

Article

Topological Classification of Admissible Reconstruction Operations

Bin Li 

Research Department, Silicon Minds Inc., Clarksville, MD 21029, USA; binli.siliconminds@gmail.com

Abstract

We develop a topological classification of admissible reconstruction operations in generative systems where extended structure is built through repeated local extension subject to compatibility constraints. Reconstruction is formalized as a feasibility-governed process rather than a dynamical or metric one, with admissibility determined by the accumulation of obstruction under composition. Using loop diagnostics, we identify global incompatibilities that are invisible to local extension rules but become unavoidable under closed composition. Under mild and realization-independent assumptions, including indefinite continuation and finite interface capacity, we show that persistent nontrivial obstruction is possible only when it is supported on codimension-2 subsets of the reconstructed domain. This result induces a small number of topological universality classes distinguished by the existence and stability of loop-detectable obstruction. The framework is model-agnostic and applies equally to discrete, combinatorial, and continuum reconstructions, providing a topological explanation for the ubiquity of codimension-2 defects in generative systems.

Keywords: topological reconstruction; admissibility; obstruction theory; loop holonomy; path groupoids; defect codimension; universality classes; linking invariants; feasibility-governed extension; coarse-graining stability

1. Introduction

A wide class of mathematical constructions may be described as generative or reconstruction procedures: extended structure is produced by iterating local extension moves subject to compatibility constraints. Canonical examples include CW complex attachment and handle decompositions [1,2], rewriting systems on combinatorial complexes [3], and gluing or descent constructions in sheaf theory [4,5]. In such settings, local admissibility does not guarantee global coherence. Extension rules may be satisfiable on bounded regions while becoming incompatible when iterated, composed, or transported around nontrivial loops. This phenomenon appears in many areas of topology, geometry, and combinatorial construction, where local compatibility does not imply global extendability.

This observation naturally separates two aspects of any reconstruction scheme: (i) the specification of local extension moves, and (ii) the global feasibility of indefinite continuation under composition and gluing. The present work is concerned with the second aspect, namely the structural conditions under which indefinite continuation remains possible.

Central question. We isolate and study the following structural problem:

Which kinds of reconstruction operations remain topologically admissible under indefinite compositional extension?



Academic Editor: Michel Planat

Received: 25 January 2026

Revised: 3 April 2026

Accepted: 10 April 2026

Published: 21 April 2026

Copyright: © 2026 by the author.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and conditions of the [Creative Commons Attribution \(CC BY\)](https://creativecommons.org/licenses/by/4.0/) license.

Here admissibility is not defined in terms of optimization, probability, or dynamics. Instead, it is understood as a topological notion: an extension scheme is admissible to the extent that local compatibility can be composed and iterated arbitrarily many times without forcing unavoidable global obstruction, in the sense of classical obstruction theory [6–8]. The emphasis is therefore not on whether extension is possible in a single step, but whether extension remains feasible under indefinite iteration and composition.

Admissibility as topology rather than dynamics. A guiding principle of this work is that many failures of indefinite continuation are best understood not as dynamical instabilities, but as manifestations of topological obstruction. Local extension may introduce mismatch that is repairable on contractible neighborhoods, yet becomes unavoidable when constraints are composed around closed loops. In such cases, the appropriate language is homotopical: obstruction appears as nontrivial loop holonomy, cocycle failure, or defect classes supported on subsets of definite codimension [9,10]. Conversely, when mismatch can be confined, canceled, or displaced into lower-dimensional support, reconstruction may remain feasible at arbitrarily large scales. This shifts the perspective from extension as a one-time problem to extension as an iterative and compositional process.

To formalize this viewpoint, we adopt a realization-independent framework. We do not assume a background manifold, metric, probabilistic structure, or equations of motion. Instead, reconstruction is modeled abstractly by a directed extension relation on a configuration space together with compositional data sufficient to define loop diagnostics. The central objects of study are obstruction classes arising under composition and the topological dimension of their support.

Main mathematical contribution. The contribution of this work is not the introduction of new linking invariants, homotopy groups, or obstruction classes. Rather, it is the formulation and resolution of a classification problem concerning topological obstruction under indefinite compositional extension with bounded repair. More precisely, we study the following question:

Which topological types of obstruction can persist under indefinite compositional extension when incompatibility may be locally repaired, but only with bounded resources?

Classical obstruction theory typically treats extension as a binary question: a structure either extends across a domain, or it fails to do so, with obstruction acting as a terminal veto [7,8]. Such formulations do not distinguish between obstructions that can be locally absorbed, confined, or displaced under iteration, and those that inevitably proliferate and destroy global feasibility. As a result, they do not address persistence under repeated extension or the role of bounded repair capacity.

The framework developed here makes this distinction explicit by incorporating repair as a first-class topological notion. Obstruction is allowed to accumulate under composition, but admissibility is governed by whether this accumulation can be confined using subextensive interface resources. This shift from binary extension to feasibility-governed reconstruction leads to a new classification problem, even though the underlying topological invariants are classical.

Under mild assumptions capturing (a) the locality of extension and (b) the finite interface (repair) capacity, we prove that admissible reconstruction operations fall into a small number of topological universality classes. These classes are characterized by the following:

- The codimension of defect support on which nontrivial obstruction localizes;
- The existence or absence of loop-detectable obstruction invariants;
- The stability of obstruction under admissible coarse graining.

Our central theorem shows that codimension-2 defect support is uniquely selected by these requirements: it is the minimal codimension at which nontrivial loop invariants exist and remain compatible with persistent reconstruction under finite repair capacity. While the existence of codimension-2 defects is classical, the necessity of codimension-2 support as a condition for persistence under indefinite, repair-permitting extension is, to our knowledge, new. The result may therefore be viewed as a selection theorem imposed by feasibility rather than dynamics, refining classical results on linking and complements [11,12].

Scope and independence from specific models. The aim of this work is the classification rather than the analysis of a particular reconstruction mechanism. Accordingly, the framework is deliberately abstract. Concrete constructions—combinatorial, geometric, or otherwise—may be viewed as realizations of the axioms introduced here, but the results depend only on the topological behavior of obstruction under admissible composition. This separation clarifies which features are universal and which arise from additional, model-specific structure.

Relation to classical topology. Although the framework is formulated abstractly in terms of admissible reconstruction and feasibility, its core constructions admit a direct interpretation in standard topological language. In Appendix A, reconstruction paths are organized into a path groupoid [13], and the obstruction functional is shown to define a categorical 1-cocycle whose holonomy around closed loops recovers the loop obstruction used throughout the paper. Persistent defects are identified with the support of nontrivial holonomy, yielding a representation of $\pi_1(\mathcal{M} \setminus D)$ into the obstruction monoid. This places the codimension-2 admissibility theorem in direct correspondence with classical linking and obstruction theory, while extending these results to a setting governed by feasibility, iteration, and persistence rather than single-stage extension.

Organization. Section 2 formalizes reconstruction as a feasibility-governed directed extension process and introduces reconstruction paths, the compositional accumulation of obstruction, and loop closure. Section 3 develops loop diagnostics and defect support, culminating in a codimension-based admissibility theorem that classifies persistent obstruction. Section 5 formulates universality classes of reconstruction operations and analyzes their stability under admissible coarse graining. Section 4 analyzes the robustness of the classification under different repair scaling assumptions and extends the framework to higher homotopy invariants, yielding a hierarchy of admissible persistent obstruction classes. Section 6 presents fully worked examples in low-dimensional settings, explicitly demonstrating how codimension controls loop-detectable obstruction, repair cost, and persistence. Section 7 relates the abstract classification to discrete, geometric, and physical reconstruction models. Finally, Section 8 discusses implications, limitations, and directions for future work. A categorical and topological reformulation in terms of path groupoids and obstruction cocycles is provided in Appendix A.

