

Article

On Yiu's Equilateral Triangles Associated with a Kiepert Hyperbola [†]

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[†] In memory of my father, Hsin-Hsi Perng, and my colleague, Dr. Boyd Coan.

Abstract: In 2014, Paul Yiu constructed two equilateral triangles inscribed in a Kiepert hyperbola associated with a reference triangle. It was asserted that each of the equilateral triangles is triply perspective with the reference triangle, and in each case, the corresponding three perspectors are collinear. In this note, we provide proof of his assertions. Furthermore, as an analogue of Lemoine's problem, we formulated and answered the question of how to recover the reference triangle given a Kiepert hyperbola, one of the two Fermat points and one vertex of the reference triangle.

Keywords: Kiepert hyperbola; Lemoine's problem; Yiu's equilateral triangles; triply perspective triangles on a conic; Hessian lines

MSC: 51M04; 51A05

1. Introduction

Historically, the Kiepert hyperbola was introduced by Ludwig Kiepert [1] in 1869 along with solving Lemoine's problem [2]. The question published by Lemoine asks about the following construction problem: Given one vertex of each of the equilateral triangles placed on the sides of a triangle, construct the original triangle (Wikipedia [2,3]). Kiepert solved the problem and generalized to a construction in which the equilateral triangles on the sides of the original triangle were replaced by isosceles triangles. For example, denoting the base angle of the isosceles triangles by ϕ , Kiepert showed that the lines connecting the vertices of the given triangle and the corresponding peaks of the isosceles triangles concur. By varying ϕ , the locus of the points of concurrence gives a rectangular hyperbola known as a Kiepert hyperbola. This hyperbola contains many interesting points, usually called *triangle centers*, in reference to the original triangle. In the case of Lemoine's problem, this corresponds to $\phi = \frac{\pi}{3}$ and the point of concurrence is called the First Fermat point F_1 , which arises as an interesting optimization problem [4]. For $\phi = 0$, the point of concurrence is the centroid M of the reference triangle, and for $\phi = -\frac{\pi}{3}$, one gets the second Fermat point F_2 . As an analogue of Lemoine's problem and in the setting of Yiu's problem, we may pose the following question: Given a Kiepert hyperbola, one of the two Fermat points and one vertex of the reference triangle, construct the other two vertices of the reference triangle. The question is answered directly by the main results of our note.

For a scalene triangle ABC , let F_1 (resp. F_2) be its first (resp. second) Fermat point. The Kiepert hyperbola $\mathcal{K}$ we need here is the unique conic passing through the five points A, B, C, F_1 and F_2 . Since it is well known that $\mathcal{K}$ must also pass through the centroid M of ABC , it is convenient to construct $\mathcal{K}$ from the five points A, B, C, F_1 and M . We will call ABC the *reference triangle* of $\mathcal{K}$. Yiu's problem is stated as follows. The orientation of the



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triangles may be flipped for different configurations, but this situation can be understood better in a more general setting such as in the proof of Proposition 1 or in its corollaries.

Theorem 1. Let $\mathcal{K}$ be a Kiepert hyperbola with reference triangle ABC . Let $\mathcal{C}$ (resp. $\mathcal{C}'$) be the circle centered at F_2 (resp. F_1) with radius F_2F_1 . The circle $\mathcal{C}$ (resp. $\mathcal{C}'$) intersects $\mathcal{K}$ at three points P, Q, R (resp. P', Q', R') other than F_1 (resp. F_2). Then one has (see Figure 1)

- (a) Triangle PQR (resp. $P'Q'R'$) is equilateral;
- (b) Triangle PQR (resp. $P'R'Q'$) is triply perspective with triangle ABC , in the sense that PA, QB, RC meet at a common point P_1 ; PB, QC, RA meet at a common point P_2 ; PC, QA, RB meet at a common point P_3 (resp. $P'A, Q'C, R'B$ meet at a common point P'_1 ; $P'C, Q'B, R'A$ meet at a common point P'_2 ; $P'B, Q'A, R'C$ meet at a common point P'_3); and moreover
- (c) The three perspector P_1, P_2, P_3 in (b) are collinear in a line that also passes through the three perspector P'_1, P'_2, P'_3 constructed from the triangle $P'R'Q'$.

