

Article

A Formula for the Euler Characteristic of the Fiber Product of Morse Functions

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Abstract

Let $\eta : X \rightarrow \mathbb{R}$ be a Morse function on a connected closed manifold X . We denote by $C(\eta)$ the fiber product of two copies of η . For Morse functions $f : M \rightarrow \mathbb{R}$ and $g : N \rightarrow \mathbb{R}$, we define the function $f * g : M \times N \rightarrow \mathbb{R}$ by $(f * g)(p, q) := f(p) \cdot g(q)$. The purpose of this paper is twofold: Firstly, we study the sufficient condition for which $\chi(C(f * g)) = \chi(C(f)) \cdot \chi(C(g))$ holds, where χ denotes the Euler characteristic. Secondly, for the case that f is the well-known Morse function on $\mathbb{C}P^n$, we determine $\chi(C(f * f))$.

Keywords: fiber product; Morse function; Euler characteristic; product of functions; projective space; orientable closed surface

1. Introduction

1.1. Motivation

We consider the following question: Let

$$\rho : S^n \rightarrow \mathbb{R} \quad (1)$$

be the height function defined by

$$\rho(x_1, \dots, x_{n+1}) = x_{n+1}.$$

Assume that two particles u and v on S^n move satisfying the condition $\rho(u) = \rho(v)$. Let $C(\rho)$ be the space that is formed by all possible positions of u and v . What is the topology of $C(\rho)$?

The answer is that there is a homeomorphism

$$C(\rho) \approx S(S^{n-1} \times S^{n-1}), \quad (2)$$

where S denotes the suspension.

1.2. The Essence of Morse Theory

We try to generalize the space $C(\rho)$ in Section 1.1 for the case that the function is a Morse function. For the convenience of some readers, we recall the essence of Morse theory (see, for example, [1,2]). Let

$$\eta : X \rightarrow \mathbb{R} \quad (3)$$

be a smooth function on a connected closed manifold X . The following apply:

- We say that a point $u \in X$ is a *critical point* of η if $d\eta_u$ is the zero map.



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- We say that a critical point u is *non-degenerate* if the Hessian matrix of η at u is non-singular with respect to some coordinate system at u .
- We say that Equation (3) is a *Morse function* if every critical point of η is non-degenerate.
- The underlying idea of Morse theory is to study the topology of X by analyzing the Morse function η . The theory allows one to find CW structures and handle decompositions on X .

1.3. A Generalization of $C(\rho)$ in Section 1.1

As in Equation (3), let $\eta : X \rightarrow \mathbb{R}$ be a Morse function on a connected closed manifold X . We denote by $C(\eta)$ the fiber product of two copies of η . More precisely, we set

$$C(\eta) := \{(u, v) \in X \times X \mid \eta(u) = \eta(v)\}.$$

About $C(\eta)$, the following results (i) and (ii) are already known:

(i) In [3], we proved a formula that describes $\chi(C(\eta))$ in terms of the information on the critical points and their indices of η (see Lemma 2 for more details). Here throughout this paper, for a finite CW-complex K , let $\chi(K)$ denote the Euler characteristic of K .

(ii) Using the formula in (i), we determined $\chi(C(\eta))$ when η is the well-known Morse function on the unitary group ([3]), and when η is the well-known Morse function on the Grassmannians ([4]).

1.4. Product of Morse Functions

Let $f : M \rightarrow \mathbb{R}$ and $g : N \rightarrow \mathbb{R}$ be Morse functions, where M and N are connected closed manifolds. We define the map

$$f * g : M \times N \rightarrow \mathbb{R}$$

by

$$(f * g)(p, q) = f(p) \cdot g(q), \quad \forall (p, q) \in M \times N.$$

Remark 1. (i) It is to be noted that the notation $f * g$ is unrelated to the convolution in analysis.

(ii) It is well-known that if $\min f > 0$ and $\min g > 0$, then $f * g$ is a Morse function (see Lemma 1 for more details).

(iii) It is clear that there is a homeomorphism

$$C(f * g) \approx C(g * f).$$

1.5. A Famous Example of $f * f$

We recall a famous example of $f * f$ below. We define $f : S^1 \rightarrow \mathbb{R}$ by $f(e^{i\theta}) = 2 + \cos \theta$. Then $f * f$ is given by

$$(f * f)(e^{i\theta_1}, e^{i\theta_2}) = (2 + \cos \theta_1)(2 + \cos \theta_2). \quad (4)$$

Note that Equation (4) is the well-known Morse function on the torus such that the critical values of the two index-one critical points are the same (see, for example, (Ref. [1] p. 72, Exercise 2.4 and its answer in p. 205).)