Notation. Throughout, $\mathcal{X}$ denotes the configuration space, $\prec$ the directed extension relation, and $H(\gamma)$ the loop obstruction associated with a closed reconstruction loop γ . All topological statements concern obstruction classes and their supports and do not presuppose smoothness unless explicitly stated.

2. Reconstruction as a Feasibility-Governed Process

We formalize reconstruction as a directed process of structural extension governed by feasibility rather than time evolution, optimization, or energy minimization. The central object is not a dynamical trajectory but an indefinitely extensible pattern whose continued growth depends on the controlled accumulation of incompatibility. Related viewpoints appear implicitly in classical obstruction theory and in categorical approaches

to composition and gluing [7,8,13]. The novelty of the present formulation lies in treating obstruction not as a binary veto on extension, but as a compositional quantity whose large-scale behavior determines admissibility under indefinite iteration.

2.1. Axiomatic Structure of Reconstruction

We formalize a reconstruction process as a quadruple

$$(\mathcal{X}, \prec, \mathcal{A}, \mathcal{O}),$$

where

- $\mathcal{X}$ is a configuration space;
- $\prec$ is a directed admissible extension relation;
- $\mathcal{A}$ is an abelian obstruction monoid;
- $\mathcal{O}$ is a compositional obstruction map.

The structure is required to satisfy the following axioms.

1. **Irreflexivity.** $\Phi \not\prec \Phi$ for all $\Phi \in \mathcal{X}$.
2. **Locality.** Whether $\Phi \prec \Phi'$ holds depends only on data localized to the extension interface.
3. **Conditional composability.** If $\Phi \prec \Phi'$ and $\Phi' \prec \Phi''$, then the composite extension $\Phi \prec \Phi''$ is defined whenever interface data are compatible; failure of compatibility is recorded as obstruction.
4. **Compositional obstruction.** There exists a map

$$\mathcal{O} : \{\Phi \prec \Phi'\} \longrightarrow \mathcal{A}$$

such that obstruction accumulates under the following composition:

$$\mathcal{O}(\gamma_1 \circ \gamma_2) = \mathcal{O}(\gamma_1) \oplus \mathcal{O}(\gamma_2).$$

These axioms are satisfied by a broad class of generative constructions, including CW complex attachment, sheaf gluing, rewriting systems, and categorical composition [2,4,5,13,14].

The obstruction map extends uniquely to reconstruction paths by compositional accumulation, making obstruction a path functional on the reconstruction path category.

2.2. Configuration Space and Directed Extension

Definition 1 (Configuration space). $\mathcal{X}$ is a set whose elements represent partial or intermediate configurations of a generative construction. No metric, topology, probability measure, or dynamical law is assumed on $\mathcal{X}$ a priori.

Elements of $\mathcal{X}$ are not interpreted as states evolving in time. Instead, they represent partially realized structures that may or may not admit further extension, as in inductive constructions on complexes or sheaf-theoretic gluing [2,4,5].

Definition 2 (Admissible extension). An admissible extension relation is a directed relation,

$$\prec \subset \mathcal{X} \times \mathcal{X},$$

where $\Phi \prec \Phi'$ is read as “ Φ' is a feasible extension of Φ .”

2.3. Reconstruction Paths

Definition 3 (Reconstruction path). *A reconstruction path is a finite or infinite directed sequence*

$$\gamma : \Phi_0 \prec \Phi_1 \prec \Phi_2 \prec \cdots$$

of admissible extensions.

A reconstruction path represents a particular mode of structural growth. It is not a time-ordered trajectory and carries no intrinsic parametrization. Such compositional path structures are closely related to homotopy composition and higher categorical composition in algebraic topology [13,14].

2.4. Obstruction and Compositional Accumulation

Definition 4 (Obstruction monoid). *The obstruction values lie in an abelian monoid $(\mathcal{A}, \oplus, 0)$. We assume that $\mathcal{A}$ is equipped with a norm or order*

$$\|\cdot\| : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0}$$

satisfying

$$\|a \oplus b\| \leq \|a\| + \|b\|, \quad \|0\| = 0.$$

Definition 5 (Local obstruction). *For each admissible extension $\Phi \prec \Phi'$, the local obstruction is*

$$\mathcal{O}(\Phi \rightarrow \Phi') \in \mathcal{A}.$$

Definition 6 (Pathwise obstruction). *For a reconstruction path,*

$$\gamma : \Phi_0 \prec \Phi_1 \prec \cdots \prec \Phi_n,$$

the accumulated obstruction is

$$\mathcal{O}(\gamma) = \bigoplus_{i=0}^{n-1} \mathcal{O}(\Phi_i \rightarrow \Phi_{i+1}).$$

This compositional law makes obstruction a categorical 1-cocycle on the reconstruction path category, analogous to obstruction cocycles and holonomy constructions in bundle theory and obstruction theory [7–9].

We do not require $\mathcal{A}$ to be a group, since obstruction may accumulate without admitting a global inverse; this reflects the fact that local repair may neutralize obstruction only locally rather than globally.

2.5. Loop Closure and Holonomy

Definition 7 (Closed reconstruction loop). *A closed reconstruction loop is a finite reconstruction path*

$$\gamma : \Phi_0 \prec \Phi_1 \prec \cdots \prec \Phi_n$$

together with an admissible identification of the terminal interface of Φ_n with the initial interface of Φ_0 .

Definition 8 (Loop obstruction). *The loop obstruction associated with a closed reconstruction loop γ is*

$$H(\gamma) := \mathcal{O}(\gamma) \in \mathcal{A}.$$

Loop obstruction is invariant under the insertion or removal of subpaths whose total obstruction vanishes and therefore defines a homotopy-level invariant of admissible reconstruction loops. It may be interpreted as the holonomy of the obstruction cocycle, in close analogy with holonomy in fiber bundles and flat connections [9,15].

2.6. Feasibility and Persistent Reconstruction

Admissibility is determined not by the absence of obstruction but by whether obstruction growth remains compatible with indefinite continuation.

Definition 9 (Bounded obstruction growth). *Let γ_R denote a family of reconstruction loops of characteristic size R . A reconstruction process has bounded obstruction growth if there exists a function $f(R)$ with subextensive growth*

$$\lim_{R \rightarrow \infty} \frac{f(R)}{R^d} = 0$$

such that

$$\|H(\gamma_R)\| \leq f(R)$$

for all admissible loops γ_R .

Definition 10 (Persistent reconstruction). *A reconstruction process is persistent if any nonzero obstruction can be confined, under admissible coarse graining, to a subset of strictly lower topological dimension than the reconstructed domain.*

Equivalently, persistent reconstruction requires that obstruction support has a codimension of at least one and remains localized under indefinite extension.

Persistence is therefore a global feasibility condition: reconstruction processes that produce superextensive obstruction growth are eliminated under indefinite continuation, while those whose obstruction can be confined to lower-dimensional support remain admissible. This perspective generalizes classical obstruction theory from single-stage extension problems to asymptotic admissibility under indefinite iteration [7,8,10].