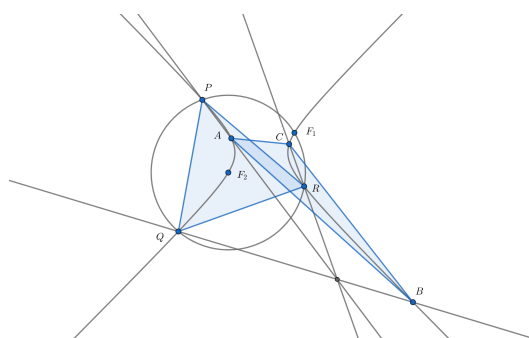


Figure 1. An equilateral triangle inscribed in a Kiepert hyperbola.

The author first learned Paul Yiu's problem in July 2018 from the Mathoverflow website ([5]; See also [6]), where Dao Thanh Oai posted and promoted Yiu's problem. According to Dao, this result is comparable to the case of Morley triangles or Napoleon triangles, and Dao placed a great importance to this construction as Kiepert hyperbola is a very special conic which passes through many triangle centers. The author established the truth of all the statements in Yiu's problem in November 2018, using analytic method with the help of computer algebra system, SAGE. The idea of this method involves a direct computation by introducing the slope variable $m = (y - y_2)/(x - x_2)$, where (x_2, y_2) is the second Fermat point. By elimination, m will need to satisfy a cubic equation $p(m) = 0$. Then by manipulations with symmetric functions on the roots of $p(m)$, one can prove all the statements of the problem. In this note, we tried to solve the problem using synthetic methods as much as possible. The statements in Theorem 2 allow one to see the general situation more transparently. Once the uniqueness of the construction is verified, this can be used to reconstruct the reference triangle, assuming that the Kiepert hyperbola, one of the two Fermat points, and one of the vertices of the reference triangle are given.

2. Preliminaries

We gather some definitions and known facts here. These are to be referred to only when necessary. The results from Lemma 1 to Proposition 1 and its corollaries may be classical, but it is not easy to sieve through the results (cf. [7–10]) to get what we need here; therefore, we provide direct proofs.

1. The first Fermat point F_1 (or the first isogonic center) of a triangle is constructed by erecting an equilateral triangle from the outside of each side of the triangle: The lines joining each outer vertex to the opposite vertex of the triangle concur at F_1 .

2. Similarly, the second Fermat point F_2 (or the second isogonic center) of a triangle is constructed by erecting an equilateral triangle from the inside of each side of the triangle.
3. The Nine Point Circle (or the Feuerbach circle) for a triangle is the circle that passes through the following nine points: the midpoint of each side of the triangle, the foot of each altitude, and the midpoint of the line segment from each vertex of the triangle to the orthocenter.
4. The Kiepert hyperbola defined in the beginning of the introduction is a rectangular hyperbola (meaning that the asymptotes are perpendicular to each other). It is known that for a triangle inscribed in a rectangular hyperbola, the orthocenter of the triangle lies on the rectangular hyperbola.
5. A version of the Feuerbach Conic Theorem we will use is the following: Let ABC be a triangle inscribed in a rectangular hyperbola. Then the Feuerbach circle of ABC passes through the center of the rectangular hyperbola. Furthermore, the center of the Kiepert hyperbola associated with ABC is midway between the two Fermat points (isogonic centers) ([11,12]).
6. Any two conics are related by a projective collineation and a projective correlation: Any three distinct points on the first conic can be made to correspond to any three distinct points of the second (Ex.4 on page 79 of [13], or Theorem 6.4.1 of [14]).

Lemma 1. Let ABC and $A'B'C'$ be two triangles inscribed in a conic $\mathcal{K}$ such that ABC is perspective with $A'B'C'$ with perspector P , and ABC is perspective with $B'C'A'$ with perspector Q . Then there exists R in line PQ such that ABC is perspective with $C'A'B'$ with perspector R .

Proof. The existence of R such that ABC is perspective with $C'A'B'$ is guaranteed since it is well known that doubly perspective implies triply perspective. It suffices to show that the three perspectors P, Q, R are collinear. Applying Pascal's theorem (see Figure 2) to the ordered lists $\{C', B', A'\}$ and $\{A, B, C\}$ shows that Q, R and P are collinear. $\square$

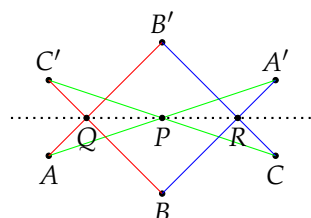


Figure 2. A Line through the perspectors.