1.6. A Question

From Section 1.4, we can construct a new Morse function $f * g$ from known Morse functions f and g . Assume that the following equation holds:

$$\chi(C(f * g)) = \chi(C(f)) \cdot \chi(C(g)) \quad (5)$$

Then we can compute $\chi(C(f * g))$ from $\chi(C(f))$ and $\chi(C(g))$. But the following example shows that Equation (5) does not hold in general.

Example 1. For $i = 1$ or 2 , we define $\zeta_i : S^1 \rightarrow \mathbb{R}$ as follows:

$$\zeta_1(x_1, x_2) = x_1 + 2 \quad \text{and} \quad \zeta_2(x_1, x_2) = 2x_1 + 3. \quad (6)$$

It is easy to see that

$$\chi(C(\zeta_1 * \zeta_1)) = \chi(C(\zeta_2 * \zeta_2)) = 6 \quad \text{but} \quad \chi(C(\zeta_1 * \zeta_2)) = 4 \quad (7)$$

(see also Example 3). The difference in Equation (7) is due to the fact that the critical values of the two index-one critical points of $\zeta_i * \zeta_i$ are the same, but that of $\zeta_1 * \zeta_2$ are different.

1.7. Purpose

From Example 1, we naturally encounter the following Questions 1 and 2: Let f and g be as in Section 1.4.

Question 1. Does there exist a sufficient condition for which Equation (5) holds?

Question 2. Let f and g be the well-known Morse functions which do not satisfy the sufficient condition in Question 1. Then, what are the both sides of Equation (5)?

The purpose of this paper is to study Questions 1 and 2.

1.8. The Organization of This Paper

This paper is organized as follows. In Section 2 we recall auxiliary lemmas. In Section 3 we state our main results. In Section 4 we prove the main results. In Section 5 we state further numerical results.

2. Auxiliary Lemmas

Definition 1. Let η be a Morse function as in Equation (3).

- (i) We denote by $\mathcal{V}(\eta)$ the set of critical values of η .
- (ii) If u is a critical point of η , then we denote the index of η at u by $\mu_\eta(u)$.

Lemma 1. Let $f : M \rightarrow \mathbb{R}$ and $g : N \rightarrow \mathbb{R}$ be Morse functions, where M and N are connected closed manifolds. We assume that

$$\min f > 0 \quad \text{and} \quad \min g > 0. \quad (8)$$

Then the following results hold:

- (i) A point $(p, q) \in M \times N$ is a critical point of $f * g$ if and only if p is a critical point of f and q is a critical point of g .
- (ii) Let (p, q) be a critical point of $f * g$. Then

$$\mu_{f*g}(p, q) = \mu_f(p) + \mu_g(q).$$

Proof. The lemma is well-known (see, (Ref. [1], p.72, Exercise 2.3 and its answer in p.204)). $\square$

Lemma 2 ([3] Theorem A). Let η be a Morse function as in Equation (3). For $\xi \in \mathcal{V}(\eta)$, let k be the number of the critical points contained in $\eta^{-1}(\xi)$. We write the critical points contained in $\eta^{-1}(\xi)$ by u_i for $1 \leq i \leq k$.

We write $\mu_\eta(u_i)$ by λ_i and we define the integer $\Omega(\eta, \xi)$ by

$$\Omega(\eta, \xi) := \sum_{i=1}^k (-1)^{\lambda_i}.$$

Then we have

$$\chi(C(\eta)) = (-1)^{\dim X} \sum_{\xi \in \mathcal{V}(\eta)} (\Omega(\eta, \xi))^2.$$

Example 2. (i) Let $\rho : S^n \rightarrow \mathbb{R}$ be the Morse function in Equation (1). Then

$$\chi(C(\rho)) = (-1)^n \cdot 2. \quad (9)$$

Note that Equation (9) is in agreement with Equation (2).

(ii) For $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, let $\eta : \mathbb{F}P^n \rightarrow \mathbb{R}$ be the well-known Morse function (see Equation (12)). Then

$$\chi(C(\eta)) = (-1)^{dn} \cdot (n+1),$$

where $d := \dim_{\mathbb{R}} \mathbb{F}$.

3. Statement of the Main Results

Assumption 1. Hereafter, we always assume that f and g satisfy Equation (8).

Firstly, modifying Lemma 2, we give a formula to compute $\chi(C(f * g))$.

