of the regular decagon inscribed within Γ (see Figure 11).

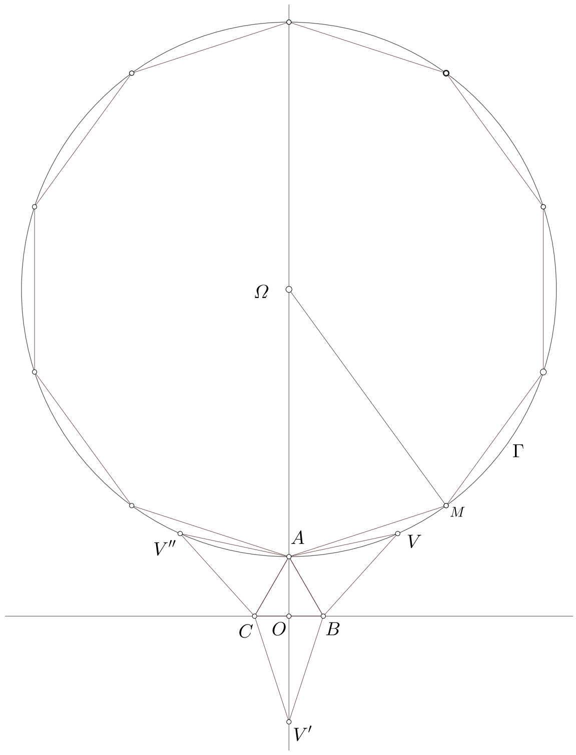

Figure 11. The regular decagon.

Moreover note that $\Omega V'/\Omega A = \varphi$. In fact:

$$\Omega V' = \Omega A + AV'$$

$$\frac{\Omega V'}{\Omega A} = 1 + \frac{AV'}{\Omega A} = 1 + \frac{1}{\varphi} = \frac{\varphi + 1}{\varphi} = \varphi.$$

If we draw the t-axis of side AB of triangle $\triangle ABC$ (see Figure 12), one can analytically verify that it intersects the circumference Γ precisely at the point M. This is a hidden property of the entire construction.

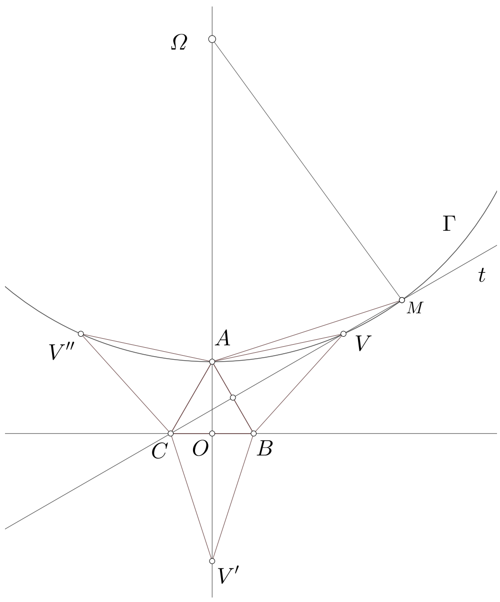

Figure 12. The property of t axes.

Furthermore, given the size of the angles marked in the figure (see Figure 13), AV is the side of the regular pentadecagon (a 15-sided polygon) inscribed in Γ (see Figure 14).

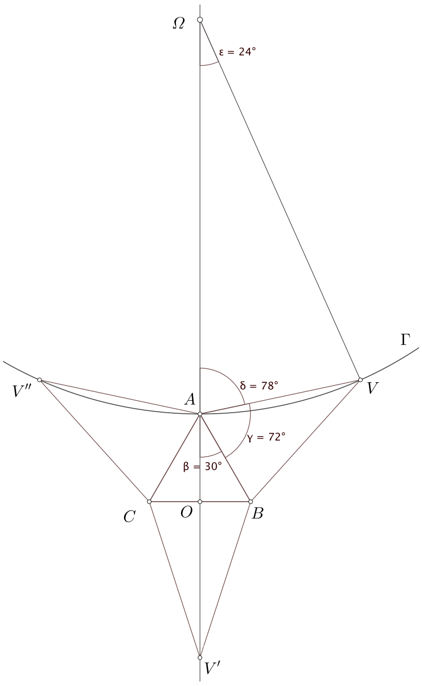

Figure 13. The side AV of the regular pentadecagon inscribed in Γ.