Definition 1. Let ABC be a triangle inscribed in a conic $\mathcal{K}$. Let the line tangent to $\mathcal{K}$ at A meet line BC at P , the line tangent to $\mathcal{K}$ at B meet line CA at Q , and the line tangent to $\mathcal{K}$ at C meet line AB at R . Then by a limiting case of Pascal's theorem, it can be shown that P, Q, R are collinear. We call this line the **Hessian line** of ABC with respect to $\mathcal{K}$ (cf. p.130 of [15]).

Lemma 2. Let ABC and $A'B'C'$ be two triply perspective triangles inscribed in a conic. Then the line passing through the three perspectors (see Lemma 1) coincides with the Hessian line of ABC with respect to the conic.

Proof. Let the tangent line at A (resp. B) meet line BC (resp. line CA) at P_1 (resp. P_2). Let ABC be perspective with $A'B'C'$ (resp. $C'A'B'$) of perspector Q_1 (resp. Q_3). Applying Pascal's theorem (see Figure 3) to the ordered lists $\{A, B, C'\}$ and $\{C, A, A'\}$ (resp. $\{B, C, A'\}$ and $\{A, B, B'\}$) shows that P_1 (resp. P_2) lies on line Q_1Q_3 . Hence lines P_1P_2 and Q_1Q_3 coincide. $\square$

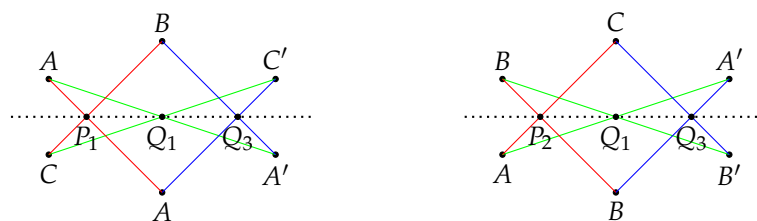


Figure 3. Agreement of the Hessian line and the line through perspectors.

The following statement is simple and useful; it may have been known to A. Cayley (1851) in his article on triply perspective triangles inscribed in a conic, published in the *Philosophical Transactions of the Royal Society*. As the author has not found a specific reference, we state the result here and provide a proof.

Proposition 1. *Two inscribed triangles on a conic are triply perspective if and only if they share the same Hessian line.*

Proof. The necessity follows from Lemma 2. To prove the other implication, assume that two inscribed triangles ABC and $A'B'C'$ on a conic share the same Hessian line. We now prove that they are triply perspective.

By 2.6, we can apply a projective collineation and a projective correlation such that the result gives an equilateral triangle ABC (still using the same letters) on a circle. In this case, the Hessian line is the line at infinity. Since ABC and $A'B'C'$ share the Hessian line, the Hessian line for $A'B'C'$ is also the line at infinity. It follows that the triangle $A'B'C'$ is also an equilateral triangle. It is clear now that the two triangles are triply perspective when the vertices are denoted with different orientations. Mapping back by the inverse collineation and correlation, we conclude that the original triangles are triply perspective. $\square$

Corollary 1. *Let ABC and $A'B'C'$ be two inscribed triangles on a conic and let L be the Hessian line of ABC . If ABC and $A'B'C'$ are perspective with a perspector $P \in L$. Then the two triangles are triply perspective.*

Proof. As in the proof of Proposition 1, we may assume that both triangles lie on a circle, and ABC is equilateral. Now the Hessian line L is the line at infinity, and the condition shows that both triangles are perspective at a point at infinity. This forces AA', BB', CC' to be three parallel lines, and we obtain an equilateral triangle $A'B'C'$ on the same circle with a different orientation. Clearly the two triangles are triply perspective. $\square$

Corollary 2. *Let $ABC, A'B'C', A''B''C''$ be three inscribed triangles on a conic and let L be the Hessian line of ABC . Assume that ABC is perspective with $A'B'C'$ (resp. $A''B''C''$) of a perspector $P \in L$ (resp. $Q \in L$). Then $A'B'C'$ is triply perspective with $A''C''B''$, i.e., they are reverse triply perspective.*

Proof. In the proof of Corollary 1, ABC has a different orientation with respect to the equilateral triangles $A'B'C'$ and $A''B''C''$ on a circle. Thus $A'B'C'$ and $A''B''C''$ have the same orientation. It follows that $A'B'C'$ is triply perspective with $A''C''B''$. $\square$

3. Proof of Theorem 1

Part (a) of Theorem 1 follows immediately from a more general result, stated as Proposition 2, by taking $\mathcal{C}$ to be the Kiepert hyperbola $\mathcal{K}$ associated with the reference triangle ABC , and F_1 and F_2 to be the two Fermat points of ABC . The formulation of this

result was inspired by a comment made by Elias M. Hagos [16] on 8 February 2021, upon his reading of the preprint of this note.

