International Journal of
*Topology*

MDPI

*Article*

# Chains of Dense $G_\delta$ Sets in Perfect Polish Spaces

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**Abstract**

We prove that in every nonempty perfect Polish space, every dense $G_\delta$ subset contains strictly decreasing and strictly increasing chains of dense $G_\delta$ subsets of length $\mathfrak{c}$, the cardinality of the continuum. As a corollary, this holds in $\mathbb{R}^n$ for each $n \geq 1$. This provides an easy answer to a question of Erdős since the set of Liouville numbers admits a descending chain of cardinality $\mathfrak{c}$, each member of which has the Erdős property. We also present counterexamples demonstrating that the result fails if either the perfection or the Polishness assumption is omitted. Finally, we show that the set $\mathcal{T}$ of real Mahler $T$-numbers is a dense Borel set and contains a strictly descending chain of length $\mathfrak{c}$ of proper dense Borel subsets.

**Keywords:** baire category; dense $G_\delta$-set; perfect Polish spaces; Borel set; Liouville numbers; Erdős property; Mahler $T$-numbers; descriptive set theory

## 1. Introduction

In 1844 Joseph Liouville proved the existence of transcendental numbers [1,2]. He introduced the set $\mathcal{L}$ of real numbers, now known as Liouville numbers, and showed that they are all transcendental. A real number $x$ is said to be a *Liouville number* if for every positive integer $n$, there exists a pair of integers $(p, q)$ with $q > 1$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

Liouville's first example was the Liouville constant. We give that example and two others below.

**Example 1. The Liouville Constant $\ell$**

*Define the Liouville constant:*

$$\lambda = \sum_{n=1}^{\infty} \frac{1}{10^{n!}} = \frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^6} + \frac{1}{10^{24}} + \cdots$$

*Let* $s_m = \sum_{n=1}^{m} \frac{1}{10^{n!}}$. *Each* $s_m \in \mathbb{Q}$, *with denominator* $q_m = 10^{m!}$. *The tail satisfies:*

$$|\lambda - s_m| = \sum_{n=m+1}^{\infty} \frac{1}{10^{n!}} < \frac{1}{10^{(m+1)!-1}}.$$

*Since* $10^{m!} < q_m$, *we obtain*

$$\left| \lambda - \frac{p_m}{q_m} \right| < \frac{1}{q_m^N} \quad \textit{for all } N \textit{ and large } m.$$

*Thus* $\lambda \in \mathcal{L}$.

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Academic Editor: Michel Planat

Received: 10 September 2025
Revised: 26 September 2025
Accepted: 19 November 2025
Published: 1 December 2025

**Citation:** Morris, S.A. Chains of Dense $G_\delta$ Sets in Perfect Polish Spaces. *Int. J. Topol.* **2025**, 2, 20. https://doi.org/10.3390/ijt2040020

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**Example 2. Cantor-like Liouville Number**

*Define*

$$\alpha = \sum_{n=1}^{\infty} \frac{1}{3^{n!}} = \frac{1}{3^1} + \frac{1}{3^2} + \frac{1}{3^6} + \frac{1}{3^{24}} + \cdots$$

*This number lies in the middle-third Cantor set because all digits in base 3 are 0 or 1 (no 2s). Let* $s_m = \sum_{n=1}^{m} \frac{1}{3^{n!}} \in \mathbb{Q}$, *with denominator* $q_m = 3^{m!}$. *The tail estimate gives*

$$|\alpha - s_m| < \frac{1}{3^{(m+1)!-1}} < \frac{1}{q_m^N}$$

*for all* $N$, *for sufficiently large* $m$. *So* $\alpha \in \mathcal{L}$.