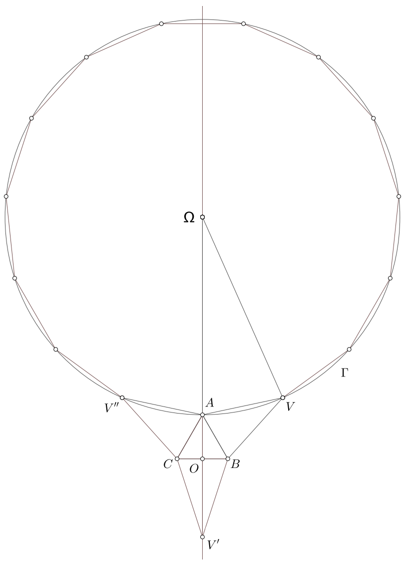

Figure 14. The regular pentadecagon.

Now, if we draw a perpendicular line from the midpoint H of segment AO (see Figure 15) to AO, it intersects the circumference Γ at points R and Q. Considering the right-angled triangle $\triangle HOQ$, we derive

$$\cos\theta = \frac{OH}{OQ} = \frac{1}{2},$$

so that

$$\theta = 60^\circ \quad \text{and} \quad \angle ROQ = 120^\circ.$$

Since it lies on the same arc of the circle where the angle at the circumference with vertex at P lies, we have

$$\angle RPQ = 60^\circ.$$

Thus, we can easily deduce that triangle $\triangle PQR$ is equilateral.

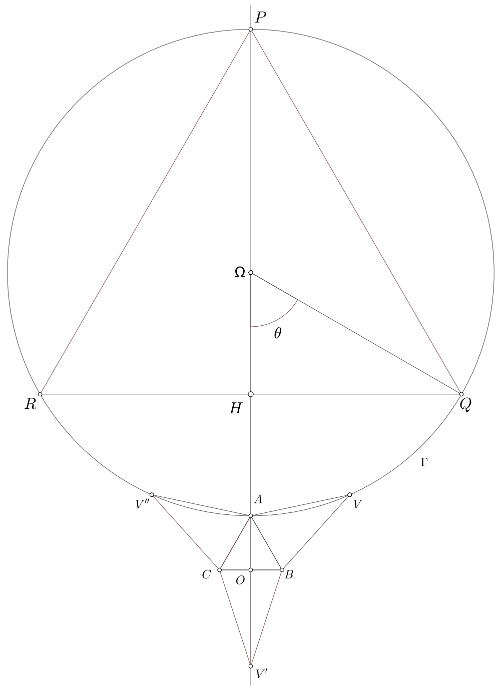

Figure 15. The equilateral triangle $\triangle PQR$.

So, AQ is the side of the regular hexagon inscribed in Γ (see Figure 16).

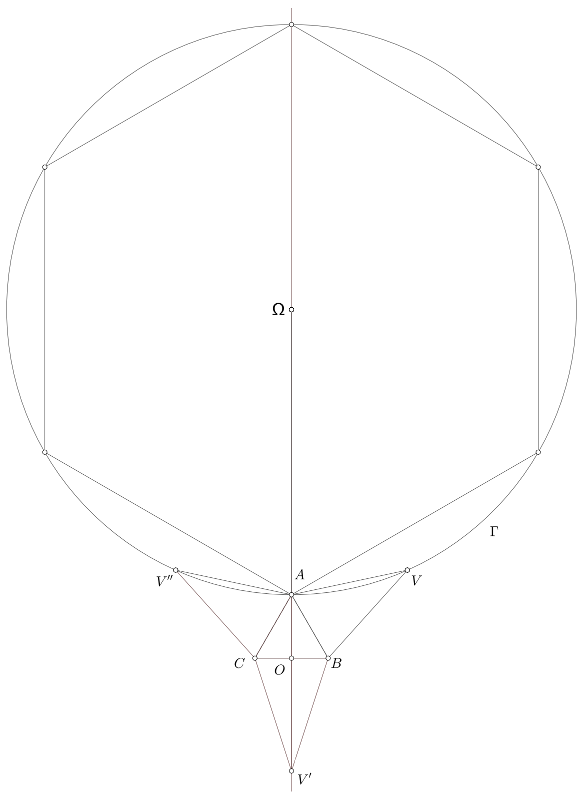

Figure 16. The regular hexagon.