Proposition 2. *Given a rectangular hyperbola $\mathcal{C}$ of center O , let F_1, F_2 be a pair of antipodal points on $\mathcal{C}$, i.e., O is the midpoint of the segment F_2F_1 . Assume that the circle centered at F_2 of radius F_2F_1 intersects $\mathcal{C}$ at P, Q, R (other than F_1). Then the triangle PQR is equilateral.*

Proof. By construction, P, Q, R, F_1 lie on a circle of center F_2 and radius F_2F_1 . Since $\mathcal{C}$ is a rectangular conic containing the points P, Q, R, F_2 , we can apply Feuerbach's theorem (2.5) to the triangles F_2PQ, F_2QR, F_2RP and PQR which are inscribed in $\mathcal{C}$. Let D, E, F be the midpoints of PQ, QR, RP , respectively, and let U, V, W be the midpoints of F_2P, F_2Q, F_2R , respectively. Note that by construction,

$$F_2U = F_2V = F_2W = \frac{1}{2}F_2P = \frac{1}{2}F_2F_1 = F_2O,$$

where O is both the center of the rectangular hyperbola $\mathcal{C}$ and the midpoint of F_2F_1 . By Feuerbach's theorem applied to the four triangles and the conic $\mathcal{C}$ with center O , it follows that

D, U, V, O are concyclic with center F_2 ,

E, V, W, O are concyclic with center F_2 ,

F, W, U, O are concyclic with center F_2 ,

and

D, E, F, O are concyclic with center F_2 ,

where the last statement follows from the first three. Now, since F_2 is the circumcenter of PQR , the points D, E, F are perpendicular feet from F_2 to the sides of the triangle PQR . Pythagorean Theorem shows now that PQR is equilateral. $\square$

Now, we reverse the process by looking at the following *model* in Theorem 2, starting with F_2, F_1 and an equilateral triangle PQR . Theorem 2 would conclude the rest of the statements in Theorem 1, provided that we can show that the candidate triangles $P''Q''R''$ yield all generic triangles ABC with F_1, F_2 as the two Fermat points (see Figure 4, where $P''Q''R''$ is labeled ABC).

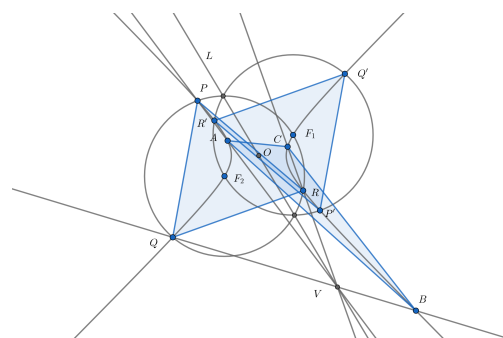


Figure 4. Construction of a reference triangle.

Theorem 2. *Let $F_2 = (-1, 0), F_1 = (1, 0)$ and let S_1 (resp. S_2) be a circle centered at F_2 (resp. F_1) with radius $F_2F_1 = 2$. Let PQR be an equilateral triangle inscribed in S_1 and let $P'Q'R'$ (an equilateral triangle inscribed in S_2) be the reflection of PQR with respect to the center $(0, 0)$ of F_2F_1 . Then*

(a) The two triangles PQR and $P'Q'R'$ are triply perspective with perspectors consisting of $(0, 0)$ and the two intersection points of circles S_1 and S_2 . Consequently, the perspectors lie on the radical axis L of S_1 and S_2 .

(b) There is a unique conic $\mathcal{K}$ passing through P, Q, R, P', Q', R', F_2 and F_1 .

(c) For any point $V \in L$, let P'' be the other point of intersection of the line PV and $\mathcal{K}$. Define Q'' and R'' in a similar way. Then PQR is triply perspective with $P''Q''R''$, and the other two perspectors lie on L . Similarly, $P'Q'R'$ is triply perspective with $P''R''Q''$ (i.e. one needs to reverse the orientation).