**Example 3. Factorial-powered Liouville Series**

*Define*

$$\theta = \sum_{k=1}^{\infty} \frac{1}{k!^k}$$

*Clearly* $s_m = \sum_{k=1}^{m} \frac{1}{k!^k} \in \mathbb{Q}$. *Note that*

$$\frac{1}{k!^k} \leq \frac{1}{k^{k^2}} \quad \text{for all } k \geq 2,$$

*so the tail satisfies*

$$|\theta - s_m| < \sum_{k=m+1}^{\infty} \frac{1}{k^{k^2}} < \frac{1}{q_m^N}$$

*for any* $N$, *using* $q_m = \operatorname{lcm}(1!^k, \ldots, m!^k)$. *Therefore* $\theta \in \mathcal{L}$.

In [3] in 1962, Paul Erdős proved that every real number is the sum of two Liouville numbers (and if non-zero is also the product of two Liouville numbers). He gave two proofs. One was a constructive proof. The other proof used the fact that the set $\mathcal{L}$ of all Liouville numbers is a dense $G_\delta$-set in $\mathbb{R}$ and showed that every dense $G_\delta$-set (defined below) in $\mathbb{R}$ has this property. Erdős asked if any proper subset of the the set of all Liouville numbers has this property, now known as the Erdős property. He conjectured that the set of strong Liouville numbers might have this property. However Petruska in [4] proved that it does not. Indeed the sum of two strong Liouville numbers is either a Liouville number or a rational number. However [5,6] did find examples of proper subsets of the set of Liouville numbers with the Erdős property. Indeed Chalebgwa and Morris [7] proved that there are $2^{\mathfrak{c}}$ subsets of the set of Liouville numbers with the Erdős property.

However, in this paper we find an easy and almost obvious solution to the problem of Erdős by proving that every dense $G_\delta$-subset of $\mathbb{R}$ contains a proper descending chain of length cardinality $\mathfrak{c}$ of dense $G_\delta$-subsets of $\mathbb{R}$. And each of these cardinality $\mathfrak{c}$ subsets of the set of Liouville numbers being a dense $G_\delta$-subset of $\mathbb{R}$ has the Erdős property.

This paper is dedicated to proving the existence of descending and ascending chains of dense $G_\delta$-subsets not just in $\mathbb{R}$ but in any perfect Polish space.

For completeless we recall some definitions. A *Polish space* is a separable topological space which is completely metrizable. Examples include not only complete metric spaces like $\mathbb{R}^n$, for any integer $n$, and any separable Banach space, but also the space of all real irrational numbers. A subset of a topological space is said to be a *$G_\delta$-subset* if it is a countable intersection of open sets of the topological space. The set of real irrational numbers is a $G_\delta$-subset of $\mathbb{R}$. The complement of a $G_\delta$-subset is said to be a *$F_\sigma$-subset*. A *perfect set* in a topological space is a set that is closed and has no isolated points. The *diameter of a set* $A$ in a metric space $(X, d)$ is defined to be $\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}$.

Dense $G_\delta$-subsets of Polish spaces play a central role in descriptive set theory and in Baire category methods. The Baire category theorem asserts that such sets are generic, and they are often used to witness "largeness" in a topological sense.

In this paper we study the internal structure of dense $G_\delta$-sets. Our main result shows that in any nonempty perfect Polish space, each dense $G_\delta$ set admits chains of proper dense $G_\delta$-subsets of continuum length, ordered either ascendingly or descendingly by inclusion. The proof proceeds by constructing a Cantor set inside the given dense $G_\delta$, then using partitions of this Cantor set to build a continuum-scale chain of meager $F_\sigma$ sets and their dense $G_\delta$ complements.

The result illustrates the ubiquity of continuum-sized order-theoretic structures inside dense $G_\delta$-sets. As an application, we derive the corresponding statement for $\mathbb{R}^n$. Finally, we show that the assumptions of perfection and Polishness are both essential by giving counterexamples.

This work builds on classical constructions of Cantor sets in perfect Polish spaces (see e.g., Oxtoby [8], Kechris [9]) and connects with the study of special subsets of the real line (see Miller [10]).