Figure 1 schematically illustrates loop-based diagnostics for defect support.

In the following sections, we show that the topology of loop obstruction—specifically its detectability by linking and the codimension of its support—completely determines which reconstruction processes satisfy these feasibility and persistence conditions.

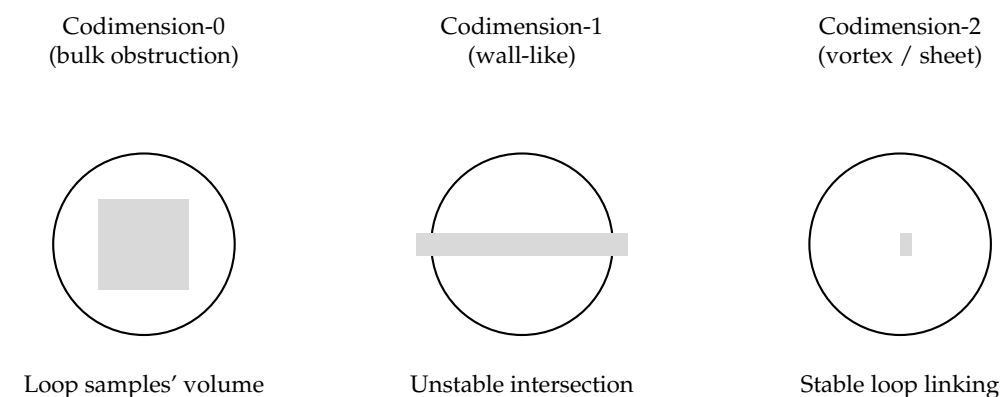


Figure 1. Loop diagnostics for defect support. Closed loops detect obstruction only through transverse linking. Bulk (codimension-0) obstruction (shown as grey square) fails to define a stable loop invariant and requires volumetric repair under extension, wall-like (codimension-1) obstruction (grey slab) yields unstable diagnostics under admissible homotopy, while codimension-2 defect (grey dot) support produces stable, loop-detectable obstruction compatible with finite interface capacity.

3. Codimension and Defect Support

We classify admissible reconstruction operations according to the codimension of the defect support on which nontrivial obstruction is localized. The classification concerns persistent obstruction classes: global incompatibilities that survive admissible local repair, coarse graining, and indefinite compositional extension.

The central result of this section is that persistent loop obstruction under finite interface capacity can be supported only on codimension-2 subsets. The special role of codimension-2 defects is well known in topology and in the theory of topological defects in ordered media [10,15,16].

3.1. Defects and Their Support

Definition 11 (Defect support). *Let $\mathcal{M}$ denote a coarse-grained topological space representing the reconstructed domain. A defect support $D \subset \mathcal{M}$ is a closed subset such that any reconstruction loop ℓ contained in $\mathcal{M} \setminus D$ has vanishing obstruction, while loops linking D may carry nontrivial obstruction.*

Definition 12 (Codimension). *A closed subset $D \subset \mathcal{M}$ has codimension k if each point of D admits a neighborhood $U \cong \mathbb{R}^d$ in which $D \cap U$ is contained in a subset homeomorphic to $\mathbb{R}^{d-k}$.*

This notion of codimension is standard in manifold topology and in the classification of defects and singular sets [10,17].

3.2. Loop Detection of Defects

Persistent obstruction is detected through loop composition and therefore depends on the topology of the complement $\mathcal{M} \setminus D$, particularly on the structure of the fundamental group of the complement [10,12].

Lemma 1 (Loop detection criterion). *Let $D \subset \mathcal{M}$ be closed. A nontrivial loop obstruction invariant*

$$H : \pi_1(\mathcal{M} \setminus D) \rightarrow \mathcal{A}$$

exists only if $\pi_1(\mathcal{M} \setminus D)$ is nontrivial.

Proof. Loop obstruction depends only on the homotopy class of loops in the complement. If $\pi_1(\mathcal{M} \setminus D) = 0$, then all loops are homotopically trivial and any compositional obstruction accumulated along a loop can be eliminated by admissible deformation. Hence, no homotopy-invariant loop obstruction exists. $\square$

Lemma 2 (Fundamental group of complements). *Let $\mathcal{M}$ be a connected d -dimensional manifold and $D \subset \mathcal{M}$ a closed subset of codimension k . Then, the following is true:*

- *If $k = 0$, the complement is either disconnected or simply connected at large scale.*
- *If $k = 1$, loops detect separation but not linking.*
- *If $k = 2$, the complement admits nontrivial π_1 generated by loops linking D .*
- *If $k \geq 3$, the complement is simply connected.*

This classification follows from classical results on complements of submanifolds and Alexander duality, which relate the topology of the complement to the codimension of the removed set [7,10–12,17].

3.3. Repair Cost and Finite Interface Capacity

Loop obstruction alone does not determine admissibility; the defect must also be compatible with bounded repair resources.

Definition 13 (Repair cost scaling). Let $C(R)$ denote the minimal admissible repair cost required to maintain compatibility in the presence of a defect enclosed by a loop of characteristic size R .

Lemma 3 (Repair scaling). If defect support has codimension k , then the minimal repair cost scales as

$$C(R) \sim R^{d-k}.$$

Proof. Repair must be applied in a tubular neighborhood of the defect support. If the defect support has codimension k , then a tubular neighborhood of fixed radius has volume scaling as R^{d-k} within a region of size R . Since repair operations are localized to this neighborhood, the minimal repair cost scales proportionally to the measure of this neighborhood, resulting in $C(R) \sim R^{d-k}$. $\square$

This scaling relation parallels classical results on defect energetics and interface scaling in topological defect theory [15,16].

Definition 14 (Finite interface capacity). A reconstruction process has finite interface capacity if

$$C(R) = o(R^d).$$

This condition excludes volumetric repair but allows repair localized on lower-dimensional subsets.

3.4. Codimension Admissibility Theorem

Theorem 1 (Codimension admissibility of persistent defects). Let $\mathcal{M}$ be a connected d -dimensional manifold admitting arbitrarily large loops. Assume reconstruction with finite interface capacity. Let $D \subset \mathcal{M}$ be defect support carrying a nontrivial loop obstruction invariant

$$H : \pi_1(\mathcal{M} \setminus D) \rightarrow \mathcal{A}.$$

Then D must have the codimension of exactly two.

Proof. Let k denote the codimension of D .

Case $k = 0$. Repair cost scales as $C(R) \sim R^d$, violating finite interface capacity. Furthermore, loop invariants cannot remain stable in bulk obstruction. Thus $k \neq 0$.

Case $k = 1$. Although the complement is disconnected, loops do not detect linking but only separation. Hence no homotopy-invariant loop obstruction map exists. Repair cost scales as R^{d-1} , which is not subextensive relative to interface growth. Thus $k \neq 1$.

Case $k \geq 3$. By Lemma 2, $\pi_1(\mathcal{M} \setminus D) = 0$. Therefore no homotopy-invariant loop obstruction can exist, and any local incompatibility can be eliminated by admissible repair without producing persistent global obstruction. Thus $k \neq k \geq 3$.

Case $k = 2$. The complement admits nontrivial π_1 generated by loops linking D . Repair cost scales as R^{d-2} , which is subextensive and compatible with finite interface capacity. Thus codimension-2 support permits persistent loop obstruction.