(d) Let $P''Q''R''$ be the triangle constructed in (c). Then $\mathcal{K}$ is the Kiepert hyperbola through the two Fermat points F_1 and F_2 with respect to the reference triangle $P''Q''R''$.

(e) For two distinct triangles $P_1Q_1R_1$ and $P_2Q_2R_2$ constructed as in (c), $P_1Q_1R_1$ is triply perspective with $P_2Q_2R_2$. Both $P_1Q_1R_1$ and $P_2Q_2R_2$ are triply perspective with $P'R'Q'$.

Proof. (a) It is clear that S_1 and S_2 intersect at $\{(0, \sqrt{3}), (0, -\sqrt{3})\}$ and L is the line with equation $x = 0$. The result follows by angle chasing and symmetry, noting that the cord joining the two intersection points above is of length $2\sqrt{3}$ corresponding to a circular angle of 60° .

(b) Denoting P (using a rational parameterization of circle S_1) by

$$P = \left(-1 + \frac{2(1-t^2)}{1+t^2}, \frac{4t}{1+t^2} \right)$$

and turning counterclockwise 120° successively, we have

$$Q = \left(\frac{-2(\sqrt{3}+1)}{t^2+1}, \frac{-(\sqrt{3}t^2+2t-\sqrt{3})}{t^2+1} \right)$$

and

$$R = \left(\frac{2(\sqrt{3}t-1)}{t^2+1}, \frac{\sqrt{3}t^2-2t-\sqrt{3}}{t^2+1} \right).$$

Then the unique conic $\mathcal{K}$ through P, Q, R, F_1, F_2 has the equation

$$(3t^2-1)x^2 + (2t^3-6t)xy + (1-3t^2)y^2 + (1-3t^2) = 0$$

and it is evident that $\mathcal{K}$ also passes through P', Q', R' , which are the reflection of P, Q, R with respect to the origin.

(c) By the result of (a) and Lemma 2, we know that L is the Hessian line of triangle PQR with respect to the conic $\mathcal{K}$. Now by the construction of $P''Q''R''$, triangle PQR is perspective with $P''Q''R''$ with perspector V , hence by Corollary 1, the two triangles are triply perspective with the other two perspectors lying on L . Since triangle PQR is triply perspective with both $P'Q'R'$ and $P''Q''R''$, it follows from (a) and Corollary 2 that $P'Q'R'$ is triply perspective with $P''R''Q''$ (i.e. one needs to reverse the orientation).

(d) This can be checked by a computer algebra system using the parameterization and equation given in (b).

(e) This follows from Corollary 2. $\square$

Now it is plausible that the model in Theorem 2 should yield all generic triangles ABC (i.e. $P''Q''R''$) up to similarity. For example, the degree of freedom is two: one from the choice of P (the position of the equilateral triangle), and one from the choice of V (the position of a perspector on L). This matches the degree of freedom for a generic triangle up to similarity. However, without further justification of the degree-of-freedom argument, the mere statement of Theorem 2 is not enough to conclude the proof of Theorem 1.

For example, what if instead of the triangle $P''Q''R''$ in $\mathcal{K}$, there exists another triangle in $\mathcal{K}$ with the same Fermat points F_2, F_1 ? If this were the case, then we could not say anything about the triply perspective property. To address this ambiguity, we need to explain why the construction in Theorem 2 allows us to recover any general scalene triangle $P''Q''R''$ as a reference triangle on the Kiepert hyperbola. To do this, we verify that the equilateral triangle from the model and the reference triangle on the Kiepert hyperbola share the same Hessian line. Since we already know that the Hessian line of the equilateral triangle is L , which is the perpendicular bisector of F_2F_1 , it remains to check that if we start with any generic triangle ABC with its associated Kiepert hyperbola, the Hessian line of this triangle must be the perpendicular bisector of the segment joining the two Fermat points of ABC . We check this directly as follows.