**Lemma 1** (Local separability is automatic)**.** *Let $X$ be a Polish space. Then every subspace $Y \subseteq X$ is second countable and hence separable. In particular, if $G_1, G_2 \subseteq X$ are dense $G_\delta$ sets, then $G_1$, $G_2$, and $G_1 \cap G_2$ are (hereditarily) locally separable.*

**Proof.** Second countability is hereditary for metrizable spaces. Since Polish spaces are metrizable and second countable, every subspace inherits a countable base, hence is second countable and therefore separable. Local separability follows because every point has a neighborhood in the subspace that is second countable (hence separable). □

## 2. Main Result on Descending Chains

**Theorem 1.** *Let $X$ be a nonempty perfect Polish space and let $G \subseteq X$ be a dense $G_\delta$-set. Then there exists a strictly decreasing chain*

$$\{H_t : t \in (0,1)\}$$

*of proper dense $G_\delta$-subsets of $X$, all contained in $G$, indexed by $(0,1)$ and ordered by inclusion: for $s < t$,*

$$H_t \subsetneq H_s \subsetneq G.$$

*In particular, the chain has length* $\mathfrak{c}$.

**Proof.** We divide the proof into four steps.

**Step 1: A Cantor set inside *G*.** Write $G = \bigcap_{k\in\mathbb{N}} U_k$ with each $U_k$ open dense. Since $X$ is perfect and Polish, we may construct a Cantor scheme of compact sets $\{F_\sigma : \sigma \in 2^{<\omega}\}$ such that:

(i) $F_\emptyset \subseteq U_0$ is nonempty compact with empty interior in $X$;
(ii) $F_{\sigma 0}, F_{\sigma 1} \subseteq F_\sigma$ are disjoint compact subsets with $\operatorname{diam}(F_{\sigma i}) \leq 2^{-|\sigma|-2}$;
(iii) $F_\sigma \subseteq \bigcap_{k<|\sigma|} U_k$.

Define

$$P = \bigcap_{n=0}^{\infty} \bigcup_{\sigma\in 2^n} F_\sigma.$$

Then $P$ is homeomorphic to the middle-third Cantor set, nowhere dense in $X$, and contained in $G$.

**Step 2: Partition into disjoint compact nowhere dense sets.** Partition $P$ into a sequence $(P_n)_{n\in\mathbb{N}}$ of pairwise disjoint, nonempty clopen (in $P$) Cantor subsets. Each $P_n$ is compact and nowhere dense in $X$.

**Step 3: A continuum chain of meager $F_\sigma$ sets.** Enumerate $\mathbb{Q}\cap(0,1)$ as $\{q_n : n\in\mathbb{N}\}$. For $t\in(0,1)$ set

$$A_t=\{n : q_n<t\},\quad K_t=\bigcup_{n\in A_t} P_n.$$

Then $K_t$ is a meager $F_\sigma$. Moreover, if $s<t$ then there exists $q_n$ with $s<q_n<t$, hence $K_s\subsetneq K_t$.

**Step 4: Dense $G_\delta$ complements.** Let $D_t=X\setminus K_t$, a dense $G_\delta$. Define

$$H_t=G\cap D_t.$$

Each $H_t$ is a dense $G_\delta$-subset of $X$, properly contained in $G$. For $s<t$ we have $H_t\subsetneq H_s$. This yields the desired chain. □

**Proposition 1** (Cardinality of the chain)**.** *In Theorem 1 the map $t\mapsto H_t$ is injective. Consequently,*

$$\big|\{H_t : t\in(0,1)\}\big|=\mathfrak{c}.$$

**Proof.** If $s<t$, Step 3 of the proof constructs $K_s\subsetneq K_t$, hence $D_t=X\setminus K_t\subsetneq X\setminus K_s=D_s$, so

$$H_t=G\cap D_t\subsetneq G\cap D_s=H_s.$$

Thus $s\neq t\Rightarrow H_s\neq H_t$, i.e., $t\mapsto H_t$ is injective. Since $(0,1)$ has cardinality $\mathfrak{c}$ and a family of $G_\delta$ subsets of a Polish space has size at most $\mathfrak{c}$, we conclude $\big|\{H_t\}\big|=\mathfrak{c}$. □