Since all other codimensions are excluded, codimension-2 is necessary. $\square$

3.5. Consequences

Persistent obstruction compatible with indefinite reconstruction must be localized on codimension-2 subsets of the reconstructed domain. This explains why filament-, vortex-, and string-like defect structures appear universally in systems governed by compositional compatibility and bounded repair capacity, consistent with the general theory of topological defects and linking invariants [10,15,16].

The result depends only on topology of the complement and scaling of the repair cost, and is therefore independent of microscopic reconstruction rules.

Figure 2 schematically shows how loop enlargement distinguishes defect codimension via the scaling of the admissible repair cost.

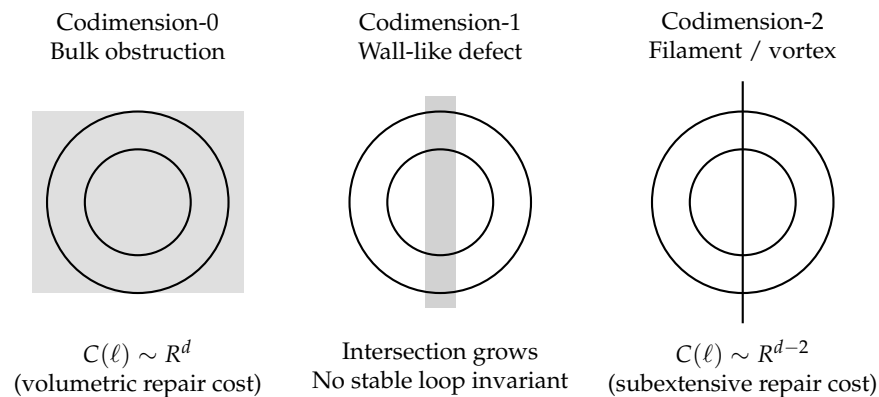


Figure 2. Loop enlargement as a diagnostic of repair cost. The loop obstruction $H(\ell)$ is a scale-independent topological invariant determined by linking. For bulk (codimension-0) defect support (grey block), maintaining admissibility requires volumetric repair. Codimension-1 defects (grey slab) intersect enlarging loops but fail to define stable loop invariants. Only codimension-2 defect support admits loop-detectable obstruction together with subextensive repair cost compatible with finite interface capacity.

4. Robustness and Extensions of the Framework

In Sections 2 and 3 we introduced an axiomatic reconstruction framework and proved the codimension admissibility theorem for persistent loop obstruction under finite interface capacity. The purpose of the present section is to examine the robustness of this classification with respect to the structural assumptions introduced in Section 2, and to discuss several natural extensions of the framework.

This section does not introduce new axioms but analyzes how the classification theorem depends on the structural assumptions already introduced, and how the result generalizes when these assumptions are modified.

The key result of Section 3 depends only on three structural ingredients: (1) compositional obstruction, (2) loop detection via π_1 , and (3) subextensive repair capacity. We now analyze how the classification depends on each of these assumptions.

4.1. Dependence on the Obstruction Monoid

In the axiomatic formulation of Section 2, obstruction values take values in an abelian monoid $(\mathcal{A}, \oplus, 0)$ equipped with a norm or order. The codimension admissibility theorem depends only on the existence of a loop obstruction map

$$H : \pi_1(\mathcal{M} \setminus D) \longrightarrow \mathcal{A},$$

and not on the detailed algebraic structure of $\mathcal{A}$. This reflects the general principle in obstruction theory that the existence of an obstruction class depends on the topology of the extension problem rather than on the specific algebraic realization of the obstruction group [7,8,10].

Proposition 1 (Algebraic robustness). *Let $(\mathcal{X}, \prec, \mathcal{A}, \mathcal{O})$ be a reconstruction system satisfying the axioms of Section 2. The codimension classification of persistent loop obstruction depends only on the existence of a nontrivial homomorphism from $\pi_1(\mathcal{M} \setminus D)$ into $\mathcal{A}$ and is therefore independent of the specific choice of obstruction monoid, provided $\mathcal{A}$ admits an identity and associative composition.*

Interpretation. The classification is topological rather than algebraic: the algebraic structure of $\mathcal{A}$ determines the type of charge or obstruction (e.g., integer, finite group, or torsion), but not the codimension at which persistent obstruction can be supported.

4.2. Robustness of the Finite Interface Capacity Assumption

The codimension admissibility theorem relies on the finite interface capacity condition, defined in Section 3 as the requirement that the repair cost grows subextensively with system size.

Let $d = \dim \mathcal{M}$ and let $C(R)$ denote the minimal repair cost required to maintain compatibility within a region of characteristic size R . By Lemma 3, a defect of codimension k requires repair cost scaling as

$$C(R) \sim R^{d-k}.$$

This scaling follows because repair must be applied in a neighborhood of the defect support, whose measure scales as R^{d-k} for a codimension- k subset.

This immediately yields a hierarchy of admissibility classes:

Repair Cost Scaling	Supported Defect Codimension
$C(R) \sim R^d$	Bulk obstruction (codimension-0)
$C(R) \sim R^{d-1}$	Wall-like defects (codimension-1)
$C(R) \sim R^{d-2}$	Filament/vortex defects (codimension-2)
$C(R) \sim R^{d-k}$	Codimension- k defects
$C(R)$ bounded	Localized defects (high codimension)

Finite interface capacity corresponds to the requirement

$$C(R) = o(R^d),$$

which excludes volumetric repair but permits repair localized on lower-dimensional subsets. Scaling relations of this type are standard in the theory of topological defects and interface energetics [15,16].

Conclusion. Under this assumption, codimension-2 support is the lowest codimension at which nontrivial loop obstruction can persist without requiring extensive repair. If the finite interface capacity assumption is relaxed, other codimension classes may become admissible. The theorem therefore applies specifically to reconstruction processes operating under subextensive repair constraints.

4.3. Extension to Higher Homotopy Invariants

Section 3 classified persistent obstruction detected by loop composition, i.e., by invariants derived from the fundamental group $\pi_1(\mathcal{M} \setminus D)$. This naturally selects codimension-2 defect support, since loops detect defects through linking.

More generally, obstruction may be detected by higher homotopy invariants. Maps

$$S^k \longrightarrow \mathcal{M} \setminus D$$

represent elements of $\pi_k(\mathcal{M} \setminus D)$ and detect higher-dimensional obstruction. Such higher homotopy obstructions play a central role in classical obstruction theory and extension problems in algebraic topology [7,8,10,14].

This suggests the following generalization.

Conjecture 4.2 (Higher-homotopy admissibility). If persistent obstruction is detected by nontrivial elements of $\pi_k(\mathcal{M} \setminus D)$, then the corresponding defect support must have codimension $k + 1$.

This generalizes the loop case ($k = 1$), which yields codimension-2 defect support:

Homotopy Invariant	Supported Defect Codimension
π_1 (loops)	codimension-2
π_2 (spheres)	codimension-3
π_k	codimension- $k + 1$

This extension suggests that the codimension-2 classification derived in this paper is the first member of a broader hierarchy of admissibility classes indexed by homotopy dimension, consistent with the general structure of obstruction theory and higher homotopy classification [7,8,10].

4.4. Example: Discrete Reconstruction with Cycle Obstruction

We now provide a concrete example illustrating the classification theorem.