Theorem 1. For $\xi \in \mathcal{V}(f * g)$, let k be the number of the critical points contained in $(f * g)^{-1}(\xi)$. We write the critical points contained in $(f * g)^{-1}(\xi)$ by (p_i, q_i) for $1 \leq i \leq k$.

We write $\mu_{f*g}(p_i, q_i)$ by λ_i and we define the integer $\Omega(f * g, \xi)$ by

$$\Omega(f * g, \xi) := \sum_{i=1}^k (-1)^{\lambda_i}.$$

Then we have

$$\chi(C(f * g)) = (-1)^{\dim(M \times N)} \sum_{\xi \in \mathcal{V}(f * g)} (\Omega(f * g, \xi))^2.$$

Remark 2. Theorem 1 is useful. In fact, using Lemma 1, we can determine (p_i, q_i) and λ_i from the information on the critical points of f and g .

Example 3. We define the Morse functions F_k and $G_k : S^k \rightarrow \mathbb{R}$ by

$$F_k(x_1, \dots, x_{k+1}) = x_1 + 2 \quad \text{and} \quad G_k(x_1, \dots, x_{k+1}) = 2x_1 + 3.$$

(Note that F_1 and G_1 equal to ζ_1 and ζ_2 which are defined in Equation (6).) Then we have

$$\chi(C(F_m * F_n)) = \begin{cases} 6, & \text{if } m + n \text{ is even,} \\ -2, & \text{if } m + n \text{ is odd,} \end{cases} \quad (10)$$

and

$$\chi(C(F_m * G_n)) = (-1)^{m+n} \cdot 4. \quad (11)$$

Proof of Example 3. Note that $\mathcal{V}(F_m * F_n) = \{1, 3, 9\}$. The indices of the critical points in $(F_m * F_n)^{-1}(3)$ are m and n . Hence

$$\Omega(F_m * F_n, 3) = (-1)^m + (-1)^n.$$

Now Equation (10) follows from Theorem 1.

On the other hand, the critical values of the critical points of $F_m * G_n$ are mutually distinct. Hence Equation (11) follows. $\square$

Secondly, in order to give an answer to Question 1 in Section 1, we introduce the following:

Assumption 2. We define the map

$$P : \mathcal{V}(f) \times \mathcal{V}(g) \rightarrow \mathbb{R}$$

by $P(a, b) = ab$. Then we assume that P is injective.

Now the following Theorem 2 is an answer to Question 1:

Theorem 2. Let $f : M \rightarrow \mathbb{R}$ and $g : N \rightarrow \mathbb{R}$ be Morse functions which satisfy Assumptions 1 and 2. Then Equation (5) holds.

Example 4. We give an example for which Assumption 2 holds. We regard $\mathbb{C}P^n = S^{2n+1}/S^1$. We define $f_n : \mathbb{C}P^n \rightarrow \mathbb{R}$ by

$$f_n([z_0, z_1, \dots, z_n]) := \sum_{i=0}^n (i+1) |z_i|^2. \quad (12)$$

On the other hand, we fix a positive transcendental number α . (For example, we can set $\alpha = e$ or π .) We define $g_n : \mathbb{C}P^n \rightarrow \mathbb{R}$ by

$$g_n([z_0, z_1, \dots, z_n]) := \sum_{i=0}^n \alpha^i |z_i|^2.$$

By [2], f_n and g_n are Morse functions. Moreover, f_n and g_n satisfy Assumption 2: In fact, note that

$$\mathcal{V}(f_n) = \{1, 2, \dots, n+1\} \quad \text{and} \quad \mathcal{V}(g_n) = \{1, \alpha, \dots, \alpha^n\}.$$

Since α is a transcendental number, Assumption 2 is in fact satisfied.

Thirdly, we consider Question 2. The following Theorems 3 and 4 are our results:

Definition 2. Let n be a positive integer.

(i) We set

$$\delta(n) := \left\{ \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \mid \text{the following conditions (a) and (b) hold} \right\}. \quad (13)$$

Here

(a) For $1 \leq i, j \leq 2$, ϕ_{ij} is an integer such that $1 \leq \phi_{ij} \leq n+1$.

(b) $\phi_{11}\phi_{22} = \phi_{12}\phi_{21}$.

(ii) We set $\Delta(n) := |\delta(n)|$, where $|\cdot|$ denotes the cardinality.

Theorem 3. For f_n in Equation (12), we have

$$\chi(C(f_n * f_n)) = \Delta(n).$$

Remark 3. (i) The values of $\Delta(n)$ for small n are given in [5]. Using the result, we obtain the following Table 1 about $\chi(C(f_n * f_n))$ for $n \leq 10$.