**Corollary 1.** *Both the descending chain $\{H_t : t\in(0,1)\}$ and the ascending chain $\{K_t : t\in(0,1)\}$ have length $\mathfrak{c}$, hence are uncountable.*

**Remark 1** (Noetherian versus Polish)**.** *A topological space is Noetherian if every ascending chain of open sets stabilizes (equivalently, every descending chain of closed sets stabilizes). Perfect Polish spaces are very far from being Noetherian. Indeed, already in $\mathbb{R}$ one has a non-stabilizing descending chain of closed sets such as $[-1,1]\supsetneq[-1/2,1/2]\supsetneq\cdots$. More structurally, the Cantor set $C\subset\mathbb{R}$ carries a canonical tree of clopen cylinders that yields infinite strictly descending chains of closed nowhere dense subsets.*

*Our main construction (Theorem 1) builds, inside any dense $G_\delta$ of a perfect Polish space, a continuum-long strictly descending chain of dense $G_\delta$ sets. This emphatically contrasts the Noetherian property: not only do closed-set chains fail to stabilize, but even within the comeager realm one can realize chains of maximal possible length $\mathfrak{c}$. In particular, no nonempty perfect Polish space is Noetherian.*

## 3. Ascending Chains

By dualizing the above argument we obtain:

**Corollary 2.** *In the setting of Theorem 1, there exists a strictly increasing chain $\{K_t : t\in(0,1)\}$ of proper dense $G_\delta$-subsets of $X$, all contained in $G$, of length $\mathfrak{c}$.*

**Corollary 3.** *Let $X$ be a nonempty locally compact perfect Polish space. Then every dense $G_\delta$ subset $G\subseteq X$ admits both a strictly increasing and a strictly decreasing chain of proper dense $G_\delta$-subsets of length $\mathfrak{c}$.*

**Proof.** By hypothesis, $X$ is perfect and Polish. Therefore Theorem 1 applies directly to $X$ and to the given dense $G_\delta$ set $G \subseteq X$, yielding a strictly decreasing chain

$$\{H_t : t \in (0,1)\}$$

of proper dense $G_\delta$ subsets of $X$ contained in $G$, with $H_t \subsetneq H_s$ for $s < t$.

For the strictly increasing chain, apply the "ascending" construction (see the corollary following Theorem 1) in the same setting to obtain a chain

$$\{K_t : t \in (0,1)\}$$

of proper dense $G_\delta$-subsets of $X$ contained in $G$ with $K_s \subsetneq K_t$ for $s < t$. In both cases, the index set $(0,1)$ has cardinality $\mathfrak{c}$, so the chains have length $\mathfrak{c}$. □

**Remark 2.** *Typical examples include any nonempty open subset of $\mathbb{R}^n$ and, more generally, any separable manifold without boundary and without isolated points; these are locally compact, perfect, and Polish, hence fall under Corollary 3.*

**Definition 1.** *A subset $X \subseteq \mathbb{R}$ has the* Erdős property *if $X + X = \mathbb{R}$; that is, for every $r \in \mathbb{R}$ there exist $x, y \in X$ with $r = x + y$.*

Erdős proved that the set of all Liouville numbers, despite being thin (Lebesgue measure zero and of Hausdorff dimension zero), has the Erdős property. In fact, he showed that every dense $G_\delta$ subset of $\mathbb{R}$ has this property. Erdős asked whether other proper subsets of the set of Liouville numbers have the property. This was answered in [7], where it was shown (with significant effort) that the set of Liouville numbers has $2^{\mathfrak{c}}$ subsets with the Erdős property. Our main theorem now gives an immediate family: since the Liouville numbers form a dense $G_\delta$-subset of $\mathbb{R}$, they admit a descending chain of length $\mathfrak{c}$ of proper dense $G_\delta$-subsets, and each of these $\mathfrak{c}$ sets has the Erdős property. For the record, there are only $\mathfrak{c}$ many $G_\delta$ (hence Borel) subsets of $\mathbb{R}$; the additional Erdős sets found in [7] are not Borel.