4. The Square

Even in the relationship between the square and the golden triangle, configurations with characteristics similar to those in the previous sections can be found.

Let $ABCD$ be a square with side length a (see Figure 17). On each side, consider four golden triangles with vertices V, V', V'', and V''', such that each side of the triangles has a length of $a\varphi$, where φ represents the golden ratio.

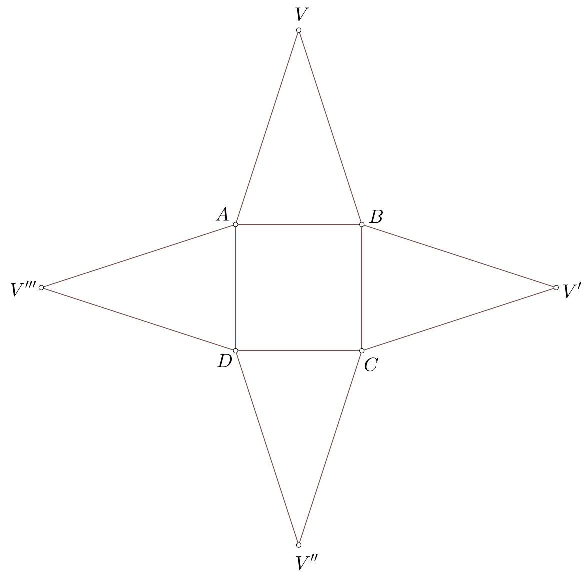

Figure 17. The square $ABCD$.

Consider the square $A'B'C'D'$ whose vertices are the midpoints of the square $ABCD$ (see Figure 18).

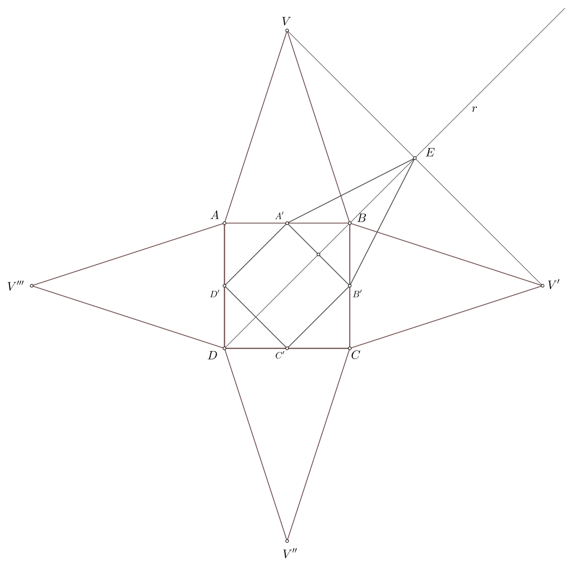

Figure 18. The square $A'B'C'D'$ and the new triangle.

Let E be the intersection point of the extended diagonal r of square $A'B'C'D'$ and the segment VV', which is perpendicular to it.

Theorem 3. *The triangle $\triangle A'B'E$ is golden.*

Proof. Consider Figure 19.

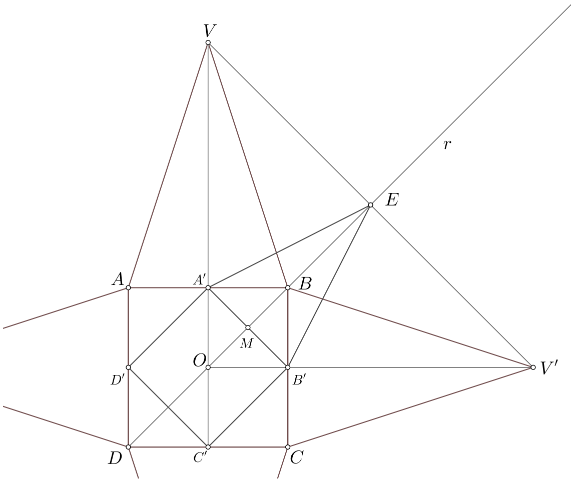

Figure 19. The golden triangle $\triangle A'B'E$.