Table 1. $\chi(C(f_n * f_n))$ for $n \leq 10$.

n	1	2	3	4	5	6	7	8	9	10
$\chi(C(f_n * f_n))$	6	15	32	49	86	111	160	209	278	319

We have from Example 2 (ii) that $\chi(C(f_n)) = n + 1$. Hence Table 1 shows that Equation (5) does not hold when $f = g = f_n$ for $n \geq 1$.

(ii) We set

$$J(n) := \chi(C(f_n * f_n)) - \frac{12}{\pi^2} n^2 \log n.$$

Then using the result in [6] (see also [5]), we obtain

$$\lim_{n \rightarrow \infty} \frac{J(n)}{n^2} = -0.51131 \dots$$

Remark 4. The $\mathbb{R}P^n$ -version of Table 1 will be given in Table 2. But the results are not so simple.

Theorem 4. For f_n in Equation (12), we have

$$\chi(C(f_1 * f_n)) = \begin{cases} 3n + 3, & \text{if } n \text{ is odd,} \\ 3n + 2, & \text{if } n \text{ is even.} \end{cases} \quad (14)$$

Remark 5. The right-hand side of Equation (14) is introduced in [7] as the sequence of integers which are congruent to 0 or 2 mod 6.

4. Proofs of the Main Results

Proof of Theorem 1. If we consider $f * g$ for η in Lemma 2, then Theorem 1 follows. $\square$

Proof of Theorem 2. Let $\xi \in \mathcal{V}(f * g)$. Recall that in Theorem 2, we assume the truth of Assumption 2. Then there exists uniquely $(a, b) \in \mathcal{V}(f) \times \mathcal{V}(g)$ such that $ab = \xi$.

We shall prove that

$$\Omega(f * g, \xi) = \Omega(f, a) \cdot \Omega(g, b). \quad (15)$$

In fact, we write the critical points in $f^{-1}(a)$ by p_i for $1 \leq i \leq \ell$. Similarly, we write the critical points in $g^{-1}(b)$ by q_j for $1 \leq j \leq m$. Then Lemma 1 tells us that the critical points in $(f * g)^{-1}(\xi)$ are given by (p_i, q_j) for $1 \leq i \leq \ell$ and $1 \leq j \leq m$.

We set $\alpha_i := \mu_f(p_i)$ and $\beta_j := \mu_g(q_j)$. Then

$$\begin{aligned} \Omega(f * g, \xi) &= \sum_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq m}} (-1)^{\alpha_i + \beta_j} \\ &= \left(\sum_{i=1}^{\ell} (-1)^{\alpha_i} \right) \cdot \left(\sum_{j=1}^m (-1)^{\beta_j} \right) \\ &= \Omega(f, a) \cdot \Omega(g, b). \end{aligned}$$

Hence Equation (15) holds.

Now we complete the proof of Theorem 2. Combining Theorem 1 and Equation (15), we have

$$\begin{aligned}\chi(C(f * g)) &= (-1)^{\dim(M \times N)} \sum_{\xi \in \mathcal{V}(f * g)} (\Omega(f * g, \xi))^2 \\ &= \left((-1)^{\dim M} \sum_{a \in \mathcal{V}(f)} (\Omega(f, a))^2 \right) \cdot \left((-1)^{\dim N} \sum_{b \in \mathcal{V}(g)} (\Omega(g, b))^2 \right) \\ &= \chi(C(f)) \cdot \chi(C(g)).\end{aligned}$$

Hence Equation (5) holds. $\square$

Proof of Theorem 3. Note that

$$\min \mathcal{V}(f_n * f_n) = 1 \quad \text{and} \quad \max \mathcal{V}(f_n * f_n) = (n + 1)^2.$$

For $\xi \in \mathcal{V}(f_n * f_n)$, we set

$$\Gamma_\xi := \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq a, b \leq n + 1 \text{ and } ab = \xi\}.$$

Since the critical values of the critical points of f_n are mutually distinct, we have

$$|\Gamma_\xi| = \Omega(f_n * f_n, \xi). \quad (16)$$

On the other hand, we set

$$\delta(n, \xi) := \left\{ \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \in \delta(n) \mid \phi_{11}\phi_{22} = \phi_{12}\phi_{21} = \xi \right\},$$

where $\delta(n)$ is defined in Equation (13).

We claim that there is a bijection

$$R : \Gamma_\xi \times \Gamma_\xi \rightarrow \delta(n, \xi). \quad (17)$$

In fact, if we define R by

$$R((a_1, b_1), (a_2, b_2)) = \begin{pmatrix} a_1 & a_2 \\ b_2 & b_1 \end{pmatrix},$$

then R is a bijection. Hence Equation (17) holds.