## 4. Applications to Euclidean Spaces

**Corollary 4.** *For each $n \geq 1$, every dense $G_\delta$-subset of $\mathbb{R}^n$ admits both a strictly increasing and a strictly decreasing chain of proper dense $G_\delta$ subsets of length $\mathfrak{c}$.*

**Proof.** The Euclidean space $\mathbb{R}^n$ is a perfect Polish space. Apply Theorem 1. □

## 5. Necessity of Hypotheses

We illustrate that both hypotheses in Theorem 1 are essential.

**Example 4** (Failure without perfection)**.** *Let $X = \mathbb{N}$ with the discrete topology. Then $X$ is Polish but not perfect. Every dense $G_\delta$ is $X$ itself, so no proper dense $G_\delta$ subset exists.*

**Example 5** (Failure without Polishness)**.** *Let $X = \mathbb{Q}$ with the usual topology. Then $X$ is not Polish (not complete). Every dense $G_\delta$ in $X$ is co-meager, but since $X$ itself is meager, no chain of proper dense $G_\delta$ subsets of continuum length can be constructed.*

*Zariski (Noetherian) Spaces*

On an algebraic variety over an algebraically closed field, the Zariski topology is Noetherian: every descending chain of closed sets stabilizes. Consequently, every ascending chain of open sets stabilizes. This fundamentally obstructs our construction.

**Proposition 2.** *Let $X$ be a Noetherian $T_1$ space. Then there is no infinite strictly descending chain of dense $G_\delta$ subsets of $X$.*

**Proof.** Suppose $\{H_\alpha\}_{\alpha\in I}$ is a strictly descending family of dense $G_\delta$ sets. Write $H_\alpha = \bigcap_n U_{\alpha,n}$ with $U_{\alpha,n}$ open and dense. Then $X \setminus H_\alpha = \bigcup_n F_{\alpha,n}$ with $F_{\alpha,n}$ closed nowhere dense. For $\alpha < \beta$, $H_\beta \subsetneq H_\alpha$ implies $X \setminus H_\alpha \subsetneq X \setminus H_\beta$, so we obtain a strictly ascending chain of closed sets. In a Noetherian space such chains stabilize, contradiction. □

Thus our results depend essentially on (non-)Noetherian behavior furnished by the perfect Polish setting (Cantor constructions, meager $F_\sigma$ complements, Baire category). In Zariski spaces the proposed continuum-length chains fail in general.

**Open Question.** Let $X$ be a nonempty perfect Polish space. Does there exist a dense $G_\delta$-set $U \subseteq X$ such that the poset

$$(\mathcal{D}(U), \subseteq),$$

where $\mathcal{D}(U)$ denotes the family of dense $G_\delta$ subsets of $U$ ordered by inclusion, is Tukey universal [9,11] among all such posets arising from dense $G_\delta$ subsets of perfect Polish spaces?

**Remark 3** (On the Open Question)**.** *A natural candidate for universality is any dense $G_\delta$ set $U \subseteq X$ that is homeomorphic to the Baire space $\omega^\omega$ (every nonempty perfect Polish space contains such a dense $G_\delta$; see, e.g., [9]). Since $\omega^\omega$ continuously surjects onto every Polish space, one tempting approach is: for any dense $G_\delta$ $V \subseteq Y$ (with $Y$ perfect Polish), fix a continuous surjection $f : U \to V$ and use the pullback map*