Consider a discrete reconstruction process on a two-dimensional cell complex in which cells are attached sequentially subject to compatibility conditions along edges. Each attachment introduces a local obstruction value in $\mathcal{A} = \mathbb{Z}$ representing mismatch accumulated along edges.

When cells are attached around a closed cycle, the accumulated mismatch defines a loop obstruction equal to the sum of local mismatches around the cycle. If this obstruction is nonzero, compatibility can be restored only by introducing a localized defect at a point enclosed by the cycle.

Under the continued growth of the complex, mismatch cannot be eliminated globally but can be confined to isolated points. These points form codimension-2 defect support in the reconstructed domain. Such discrete attachment and obstruction phenomena are closely related to CW complex extension problems and cellular obstruction theory [2,10].

This discrete model demonstrates explicitly how loop obstruction leads to codimension-2 defect localization, consistent with Theorem 1.

4.5. Summary

The codimension admissibility theorem is robust under changes in the algebraic structure of obstruction and applies to any reconstruction process governed by compositional compatibility and finite interface capacity. The classification extends naturally to higher homotopy invariants, suggesting a hierarchy of admissibility classes determined by the topology of obstruction detection.

These results indicate that persistent incompatibility in generative reconstruction systems is constrained primarily by topology and scaling of repair, rather than by microscopic reconstruction rules, in agreement with general principles of obstruction theory and topological defect classification [8,10,16].

5. Universality Classes of Reconstruction Operations

The codimension-based classification of defect support established in Section 3 induces a natural partition of reconstruction operations into universality classes. These classes capture the large-scale, realization-independent behavior of admissible reconstruction under indefinite extension and admissible coarse graining.

Universality here refers not to microscopic extension rules or local update schemes, but to the persistent topological structure of obstruction that survives feasibility constraints.

In particular, universality is determined by the codimension and topological detectability of defect support, rather than by the detailed implementation of reconstruction.

5.1. Definition of Universality Classes

Definition 15 (Universality class). *Two reconstruction processes $\mathcal{R}_1$ and $\mathcal{R}_2$ are said to belong to the same universality class if, under admissible coarse graining and indefinite extension, their persistent obstruction structures are supported on defect sets of the same codimension and induce isomorphic loop obstruction maps*

$$H_i : \pi_1(\mathcal{M} \setminus D_i) \longrightarrow \mathcal{A}, \quad i = 1, 2,$$

up to admissible deformation and homotopy.

Universality is thus defined with respect to persistent obstruction data, not with respect to the full space of local reconstruction rules. Distinct microscopic mechanisms may therefore lie in the same universality class provided their long-range obstruction behavior is topologically equivalent. This mirrors classical classification results in topology, where homotopy-invariant data determines equivalence classes independently of local models [7,14].

Remark 1. *This notion of universality parallels that of critical phenomena, but replaces dynamical renormalization flows with the feasibility-based filtering of reconstruction processes.*

5.2. Loop Diagnostics and Topological Classification

Closed reconstruction loops provide a realization-independent diagnostic of persistent topological obstruction.

Corollary 1 (Loop-detectable universality classes). *Let $\mathcal{R}$ be a reconstruction process with persistent obstruction. Then $\mathcal{R}$ admits a nontrivial loop obstruction invariant if and only if its defect support belongs to the codimension-2 universality class.*

Proof. This is an immediate consequence of Theorem 1. For codimension-2 defect support, the complement $\mathcal{M} \setminus D$ admits nontrivial fundamental group, and loop obstruction is detected by linking. For defect support of codimension different from two, either no nontrivial loop invariant exists or obstruction fails to be invariant under admissible homotopy. $\square$

Remark 2. *Loop diagnostics classify universality classes only with respect to topological obstruction. They do not distinguish reconstruction processes that share the same obstruction class but differ in microscopic realization or in the detailed form of repair mechanisms.*

5.3. Stability Under Admissible Coarse Graining

Universality classes differ in their stability under admissible coarse graining.

Definition 16 (coarse graining stability). *A universality class is said to be stable under admissible coarse graining if every reconstruction process in the class retains nontrivial persistent obstruction under arbitrary iterations of admissible coarse graining.*

Corollary 2 (coarse graining stability of universality classes). *Among reconstruction processes admitting nontrivial obstruction, only those in the codimension-2 universality class are stable under admissible coarse graining.*

Proof. By Theorem 1, only codimension-2 defect support permits nontrivial loop obstruction compatible with finite interface capacity. Admissible coarse graining suppresses local fluctuations while preserving homotopy-invariant linking. Bulk-supported (codimension-0) obstruction is eliminated, and wall-supported (codimension-1) obstruction fails to define stable homotopy invariants. Thus only codimension-2 universality classes persist under repeated coarse graining. $\square$

Remark 3. The distinction between stable and unstable universality classes is independent of any metric or dynamical assumptions and depends solely on topological and feasibility considerations.

5.4. Universality Versus Microscopic Detail

Universality classes abstract away from microscopic implementation. Reconstruction processes with distinct local rules, constraint representations, or update mechanisms may belong to the same universality class provided admissibility suppresses obstruction in the same topological manner.

Remark 4. This explains why a wide range of generative systems exhibit similar large-scale behavior despite microscopic diversity: feasibility constraints act as a universal filter that reduces a large space of possible reconstruction rules to a small number of topologically distinct outcomes, determined by persistent obstruction data [14].

5.5. Summary

Universality classes of reconstruction operations are determined by the topology of persistent obstruction and its detectability by closed loops. Among all admissible reconstruction processes, only codimension-2 defect support yields nontrivial, stable loop obstruction compatible with indefinite extension and admissible coarse graining. As shown in Figure 3, this universality provides a realization-independent framework for classifying admissible generative processes based on feasibility rather than dynamics.

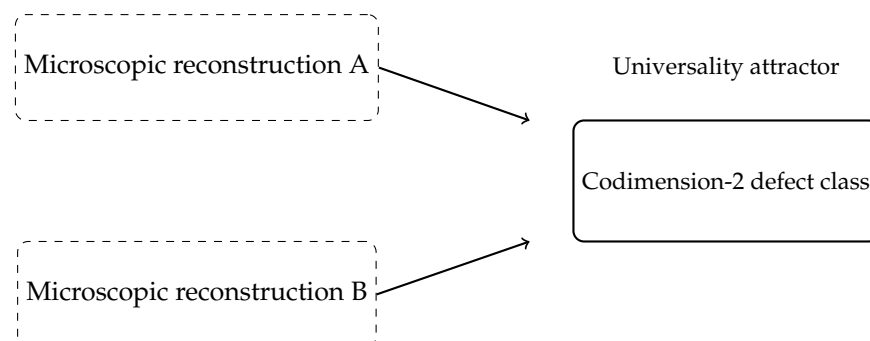


Figure 3. Universality classes of admissible reconstruction. Distinct microscopic realizations flow under coarse graining to the same admissible defect structure. Universality is determined by persistent topological support, not by microscopic dynamics.

6. Worked Examples

This section provides explicit, fully worked examples illustrating the framework and the codimension admissibility theorem. Throughout, we take the obstruction monoid to be $\mathcal{A} = \mathbb{Z}$ under addition, and interpret the loop obstruction $H(\ell)$ as a holonomy (winding) charge. The repair cost $C(\ell_R)$ is modeled by the minimal interface measure required to confine or screen the defect in a neighborhood of the defect support within the region controlled by a loop of characteristic size R .

