

Article

Extensions of Multidirected Graphs: Fuzzy, Neutrosophic, Plithogenic, Rough, Soft, Hypergraph, and Superhypergraph Variants

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Abstract

Graph theory models relationships by representing entities as vertices and their interactions as edges. To handle directionality and multiple head–tail assignments, various extensions—directed, bidirected, and multidirected graphs—have been introduced, with the multidirected graph unifying the first two. In this work, we further enrich this landscape by proposing the Multidirected hypergraph, which merges the flexibility of hypergraphs and superhypergraphs to describe higher-order and hierarchical connections. Building on this, we introduce five uncertainty-aware Multidirected frameworks—fuzzy, neutrosophic, plithogenic, rough, and soft multidirected graphs—by embedding classical uncertainty models into the Multidirected setting. We outline their formal definitions, examine key structural properties, and illustrate each with examples, thereby laying groundwork for future advances in uncertain graph analysis and decision-making.

Keywords: superhypergraph; hypergraph; fuzzy graph; neutrosophic graph; multidirected graph; multidirected hypergraph; rough graph; soft graph



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1. Introduction

1.1. Various Graph-Theoretic Frameworks

Graph theory provides a framework in which entities are represented by vertices and their pairwise interactions by edges, underpinning diverse applications in fields such as artificial intelligence, network analysis, and chemical modeling [1,2]. In this paper, we explore the following concepts:

- **Hypergraphs and Superhypergraphs:** Higher-order and hierarchical generalizations of ordinary graphs[3–7]
- **Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Graphs:** Frameworks for modeling uncertainty and contradiction[8–12]
- **Directed, Bidirected, and Multidirected Graphs:** Structures for representing asymmetric and multi-headed connections[13–18].

1.1.1. Hypergraphs and Superhypergraphs

Whereas standard graphs permit edges only between pairs of vertices, hypergraphs generalize this notion by allowing each hyperedge to connect any nonempty subset of vertices, thus capturing higher-order relationships [7,19–23]. Superhypergraphs take this further by iteratively applying the powerset operation: at each level, “supervertices” and “superedges” are drawn from the previous powerset, yielding a multi-layered hierarchy

of connections [24–27]. Table 1 contrasts the key characteristics of ordinary graphs, hypergraphs, and n -superhypergraphs.

Table 1. Comparison of basic structures: graph, hypergraph, and n -superhypergraph.

Structure	Vertex Set	Edge Set	Arity	Notes
Graph	V	$E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$	2	Binary relations only
HyperGraph	V	$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$	≥ 1	Multi-way relations
n -SuperHyperGraph	$V \subseteq \mathcal{P}^n(V_0)$	$E \subseteq \mathcal{P}^n(V_0)$	≥ 1	Hierarchical “super” structure

1.1.2. Graph Models Under Uncertainty

Several extensions of classical graph theory have been proposed to accommodate uncertainty in vertices and edges. In fuzzy graphs, each vertex and edge is assigned a membership value in $[0, 1]$ to express gradual inclusion [8,9]. Intuitionistic fuzzy graphs further distinguish non-membership degrees alongside membership [28–31], while neutrosophic graphs introduce an indeterminacy component in addition to truth and falsity levels [32–34]. Plithogenic graphs take this a step further by quantifying contradictions between attribute values [35]. Alternative approaches include rough graphs, which use lower and upper approximations to handle vagueness in both nodes and edges [36,37], soft graphs, which parameterize subsets of vertices and edges via soft set theory [38,39], and vague graphs, which model ambiguity using vague sets [40–42].

Table 2 provides an overview of these uncertainty-aware graph frameworks.

Table 2. Summary of fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic graph frameworks.

Structure	Vertex Description	Edge Description	Arity	Notes
Fuzzy Graph	V with $\sigma : V \rightarrow [0, 1]$	$E \subseteq V \times V$ with $\mu : E \rightarrow [0, 1]$	2	Models gradual membership of vertices and edges
Intuitionistic Fuzzy Graph	V with $\lambda, \nu : V \rightarrow [0, 1]$	$E \subseteq V \times V$ with $\lambda, \nu : E \rightarrow [0, 1]$	2	Captures both membership (λ) and non-membership (ν)
Neutrosophic Graph	V with $T, I, F : V \rightarrow [0, 1]$	$E \subseteq V \times V$ with $T, I, F : E \rightarrow [0, 1]$	2	Includes truth, indeterminacy, and falsity degrees
Plithogenic Graph	V with $\text{adf}, \text{acf} : V \times M_l \rightarrow [0, 1]$	$E \subseteq V \times V$ with $\text{bdf}, \text{bcf} : E \times N_m \rightarrow [0, 1]$	2	Models attribute appurtenance and contradiction

1.1.3. Directed, Bidirected, and Multidirected Graphs

To capture directional and multiplex relationships, a range of graph extensions have been introduced. Directed graphs assign a single orientation to each edge, while bidirected graphs allow each vertex at the ends of an edge to have its own independent orientation [43]. Multidirected graphs generalize both models by allowing multiple directed edges (arcs) between the same pair of vertices, each with its own direction and multiplicity [44,45].