Combining Equations (16) and (17), we have

$$(\Omega(f_n * f_n, \xi))^2 = |\delta(n, \xi)|. \quad (18)$$

Then combining Theorem 1 and Equation (18), we have

$$\begin{aligned}\chi(C(f_n * f_n)) &= \sum_{\xi=1}^{(n+1)^2} (\Omega(f_n * f_n, \xi))^2 \\ &= \sum_{\xi=1}^{(n+1)^2} |\delta(n, \xi)| \\ &= |\delta(n)| \\ &= \Delta(n).\end{aligned}$$

This completes the proof of Theorem 3. $\square$

Proof of Theorem 4. We set

$$I_n := \{1, 2, 3, \dots, n+1\} \quad \text{and} \quad J_n := \{2, 4, 6, \dots, 2(n+1)\}.$$

It is clear that

$$\mathcal{V}(f_1 * f_n) = I_n \cup J_n. \quad (19)$$

Let ξ be an element of Equation (19). Then

$$\Omega(f_1 * f_n, \xi) = \begin{cases} 2, & \text{if } \xi \in I_n \cap J_n, \\ 1, & \text{if } \xi \notin I_n \cap J_n. \end{cases}$$

(i) The case that $n = 2m + 1$.

Since $I_n \cap J_n = \{2, 4, 6, \dots, 2m+2\}$, we have from Theorem 1 that

$$\chi(C(f_1 * f_n)) = 4 \cdot (m+1) + 2m + 2 = 3n + 3.$$

(ii) The case that $n = 2m$.

Since $I_n \cap J_n = \{2, 4, 6, \dots, 2m\}$, we have from Theorem 1 that

$$\chi(C(f_1 * f_n)) = 4m + 2m + 2 = 3n + 2. \quad \square$$

5. Further Numerical Results

Proposition 1. Similarly to Equation (12), we define $h_n : \mathbb{R}P^n \rightarrow \mathbb{R}$ by

$$h_n([x_0, x_1, \dots, x_n]) := \sum_{i=0}^n (i+1)x_i^2.$$

Then the following Table 2 holds:

Table 2. $\chi(C(h_n * h_n))$ for $n \leq 10$.

n	1	2	3	4	5	6	7	8	9	10
$\chi(C(h_n * h_n))$	6	15	24	41	62	87	104	145	182	223

Proof. Using Theorem 1, we can prove Proposition 1 easily. $\square$

Remark 6. We explain why Tables 1 and 2 are different for $n = 3$. There are three critical points in $(f_3 * f_3)^{-1}(4)$ and $(h_3 * h_3)^{-1}(4)$. The critical points correspond to the expressions of 4 as a

product of two positive integers: 1×4 , 2×2 and 4×1 . The indices of these critical points are $3d, 2d$ and $3d$, where

$$d := \begin{cases} 2, & \text{for the case } f_3 * f_3, \\ 1, & \text{for the case } h_3 * h_3. \end{cases}$$

This implies that

$$\Omega(f_3 * f_3, 4) = 3 \text{ and } \Omega(h_3 * h_3, 4) = -1.$$

This is the reason why the tables are different for $n = 3$.

Proposition 2. For $n \geq 1$, let $T(n)$ be the connected sum of n copies of $(S^1)^2$. Let $k_n : T(n) \rightarrow \mathbb{R}$ be the usual height function. More precisely, k_n has $2n + 2$ critical points such that the critical values of any two critical points are distinct. Then the following Table 3 holds:

Table 3. $\chi(C(k_n * k_n))$ for $n \leq 10$.

n	1	2	3	4	5	6	7	8	9	10
$\chi(C(k_n * k_n))$	32	62	112	182	264	406	592	734	944	1326

Proof. Using Theorem 1, we can also prove Proposition 2 easily. $\square$

6. Conclusions

Let $f : M \rightarrow \mathbb{R}$ and $g : N \rightarrow \mathbb{R}$ be Morse functions. We define the function $f + g : M \times N \rightarrow \mathbb{R}$ by

$$(f + g)(p, q) := f(p) + g(q), \quad \forall (p, q) \in M \times N.$$

Then $f + g$ is also a Morse function. We can use $f + g$ when we prove the Künneth formula using the Morse homology (see, for example, [8]).

It may be interesting to study how the results in this paper are changed when we consider $f + g$ instead of $f * g$. This is our further task.

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