6.1. Example I: Codimension-2 Defect in $\mathbb{R}^3$ (Vortex Filament)

Let $\mathcal{M} = \mathbb{R}^3$ and let the defect support be the z -axis

$$D = \{(0, 0, z) : z \in \mathbb{R}\}.$$

Then D is a closed subset of codimension-2.

Complement and fundamental group.

The complement deformation retracts onto a circle:

$$\mathbb{R}^3 \setminus D \simeq (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R} \simeq S^1$$

Hence,

$$\pi_1(\mathcal{M} \setminus D) \cong \mathbb{Z},$$

is generated by loops winding once around the z -axis [7,10].

Explicit loop family and loop obstruction.

For each radius $R > 0$ and fixed height z_0 , define the loop

$$\ell_R(t) = (R \cos t, R \sin t, z_0), \quad t \in [0, 2\pi].$$

Define $H : \pi_1(\mathcal{M} \setminus D) \rightarrow \mathbb{Z}$ by

$$H([\ell]) := \text{wind}_D(\ell),$$

the winding number of the projection of ℓ to the (x, y) -plane around the origin. For ℓ_R we have $H([\ell_R]) = 1$. If ℓ is homotopic to ℓ_R in $\mathcal{M} \setminus D$, then $H([\ell]) = 1$; if ℓ is null-homotopic, then $H([\ell]) = 0$. Thus H is a well-defined homomorphism and yields a nontrivial loop invariant.

Repair cost scaling and finite interface capacity.

In this example, admissible repair can be confined to a tubular neighborhood of D within the region controlled by ℓ_R . Let $\varepsilon > 0$ be a fixed coarse graining thickness; the relevant interface is the boundary of a tube of radius ε around the segment of D intersecting the region. The measure of this interface scales like

$$C(\ell_R) \sim (\text{tube area}) \propto R^{d-2} = R \quad (d = 3),$$

which is subextensive relative to R^3 . Hence, codimension-2 defects admit nontrivial loop obstruction compatible with finite interface capacity.

This example realizes the codimension-2 universality class: nontrivial loop obstruction exists (detected by linking) and remains stable under enlargement of the loop.

6.2. Example II: Codimension-1 Defect in $\mathbb{R}^2$ (Domain Wall)

Let $\mathcal{M} = \mathbb{R}^2$ and let the defect support be the x -axis

$$D = \{(x, 0) : x \in \mathbb{R}\},$$

a closed subset of codimension-1.

Complement and loop behavior.

The complement has two connected components,

$$\mathbb{R}^2 \setminus D = \{y > 0\} \sqcup \{y < 0\},$$

each contractible. Therefore, any loop contained in $\mathbb{R}^2 \setminus D$ is null-homotopic within its component, and in particular

$$\pi_1(\mathbb{R}^2 \setminus D) = 0$$

Reference [10]. Hence there is no nontrivial homotopy-invariant loop obstruction map $H : \pi_1(\mathcal{M} \setminus D) \rightarrow \mathcal{A}$.

Why “crossing number” is not an admissible loop invariant.

One might attempt to define an obstruction by counting the number of intersections of a loop with D . However, this quantity is not a homotopy invariant of loops in $\mathcal{M} \setminus D$, because admissible homotopies are not allowed to cross D ; loops in the complement have zero intersection number by definition. Any obstruction defined using intersection with D necessarily depends on geometric choices outside the complement and therefore does not descend to $\pi_1(\mathcal{M} \setminus D)$.

Repair cost scaling.

If obstruction is associated with maintaining a coherent identification across the wall, then repair requires controlling interface mismatch along segments of D spanning the region of size R . For loops of characteristic size R whose enclosed region abuts the wall, the minimal repair interface grows like the wall length within the region as follows:

$$C(\ell_R) \sim (\text{wall length}) \propto R^{d-1} = R \quad (d = 2).$$

This is not subextensive relative to R^2 in the sense required by finite interface capacity when such repairs must be performed persistently across enlarging regions: the repair is hypersurface supported and cannot be confined to a lower-dimensional linking core. Accordingly, codimension-1 obstruction is unstable under indefinite continuation, and no nontrivial loop invariant exists in the complement.

This example demonstrates the mechanism behind the codimension admissibility theorem: wall-like defects do not support loop-detectable holonomy invariants.

6.3. Example III: Codimension- ≥ 3 Defect in $\mathbb{R}^3$ (Point Defect)

Let $\mathcal{M} = \mathbb{R}^3$ and let the defect support be a single point

$$D = \{0\} \subset \mathbb{R}^3,$$

which has codimension 3.

Complement and fundamental group.

The complement deformation retracts onto the sphere S^2 :

$$\mathbb{R}^3 \setminus \{0\} \simeq S^2,$$

and therefore

$$\pi_1(\mathbb{R}^3 \setminus \{0\}) \cong \pi_1(S^2) = 0$$

References [7,10]. Thus, there is no nontrivial homomorphism $H : \pi_1(\mathcal{M} \setminus D) \rightarrow \mathcal{A}$, i.e., no loop-detectable obstruction supported on a point defect in three dimensions.

Higher invariants versus loop invariants.

Although point defects in $\mathbb{R}^3$ can be detected by higher homotopy data (e.g., by π_2 of the complement), they are invisible to the loop obstruction framework of this paper, which is intentionally based on loop composition and π_1 -holonomy. This distinction provides an explicit example of how the framework selects codimension-2 defects as the unique carriers of loop-detectable persistent obstruction [16].

Repair cost.

Any repair localized near the point defect can be confined to a bounded neighborhood of D and therefore has a bounded cost independent of R . However, since no nontrivial loop obstruction exists, such defects do not contribute to the universality classes classified in Sections 3–5.

This example shows that codimension ≥ 3 supports cannot carry nontrivial loop holonomy, in agreement with Theorem 1.

7. Relation to Generative and Physical Models

The classification developed in this work is formulated in abstract topological terms and does not depend on any specific geometric, analytic, or dynamical assumptions. Nevertheless, it applies uniformly to a broad class of generative systems in mathematics, physics, and computer science. Any process that constructs global structure through repeated local extension subject to feasibility constraints falls within its scope, including classical gluing and assembly procedures in topology and geometry [7,14].

The purpose of this section is interpretative rather than foundational. No new results are introduced here. Instead, we indicate how the codimension-based admissibility classification derived in Sections 2–5 manifests across representative classes of reconstruction models.

7.1. Discrete and Combinatorial Growth Models

In discrete growth models—such as cellular complexes, random tilings, lattice aggregation, and combinatorial assembly—local update rules generate global structure by accretion. Examples include CW complex construction, simplicial and cubical complexes, and rewriting systems on graphs and complexes [10]. In these settings, admissibility corresponds to the absence of incompatible local constraints under repeated attachment.

Loop obstruction arises naturally when attachment rules fail to close consistently around cycles in the underlying combinatorial structure, as in obstruction-theoretic treatments of cellular extensions [8]. The results of Section 3 imply that only those growth rules whose induced obstruction can be confined to codimension-2 substructures are compatible with persistent reconstruction under finite interface capacity. Bulk- or hypersurface-supported inconsistencies are suppressed under continued growth, while codimension-2 defects persist as robust topological features of the assembled structure.

This behavior is consistent with classical results on defect localization and linking in combinatorial topology [7], but here arises as a consequence of feasibility under indefinite extension rather than as a property of any particular construction.