In this paper, we focus on multidirected graphs, as they encompass both directed and bidirected structures through explicit encoding of edge directions and counts. Table 3 summarizes the differences among directed, bidirected, and multidirected graph variants.

Table 3. Comparison of directed, bidirected, and multidirected graph variants.

Structure	Vertex Set	Edge Description	Key Feature
Directed Graph	V	$E \subseteq V \times V$ (ordered pairs (u, v))	Each edge has a single orientation from source to target [46]
Bidirected Graph	V	$E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$ with $\tau : V \times E \rightarrow \{-1, 0, 1\}$	Each endpoint of an undirected edge carries its own local direction [43]
Multidirected Graph	V	Multiset of arcs with $s, t : E \rightarrow V$ and $m : V \times V \rightarrow \mathbb{N}_0$	Allows multiple parallel edges; records source, target, and multiplicity [44]

1.2. Our Contributions

Building on these developments, this paper defines a suite of new Multidirected structures by combining multidirected graphs with existing graph and hypergraph extensions. We introduce the fuzzy multidirected graph, neutrosophic multidirected graph, plithogenic multidirected graph, rough multidirected graph, soft multidirected graph, Multidirected hypergraph, and Multidirected superhypergraph. In addition to these, we also explore several new graph classes, such as the bimixed graph and the hyperdirected graph. For each of these classes, we present formal definitions, investigate their core mathematical properties, and illustrate them with concrete examples.

It should be noted that this paper is a theoretical investigation conducted entirely at the conceptual level. Throughout this paper, we assume that all graphs are finite. We anticipate that these contributions will enhance both the theoretical foundations and potential applications of graph theory and its diverse extensions.

1.3. Structure of This Paper

This subsection outlines the structure of the paper. Section 2 introduces a range of foundational graph structures, including hypergraph, superhypergraph, directed graph, directed hypergraph, directed superhypergraph, fuzzy graph, neutrosophic graph, plithogenic graph, fuzzy directed graph, neutrosophic directed graph, plithogenic directed graph, bidirected graph, and multidirected graph. Section 3 examines the structural properties of advanced variants such as the fuzzy multidirected graph, neutrosophic multidirected graph, plithogenic multidirected graph, rough multidirected graph, soft multidirected graph, Multidirected hypergraph, and Multidirected superhypergraph. It also explores the characteristics of newly proposed graph classes, including the bimixed graph. Section 4 provides a brief summary of the conclusions and discusses directions for future research.

2. Preliminaries

In this section, we review the fundamental notions and notation that underpin the developments in this paper. We restrict our attention to finite graphs throughout. The symbol n will always denote a nonnegative integer unless specified otherwise.

2.1. Graphs and Hypergraphs

A hypergraph extends the concept of an ordinary graph by permitting hyperedges—subsets of the vertex set of size at least two—thereby capturing multiway relationships among elements [47–54]. Below are the formal definitions of a graph, its subgraph, and a hypergraph.

Definition 1 (Graph [55]). A graph G consists of a pair

$$G = (V, E),$$

where V is a nonempty set of vertices and $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$ is a set of edges, each edge joining two distinct vertices.

Definition 2 (Hypergraph [7,56]). A hypergraph H is a pair

$$H = (V(H), E(H)),$$

where:

- $V(H)$ is a nonempty set of vertices.
- $E(H)$ is a collection of nonempty subsets of $V(H)$, called hyperedges.

Each hyperedge may connect any number of vertices, allowing the model to represent higher-order relationships. In this work, we assume $V(H)$ and $E(H)$ are both finite.