$$\Phi : \mathcal{D}(V) \to \mathcal{D}(U), \qquad \Phi(H) = f^{-1}(H).$$

*This $\Phi$ is monotone and sends dense $G_\delta$ sets to dense $G_\delta$ sets. Thus to obtain a Tukey reduction $(\mathcal{D}(V), \subseteq) \leq_T (\mathcal{D}(U), \subseteq)$ it would suffice to ensure* cofinality*: for every $G \in \mathcal{D}(U)$ there exists $H \in \mathcal{D}(V)$ with $f^{-1}(H) \subseteq G$. Establishing such cofinality uniformly appears to require a mild "category-preserving" hypothesis on $f$ (e.g., openness, or a comeager domain on which $f$ admits a Baire-measurable right inverse), and this is the main technical obstacle. We note that a closely related quotient version* modulo meager sets *is straightforward: if one orders comeager subsets by inclusion modulo the meager ideal, all perfect Polish spaces yield the same Tukey type (cf. [9,11]). Our question is finer, because it insists on working inside $\mathcal{D}(U)$ with raw inclusion rather than modulo meager. In short, a positive solution seems within reach via a universal $U \cong \omega^\omega$, provided one can verify a general category-preserving/uniformization lemma that forces the pullback map to be cofinal.*

## 6. Connections with Mahler's *T*-Numbers

Mahler's classification of transcendental numbers divides them into classes $\mathcal{S}$, $\mathcal{T}$, and $\mathcal{U}$, according to the growth of the approximation exponents $w_d(\xi)$. A real number $\xi$ is a $T$-number precisely when

$$0 < \sup_d w_d(\xi) = \infty \quad \text{while every } w_d(\xi) < \infty.$$

The set $\mathcal{T}$ of all $T$-numbers is known to be nonempty and uncountable by Schmidt's seminal work [12,13].

From the perspective of descriptive set theory, Ki established that $\mathcal{T}$ has precise Borel complexity:

$$\mathcal{T} \in \Pi^0_4 \setminus \Sigma^0_3,$$

so $\mathcal{T}$ is a Borel set, in fact at a relatively low level of the Borel hierarchy [14].

A striking structural property of $\mathcal{T}$ is its density. By Mahler's invariance under algebraic dependence, if $\xi \in \mathcal{T}$ and $q \in \mathbb{Q}$, then $\xi + q \in \mathcal{T}$. Since $\xi + \mathbb{Q}$ is dense in $\mathbb{R}$, it follows that $\mathcal{T}$ itself is dense. Consequently, results such as Theorem 1, which produce continuum-length chains of descending dense analytic sets, apply to $\mathcal{T}$ as well. In particular, $\mathcal{T}$ contains strictly descending chains of proper dense Borel subsets of length $\mathfrak{c}$.

These parallels emphasize that both in number theory (via Mahler's classification) and in descriptive set theory (via dense $G_\delta$-sets in Polish spaces), large sets naturally accommodate long descending chains of dense subsets.

## 7. Mahler's Classification and the Set $\mathcal{T}$

For a real (or complex) number $\xi$, let $w_d(\xi)$ be the supremum of exponents $w$ such that

$$0 < |P(\xi)| \leq H(P)^{-w}$$

for infinitely many integer polynomials $P$ of degree at most $d$, where $H(P)$ denotes the height. Mahler's classes $A, S, T, U$ are defined by the growth of the sequence $d \mapsto w_d(\xi)$. A number $\xi$ is a *T-number* iff

$$0 < \sup_d w_d(\xi) = \infty \quad \text{while each } w_d(\xi) < \infty.$$

Mahler proved the following fundamental property.

**Theorem 2** (Invariance under algebraic dependence)**.** *If $\xi, \eta$ are algebraically dependent over $\mathbb{Q}$, then they belong to the same Mahler class.*

In particular, for any rational $q$, $\xi$ and $\xi + q$ are algebraically dependent; hence $\xi \in \mathcal{T} \Rightarrow \xi + q \in \mathcal{T}$ [15] (Theorem 2.3).

The set $\mathcal{T}$ of real $T$-numbers is known to be nonempty (indeed uncountable) by Schmidt's work [12,13]. Moreover, its descriptive set-theoretic complexity is sharp: $\mathcal{T} \in \Pi^0_4$ but $\mathcal{T} \notin \Sigma^0_3$ [14].