7.2. Geometric and Manifold Reconstruction

Geometric reconstruction schemes—including atlas gluing, patchwise manifold construction, and discrete-to-continuum limits—also fall naturally under the admissibility framework. Here, reconstruction proceeds by extending local coordinate patches subject to overlap compatibility, as in standard constructions of smooth and topological manifolds [17].

In this setting, loop obstruction corresponds to transition function holonomy, a familiar topological diagnostic in differential and geometric topology [9]. From the perspective developed here, the appearance of curvature, torsion, and related geometric invariants reflects the presence of codimension-2 obstruction detected by loops encircling regions of mismatch. The codimension admissibility theorem isolates the topological conditions under which such obstruction can persist under refinement and admissible re-gluing.

Importantly, no geometric dynamics are assumed. The classification concerns only the topological support and detectability of obstruction under composition, independent of any metric or variational structure.

7.3. Reconstruction-Based Physical Frameworks

Reconstruction-first approaches in physics provide natural realizations of the abstract framework considered here. In such approaches, spacetime and fields are not assumed a priori but emerge from the admissible gluing of local structures. Consistency conditions are transported across interfaces, and obstruction accumulation plays a central role in selecting viable large-scale configurations, closely paralleling topological formulations of gauge theory and defect classification [15].

Remark 5. *In many physical models, physically relevant excitations—including gauge fluxes, vortices, and confinement-inducing defects—are supported on codimension-2 substructures. The present results do not explain the dynamics or energetics of such excitations, but clarify the topological reason why codimension-2 support is compatible with persistent, loop-detectable obstruction under feasibility constraints [16].*

From this viewpoint, the recurrence of codimension-2 structures across physical reconstruction models is not accidental, but reflects a general topological constraint on admissible extension under finite interface capacity.

7.4. Universality Across Implementations

A key consequence of the topological formulation is that the admissibility classes identified here are insensitive to microscopic detail. Distinct realizations—discrete or continuous and deterministic or stochastic—may belong to the same universality class provided their extension rules suppress obstruction in the same topological manner.

This universality explains why disparate generative systems often exhibit similar large-scale defect structures and stability properties. Feasibility acts as a coarse graining filter that reduces a large space of possible reconstruction rules to a small number of topologically distinct outcomes, classified by the codimension and detectability of persistent obstruction [14].

7.5. Summary

The codimension-based classification of admissible reconstruction operations provides a unifying topological language for understanding stability and obstruction across generative models. Without invoking dynamics or model-specific assumptions, it constrains the possible large-scale behaviors of reconstruction processes in mathematics and physics alike, identifying codimension-2 defect support as the unique carrier of nontrivial, loop-detectable structure compatible with indefinite extension.

8. Discussion and Outlook

This work establishes that reconstruction processes governed by feasibility rather than dynamics admit a sharply constrained topological classification. When global structure is built through repeated local extension, admissibility suppresses incompatible compo-

sitions and selects only those obstruction types that remain compatible with indefinite continuation. Under minimal and realization-independent assumptions—indefinite reconstruction and finite interface capacity—we have shown that only a restricted class of obstruction can persist. In particular, codimension-2 defect support emerges as the unique carrier of nontrivial, loop-detectable obstruction compatible with persistent large-scale reconstruction. This result refines classical insights from obstruction theory and linking by identifying codimension-2 support not merely as possible, but as necessary under feasibility constraints [7–9].

A central conceptual outcome is that admissibility is inherently topological. It is determined by how obstruction accumulates under composition and how it is detected by closed reconstruction loops, rather than by rates, energies, or equations of motion. Within this framework, familiar notions such as holonomy, curvature, flux, and vortex-like defects are interpreted as manifestations of feasibility constraints imposed by reconstruction itself. From this perspective, the ubiquity of codimension-2 structures across generative systems is not accidental but follows from general constraints on how local compatibility can be extended globally, consistent with classical treatments of topological defects [15,16].

Scope and interpretation. The results of this paper are purely topological. No assumptions are made about dynamics, geometry, physical laws, or empirical mechanisms, and no new physical claims are introduced. Section 7 should be understood as interpretative, illustrating how the classification applies to representative reconstruction models rather than proposing new model-specific conclusions. The codimension admissibility theorem identifies a structural constraint on admissible extension that holds whenever persistent obstruction is detected by closed composition, independent of realization.

The analysis deliberately abstracts away from metric, analytic, and probabilistic structure. As a result, the conclusions apply uniformly to discrete, combinatorial, and continuum realizations, provided that admissible extension and loop diagnostics are well defined. The classification depends only on the topological behavior of obstruction under admissible homotopy, as formalized through reconstruction paths, loop obstruction, and the associated groupoid and cocycle structures summarized in Appendix A [13].

Several directions for future work naturally follow. First, the codimension-2 universality class admits further internal refinement. Distinct subclasses may be distinguished by the algebraic structure of the obstruction monoid, the presence of torsion, or the behavior of higher-order loop and surface invariants, suggesting connections with classical obstruction theory and characteristic classes [8,14]. Second, extending the framework to reconstruction processes with hierarchical or nested structure may reveal multi-scale admissibility phenomena and secondary universality classes, paralleling stratified and filtered topological spaces. Third, explicit realizations in geometric or algebraic settings may be used to test how strongly feasibility alone constrains emergent symmetries or effective descriptions.

More broadly, the results support a unifying viewpoint: many robust features of emergent structure are dictated not by detailed microscopic rules, but by topological constraints on how local compatibility can be coherently extended. Admissibility acts as a universal filter on generative processes, sharply limiting the space of viable large-scale organizations. In this sense, topology serves not merely as a descriptive language for reconstruction, but as a foundational principle governing which structures can persist under indefinite extension, in accordance with its central role as a classifier of global structure across mathematics [7,14].

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: Author Bin Li was employed by the company Silicon Minds Inc. The author declares that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

Appendix A. Reconstruction as a Path Groupoid with Obstruction Cocycle

This appendix provides a concrete topological reformulation of the reconstruction framework used in the main text. Its purpose is twofold: (i) to embed feasibility-governed reconstruction into standard categorical and topological language, and (ii) to show that loop obstruction arises naturally as a holonomy-type invariant. No assumptions are introduced beyond those stated in Sections 2 and 3. All results in this appendix are reformulations of statements proved in the main text.

Relation to standard topology and category theory. The constructions in this appendix are reformulations of standard notions in algebraic topology and category theory. In particular, reconstruction paths form a path category or path groupoid [13], obstruction accumulation defines a 1-cocycle on this category [7,8], and loop obstruction corresponds to the holonomy of this cocycle around closed paths, analogous to holonomy constructions in bundle theory [9]. From this perspective, persistent defect support corresponds to the support of nontrivial holonomy in the complement of the defect set, placing the framework within classical obstruction and cohomology theory.

Appendix A.1. The Reconstruction Path Category

Let $\mathcal{X}$ be the configuration space introduced in Section 2, equipped with the admissible extension relation $\prec$. We define a small category $\mathcal{G}$, called the reconstruction path category, as follows:

- Objects of $\mathcal{G}$ are configurations $\Phi \in \mathcal{X}$.
- Morphisms are finite admissible reconstruction paths:

$$\gamma : \quad \Phi_0 \prec \Phi_1 \prec \cdots \prec \Phi_n.$$

- Composition is given by concatenation of paths whenever interface compatibility permits.
- Identity morphisms are trivial paths of length zero.