In the isosceles right triangle $\triangle VOV'$, the segment OE serves as the altitude to the hypotenuse VV', so we have

$$OE = \frac{OV}{\sqrt{2}} = \frac{\sqrt{2}}{2}OV.$$

$$ME = OE - OM = \frac{\sqrt{2}}{2}OV - \frac{\sqrt{2}}{4}a = \frac{\sqrt{2}}{2}\left(OV - \frac{a}{2}\right).$$

Since

$$OV = OA' + A'V = \frac{a}{2}(1 + \sqrt{4\varphi + 3}),$$

one finds

$$ME = \frac{a\sqrt{8\varphi + 6}}{4}.$$

Therefore,

$$A'E = \sqrt{ME^2 + A'M^2} = \frac{\sqrt{2}}{2}a\varphi.$$

Since

$$A'B' = \frac{\sqrt{2}}{2}a,$$

one has

$$\frac{A'E}{A'B'} = \varphi.$$

□

We obtain an initial configuration of golden triangles similar to the one previously obtained with the equilateral triangle (see Figure 20).

From this configuration, we can derive infinitely many configurations for the square. However, similar configurations can also be obtained with other regular polygons, such as in the regular hexagon (see Figure 21).

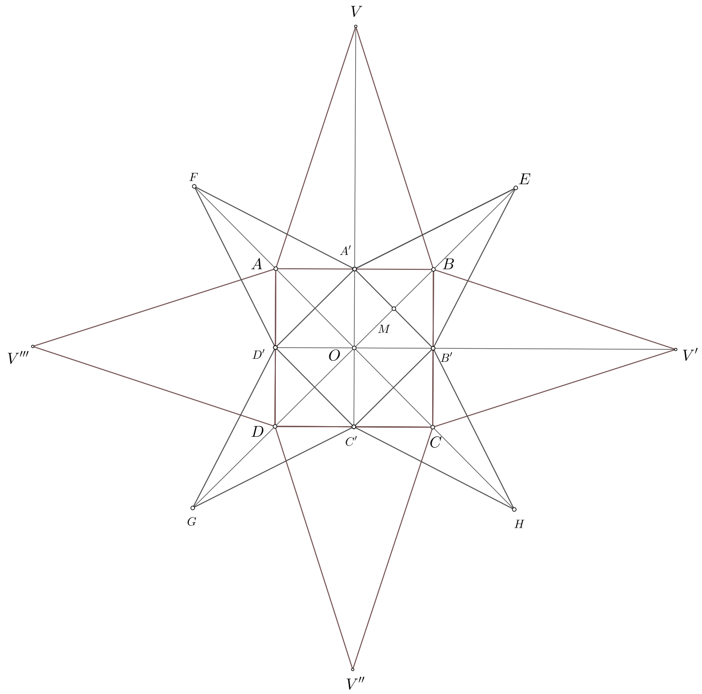

Figure 20. A first configuration of golden triangles.

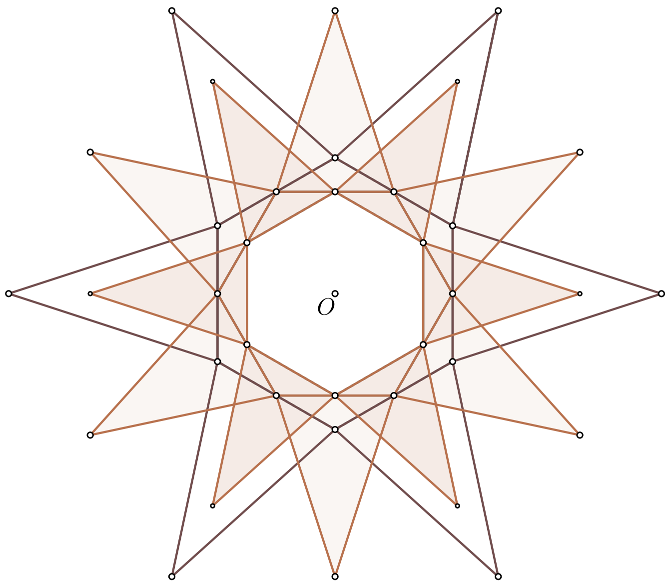

Figure 21. A configuration of golden triangles in the regular hexagon.

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