## 8. Density of $\mathcal{T}$

**Theorem 3.** *The set $\mathcal{T} \subseteq \mathbb{R}$ is dense.*

**Proof.** By Schmidt's seminal work on $T$-numbers [12,13], there exists $\tau \in \mathcal{T}$. By Theorem 2, $\tau + q \in \mathcal{T}$ for all $q \in \mathbb{Q}$ [15] (Theorem 2.3). Since $\tau + \mathbb{Q}$ is dense in $\mathbb{R}$, and $\tau + \mathbb{Q} \subseteq \mathcal{T}$, it follows that $\mathcal{T}$ is dense, because any superset of a dense set is dense. □

## 9. Complexity Remark

From Ki's work [14] we have $\mathcal{T} \in \Pi^0_4 \setminus \Sigma^0_3$. In particular, $\mathcal{T}$ is Borel and hence automatically analytic ($\mathbf{\Sigma}^1_1$).

## 10. Perfect-Set Input for Analytic Sets

We use the standard perfect-set theorem: every uncountable analytic subset of a Polish space contains a nonempty perfect subset (hence a homeomorphic copy of the Cantor set) [9] (Perfect Set Theorem).

**Proposition 3.** *Let $A \subseteq \mathbb{R}$ be a dense analytic set. Then there exists a strictly descending chain $\{A_\alpha : \alpha < \mathfrak{c}\}$ of* proper *dense analytic subsets of $\mathbb{R}$ with $A_\beta \subsetneq A_\alpha \subseteq A$ for $\alpha < \beta < \mathfrak{c}$. Moreover, if $A$ is Borel then each $A_\alpha$ can be chosen Borel.*

**Construction.** Since $A$ is dense it is uncountable. By the perfect-set theorem, fix a Cantor set $P \subseteq A$ [9]. Let $h : 2^\omega \to P$ be a homeomorphism (so $P$ is closed, perfect, and nowhere dense in $\mathbb{R}$).

Equip $2^\omega$ with the lexicographic order $\leq_{\text{lex}}$. Choose a well-order $(s_\alpha)_{\alpha<\mathfrak{c}}$ of $2^\omega$ that is cofinal in $(2^\omega, \leq_{\text{lex}})$. For each $\alpha < \mathfrak{c}$, set

$$K_\alpha = h\big(\{x \in 2^\omega : x \leq_{\text{lex}} s_\alpha\}\big).$$

Each $K_\alpha$ is closed in $P$, hence closed in $\mathbb{R}$, and $K_\alpha \subseteq P \subseteq A$. Moreover $\alpha < \beta \Rightarrow K_\alpha \subsetneq K_\beta$, and no $K_\alpha$ has interior in $\mathbb{R}$ (since $K_\alpha \subseteq P$ and $P$ has empty interior), so each $K_\alpha$ is nowhere dense.

Now define

$$A_\alpha = A \setminus K_\alpha.$$

Then:

- $A_\alpha$ is analytic (analytic minus closed is analytic); if $A$ is Borel then $A_\alpha$ is Borel.
- $A_\alpha$ is dense: removing a closed nowhere dense set from a dense set preserves density.
- The chain is strict and descending since $K_\alpha \subsetneq K_\beta \subseteq A$ implies $A_\beta = A \setminus K_\beta \subsetneq A \setminus K_\alpha = A_\alpha$.

This gives the required chain of length $\mathfrak{c}$. □

## 11. Descending Chain of Borel Sets in $\mathcal{T}$

**Theorem 4.** *Let $\mathcal{T} \subseteq \mathbb{R}$ be the set of Mahler T-numbers. Then $\mathcal{T}$ is dense and Borel ($\Pi^0_4$), and $\mathcal{T}$ contains a strictly descending chain of length $\mathfrak{c}$ of proper dense Borel (hence analytic) subsets.*

**Proof.** Density follows from Theorem 3, and the Borel complexity from [14]. Applying Proposition 3 with $A = \mathcal{T}$, we obtain the chain of dense Borel subsets. □

**Funding:** This research received no external funding.

**Data Availability Statement:** No new data were created or analyzed in this study.

**Conflicts of Interest:** The author declares no conflicts of interest.

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