Example 1 (Recipe Ingredient Hypergraph). Consider a cookbook where each recipe combines several ingredients. Let

$$V(H) = \{\text{Flour, Sugar, Eggs, Milk, Butter, Vanilla}\}$$

be the set of all ingredients. Define the set of hyperedges $E(H)$ by letting each recipe correspond to the subset of ingredients it uses, for example:

$$\begin{aligned} e_{\text{Pancake}} &= \{\text{Flour, Eggs, Milk, Butter}\}, \\ e_{\text{Cookie}} &= \{\text{Flour, Sugar, Butter, Vanilla}\}, \\ e_{\text{Omelette}} &= \{\text{Eggs, Milk, Butter}\}, \\ e_{\text{Cake}} &= \{\text{Flour, Sugar, Eggs, Milk, Butter, Vanilla}\}. \end{aligned}$$

Then

$$H = (V(H), E(H))$$

is a hypergraph in which each hyperedge represents a recipe connecting multiple ingredients simultaneously. This structure naturally models the higher-order relationships among ingredients in culinary applications.

2.2. Powerset and nth-Powerset

In this paper, we use the concepts of powerset and nth-powerset. The powerset of a set S is the collection of every subset of S , including both the empty set and S itself. The n -th powerset $P_n(S)$ arises by applying the powerset operator n times successively, building nested collections of subsets at each level [57–59]. Below, we formally define the n -th powerset as a foundation for these structures.

Definition 3 (Base Set [60]). A base set S serves as the fundamental universe from which all derived constructions—such as powersets and hyperstructures—are built. Concretely,

$$S = \{x \mid x \text{ belongs to the domain under consideration}\}.$$

Every element of any higher-order construct like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ is drawn from this original set S .

Definition 4 (Powerset). For any set S , its powerset, written $\mathcal{P}(S)$, is the set of all subsets of S , including both the empty set and S itself. Formally,

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 5 (n -th Powerset [61,62]). Starting from a base set H , one defines its iterated powersets by

$$\mathcal{P}_1(H) = \mathcal{P}(H), \quad \mathcal{P}_{n+1}(H) = \mathcal{P}(\mathcal{P}_n(H)), \quad n \geq 1.$$

If one wishes to exclude the empty set at each stage, the non-empty n -th powerset is given by

$$\mathcal{P}_1^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}, \quad \mathcal{P}_{n+1}^*(H) = \mathcal{P}(\mathcal{P}_n^*(H)) \setminus \{\emptyset\}.$$

Example 2 (Meal Planning via n -th Powerset). Let

$$H = \{\text{Breakfast}, \text{Lunch}\}.$$

Then the first powerset

$$\mathcal{P}_1(H) = \{\emptyset, \{\text{Breakfast}\}, \{\text{Lunch}\}, \{\text{Breakfast}, \text{Lunch}\}\}$$

is the set of all possible daily menus. The second powerset

$$\mathcal{P}_2(H) = \mathcal{P}(\mathcal{P}_1(H))$$

consists of all two-day meal sequences—for example, $\{\{\text{Breakfast}\}, \{\text{Lunch}\}\}$ (represents “Breakfast” on day one, “Lunch” on day two) or $\{\{\text{Breakfast}, \text{Lunch}\}, \{\text{Breakfast}\}\}$ (represents a full-day meal followed by “Breakfast” only). Likewise, the third powerset

$$\mathcal{P}_3(H) = \mathcal{P}(\mathcal{P}_2(H))$$

models weekly menus as collections of two-day plans. In general, $\mathcal{P}_n(H)$ encodes an n -layered planning framework:

- $\mathcal{P}_1(H)$: individual days,
- $\mathcal{P}_2(H)$: multi-day sequences,
- $\mathcal{P}_3(H)$: weekly bundles,
- ...

2.3. Hyperstructures and Superhyperstructures

In order to build a unified theoretical framework for hyperstructures [63–68] and their higher-order counterparts, superhyperstructures [61], we recall several essential definitions. In particular, any hypergraph can be viewed as a hyperstructure, and likewise, a superhypergraph naturally induces a superhyperstructure.

Definition 6 (Classical Structure [61,69]). A classical structure consists of a nonempty set H together with one or more operations—called classical operations—that map tuples of elements back into H and satisfy prescribed axioms. Formally, an m -ary classical operation is a function

$$\#_0 : H^m \longrightarrow H,$$

where H^m denotes the Cartesian product of m copies of H . Examples include addition or multiplication in familiar algebraic systems.

Definition 7 (Hyperoperation [70,71]). A hyperoperation generalizes a binary operation by allowing its output to be a subset rather than a single element. For a set S , a hyperoperation $\circ$ is a map

$$\circ : S \times S \longrightarrow \mathcal{P}(S),$$

where $\mathcal{P}(S)$ is the power set of S .

Definition 8 (Hyperstructure [61,69]). A hyperstructure is obtained by replacing the underlying set of a classical structure with its power set. Concretely, given a base set S , a hyperstructure is the pair

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where $\circ$ is a hyperoperation acting on subsets of S .