Hessian Line for a Generic Reference Triangle ABC

Here we consider the generic case for ABC (corresponding to $P''Q''R''$ in the statement of Theorem 2). Without loss of generality, let $A = (a, b), B = (0, 0), C = (1, 0)$. We compute the two Fermat points and the Kiepert hyperbola $\mathcal{C}$ as follows:

$$F_1 = \left(\frac{\sqrt{3}a^2 + \sqrt{3}b^2 + (4a+1)b + \sqrt{3}a}{2(\sqrt{3}a^2 + \sqrt{3}b^2 - \sqrt{3}a + 3b + \sqrt{3})}, -\frac{3a^2 - b^2 - \sqrt{3}b - 3a}{2(\sqrt{3}a^2 + \sqrt{3}b^2 - \sqrt{3}a + 3b + \sqrt{3})} \right)$$

$$F_2 = \left(\frac{\sqrt{3}a^2 + \sqrt{3}b^2 - (4a+1)b + \sqrt{3}a}{2(\sqrt{3}a^2 + \sqrt{3}b^2 - \sqrt{3}a - 3b + \sqrt{3})}, \frac{3a^2 - b^2 + \sqrt{3}b - 3a}{2(\sqrt{3}a^2 + \sqrt{3}b^2 - \sqrt{3}a - 3b + \sqrt{3})} \right)$$

and

$$\mathcal{C} : (2ab - b)x^2 + (2b^2 - 2a^2 + 2a - 2)xy - (2ab - b)y^2 - (2ab - b)x + (a^2 - b^2 + a)y = 0.$$

We verify directly that the tangent line at A intersects line BC at

$$P_2 = \left(\frac{a^2 + b^2 - 2a}{2a^2 + 2b^2 - 2a - 1}, 0 \right).$$

And similarly, the tangent line at C intersects line AB at

$$P_1 = \left(\frac{2a^2 - a}{a^2 + b^2 + 2a - 2}, \frac{2ab - b}{a^2 + b^2 + 2a - 2} \right).$$

Note that line P_1P_2 is the Hessian line associated with triangle ABC . By checking the slopes of line F_2F_1 and line P_1P_2 , we see that they are perpendicular to each other. Furthermore, the common intersection of line F_2F_1 and line P_1P_2 is

$$\left(\frac{a^4 + b^4 + 2(a^2 - 2a)b^2 + a}{2(a^4 + b^4 - 2a^3 + (2a^2 - 2a - 1)b^2 + 3a^2 - 2a + 1)}, \frac{(4a^2 - 4a + 1)b}{2(a^4 + b^4 - 2a^3 + (2a^2 - 2a - 1)b^2 + 3a^2 - 2a + 1)} \right),$$

which is the midpoint of F_2F_1 . Hence the Hessian line P_1P_2 of ABC is the perpendicular bisector of F_2F_1 . It follows that the Hessian line associated with PQR and ABC agree, being the perpendicular bisector of F_2F_1 . This concludes the proof of Theorem 1.

In the spirit of Lemoine's problem, we may ask how to recover the reference triangle used to construct the Kiepert hyperbola. To this end, we have the following result.

Corollary 3. *Given a Kiepert hyperbola $\mathcal{K}$, one of the two Fermat points and a vertex of the reference triangle on $\mathcal{K}$, we can recover the other two vertices of the reference triangle.*

Proof. Note that the center O of $\mathcal{K}$ can be constructed by using 2.5. Reflecting one Fermat point with respect to O gives another Fermat point. With $\mathcal{K}$ and the two Fermat points, we follow the construction as in the statements of Theorem 2 and the proof for (a) of Theorem 1 to get an equilateral triangle PQR . Note also that the reference triangle ABC plays the same role as $P''Q''R''$ in Theorem 2. Using a construction similar to (c) of Theorem 2, join any vertex of PQR , say P , with a vertex of the reference triangle, say A ; let line PA intersect L at a point V (See Figure 4). Then follow the steps in (c) to recover B and C . $\square$

4. Discussion

In this article, we first develop synthetic proofs of Proposition 1 and Proposition 2, and then, by combining these results with analytic techniques, we obtain a complete solution to Yiu's problem. To facilitate the proofs, it is appropriate to address the following auxiliary results. First, establish statement (d) of Theorem 2 by a synthetic argument; that is, demonstrate that F_1 and F_2 are the Fermat points of the triangle $P''Q''R''$. Second, given a generic triangle ABC with Fermat points F_1 and F_2 and associated Kiepert hyperbola $\mathcal{K}$, prove by a synthetic method that the Hessian line of ABC with respect to $\mathcal{K}$ coincides with the perpendicular bisector of the segment F_1F_2 .

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