This construction parallels the standard path category or path groupoid associated with a topological space, but is defined purely in terms of admissible compositional extension rather than continuous paths [13]. The invertibility of morphisms is not assumed; when admissible local inverses exist up to coarse graining, $\mathcal{G}$ restricts to a groupoid, but the categorical formulation suffices for all purposes here.

Appendix A.2. Obstruction as a Categorical 1-Cocycle

Let $\mathcal{A}$ denote the abelian monoid of obstruction data introduced in Section 2.

Definition A1 (Obstruction cocycle). *An obstruction cocycle is a map*

$$\mathcal{O} : \text{Mor}(\mathcal{G}) \longrightarrow \mathcal{A}$$

satisfying the following:

1. *Locality:* $\mathcal{O}(\gamma)$ depends only on the local extension interfaces traversed by γ .
2. *Cocycle condition:*

$$\mathcal{O}(\gamma_1 \circ \gamma_2) = \mathcal{O}(\gamma_1) \oplus \mathcal{O}(\gamma_2),$$

whenever composition is defined.

This definition is formally identical to that of a 1-cocycle on a small category or groupoid with values in an abelian monoid, as in classical cohomology and obstruction theory [7–9]. No linear, metric, or differential structure is required.

The pathwise accumulation of obstruction defined in Section 2 is precisely the evaluation of this cocycle on morphisms of $\mathcal{G}$.

Appendix A.3. Loop Obstruction as Holonomy

A closed reconstruction loop corresponds to a morphism $\gamma : \Phi \rightarrow \Phi$ in $\mathcal{G}$.

Definition A2 (Loop holonomy). *The loop obstruction associated with a closed reconstruction loop γ is the holonomy*

$$H(\gamma) := \mathcal{O}(\gamma) \in \mathcal{A}.$$

Proposition A1. *The loop obstruction $H(\gamma)$ depends only on the admissible homotopy class of γ and defines a homomorphism on loop classes.*

Proof. This follows directly from the cocycle condition and from invariance under insertion or removal of subpaths with vanishing total obstruction, as established in Section 2. $\square$

Thus, loop obstruction is precisely the holonomy of the obstruction cocycle around closed morphisms. This places the reconstruction framework within standard holonomy theory, generalized from continuous connections to compositional categories [9].

Appendix A.4. Defect Support and Codimension

Assume that reconstruction admits a coarse-grained description on a topological space $\mathcal{M}$, as in Section 3. Let $D \subset \mathcal{M}$ denote the minimal closed subset such that all reconstruction loops contained in $\mathcal{M} \setminus D$ have vanishing loop obstruction.

Definition A3 (Defect support). *A subset $D \subset \mathcal{M}$ is called a defect support if the restriction of the obstruction cocycle to loops in $\mathcal{M} \setminus D$ is trivial.*

In this case, loop obstruction factors through the fundamental group of the complement:

$$H : \pi_1(\mathcal{M} \setminus D) \longrightarrow \mathcal{A}.$$

Proposition A2. *If H is nontrivial and stable under admissible coarse graining, then the defect support D must have a codimension of exactly two in $\mathcal{M}$.*

Proof. This statement is equivalent to Theorem 1. For defect support of codimension-0 or -1, loop obstruction fails to define a homotopy invariant or requires repair cost incompatible with finite interface capacity. For codimension ≥ 3 , the complement $\mathcal{M} \setminus D$ is simply connected and admits no nontrivial loop invariant. Only codimension-2 support permits nontrivial holonomy detected by linking and compatible with persistent reconstruction, consistent with classical results on complements and linking in algebraic topology [10,12]. $\square$

This reformulation makes explicit that the codimension admissibility theorem is a statement about the topology of complements and holonomy, not about specific reconstruction rules.

Appendix A.5. Illustrative Example: CW Complex Attachment

As a concrete instantiation, consider a CW complex constructed by attaching two cells to a one-skeleton subject to a cocycle constraint on overlaps [2,10]. Local attachment may be admissible on contractible subcomplexes, while the failure of the cocycle condition around a loop in the one-skeleton produces a nontrivial obstruction class.

In this case, the obstruction is supported on codimension-2 subsets of the CW complex (points in dimension two; lines in dimension three) and is detected precisely by loop holonomy in the complement. This example illustrates how classical obstruction phenomena are recovered as special cases of the feasibility-governed reconstruction framework.

Appendix A.6. Relation to Classical Obstruction Theory

Classical obstruction theory studies the extension of sections, maps, or bundles over skeleta and typically treats obstruction as a binary obstruction to further extension [7,8]. The framework developed in this paper generalizes this viewpoint in the following ways:

1. Treating extension as a directed compositional process rather than a single step;
2. Allowing obstruction to accumulate under composition;
3. Classifying admissibility in terms of persistence under indefinite extension.

From this perspective, feasibility-governed reconstruction may be viewed as an abstraction of obstruction theory adapted to generative and iterative settings. The groupoid and cocycle formulation provided here demonstrates that the main results of the paper are grounded in standard topological structures, while extending their scope to problems of persistence, feasibility, and universality.

References

1. Milnor, J. *Lectures on the h-Cobordism Theorem*; Princeton University Press: Princeton, NJ, USA, 1965.
2. Whitehead, J.H.C. Combinatorial homotopy. II. *Bull. Am. Math. Soc.* **1949**, *55*, 453–496. [[CrossRef](#)]
3. Lyndon, R.C.; Schupp, P.E. *Combinatorial Group Theory*; Springer: Berlin, Germany, 1977.
4. Bredon, G.E. *Sheaf Theory*, 2nd ed.; Springer: New York, NY, USA, 1997.
5. Kashiwara, M.; Schapira, P. *Categories and Sheaves*; Springer: Berlin, Germany, 2006.
6. Mosher, R.; Tangora, M. *Cohomology Operations and Applications in Homotopy Theory*; Harper & Row: New York, NY, USA, 1968.
7. Spanier, E.H. *Algebraic Topology*; McGraw-Hill: New York, NY, USA, 1966.
8. Steenrod, N. *The Topology of Fibre Bundles*; Princeton University Press: Princeton, NJ, USA, 1951.
9. Bott, R.; Tu, L.W. *Differential Forms in Algebraic Topology*; Springer: New York, NY, USA, 1982.
10. Hatcher, A. *Algebraic Topology*; Cambridge University Press: Cambridge, UK, 2002.
11. Alexander, J.W. Topological invariants of knots and links. *Trans. Am. Math. Soc.* **1928**, *30*, 275–306. [[CrossRef](#)]
12. Milnor, J. Infinite cyclic coverings. In *Proceedings of the Conference on the Topology of Manifolds*; Prindle, Weber & Schmidt: Boston, MA, USA, 1968; pp. 115–133.
13. Brown, R. *Topology and Groupoids*; BookSurge Publishing: Charleston, SC, USA, 2006.
14. May, J.P. *A Concise Course in Algebraic Topology*; University of Chicago Press: Chicago, IL, USA, 1999.
15. Nakahara, M. *Geometry, Topology and Physics*, 2nd ed.; CRC Press: Boca Raton, FL, USA, 2003.
16. Mermin, N.D. The topological theory of defects in ordered media. *Rev. Mod. Phys.* **1979**, *51*, 591–648. [[CrossRef](#)]
17. Lee, J.M. *Introduction to Topological Manifolds*, 2nd ed.; Springer: New York, NY, USA, 2011.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.