Example 3 (Collaborative Project Planning as a Hyperstructure). Suppose a research lab has a set of distinct tasks.

$$S = \{\text{Literature Review, Experimentation, Data Analysis, Manuscript Writing}\}.$$

We form the hyperstructure

$$\mathcal{H} = (\mathcal{P}(S), \circ)$$

by letting each element of $\mathcal{P}(S)$ represent a possible “mini-project” (any subset of tasks). Define the hyperoperation $\circ$ on two mini-projects $A, B \subseteq S$ by

$$A \circ B = A \cup B,$$

which corresponds to merging two partial plans into a combined plan. For instance, if

$$A = \{\text{Literature Review, Experimentation}\}, \quad B = \{\text{Data Analysis, Manuscript Writing}\},$$

then

$$A \circ B = \{\text{Literature Review, Experimentation, Data Analysis, Manuscript Writing}\}.$$

Thus, $\mathcal{H}$ models how independent sub-projects can be dynamically joined, and the union operation on subsets of tasks is precisely the hyperoperation giving $\mathcal{H}$ its hyperstructure character.

Definition 9 (SuperHyperOperation [61]). Let H be a nonempty set and define its iterated powersets by

$$P^0(H) = H, \quad P^{k+1}(H) = \mathcal{P}(P^k(H)) \quad (k \geq 0).$$

An (m, n) -SuperHyperOperation is an m -ary mapping

$$\circ^{(m,n)} : H^m \longrightarrow P_*^n(H),$$

where $P_*^n(H)$ denotes either the full n th powerset $P^n(H)$ (in the neutrosophic variant) or its nonempty subcollection (in the classical variant). These operations extend hyperoperations by producing nested subsets at level n .

Definition 10 (n -Superhyperstructure [61,69,72]). An n -superhyperstructure arises when one replaces the base set of a hyperstructure by its n th iterated powerset. Formally, for a set S ,

$$\mathcal{SH}_n = (P^n(S), \circ),$$

where $\circ$ is a mapped hyperoperation on $P^n(S)$.

Example 4 (Organizational Hierarchy as a 2-Superhyperstructure). Let

$$S = \{\text{Taro, Jiro, Saburo}\}.$$

The first iterated powerset is

$$\begin{aligned} P^1(S) = \{ & \emptyset, \{\text{Taro}\}, \{\text{Jiro}\}, \{\text{Saburo}\}, \{\text{Taro, Jiro}\}, \\ & \{\text{Taro, Saburo}\}, \{\text{Jiro, Saburo}\}, \\ & \{\text{Taro, Jiro, Saburo}\} \}, \end{aligned}$$

which corresponds to all possible committees. Taking the powerset once more,

$$\mathcal{P}^2(S) = \mathcal{P}(P^1(S)),$$

yields collections of these committees, i.e., councils. For example, one might choose

$$U = \{\{\text{Taro}\}, \{\text{Jiro, Saburo}\}\}, \quad V = \{\{\text{Jiro}\}, \{\text{Taro, Jiro}\}\},$$

two distinct councils in $\mathcal{P}^2(S)$. Defining the hyperoperation $\circ: \mathcal{P}^2(S) \times \mathcal{P}^2(S) \rightarrow \mathcal{P}^2(S)$ by union,

$$U \circ V = U \cup V,$$

turns $(\mathcal{P}^2(S), \circ)$ into a 2-superhyperstructure. Here each “supervertex” is a council of committees, and $\circ$ models the merger of councils.

Example 5 (Academic Curriculum as a 3-Superhyperstructure). Consider the universe of individual courses.

$$S = \{\text{CS101, MATH101, ENG101, HIST201}\}.$$

- The first powerset $\mathcal{P}^1(S) = \mathcal{P}(S)$ consists of all possible subsets of courses (e.g., $\{\text{CS101, MATH101}\}$, $\{\text{ENG101}\}$, etc.), each representing a single semester course selection.
- The second powerset $\mathcal{P}^2(S) = \mathcal{P}(P^1(S))$ gathers collections of semester plans into academic years. For example,

$$Y_1 = \{\{\text{CS101, MATH101}\}, \{\text{ENG101, HIST201}\}\},$$

$$Y_2 = \{\{\text{CS101, ENG101}\}, \{\text{MATH101, HIST201}\}\}.$$

Each Y_i is a set of two semester-course-sets.

- The third powerset $\mathcal{P}^3(S) = \mathcal{P}(\mathcal{P}^2(S))$ collects these year plans into a multi-year curriculum structure. For example, one element of $\mathcal{P}^3(S)$ might be

$$C = \{Y_1, Y_2\},$$

representing a two-year program consisting of Year 1 and Year 2.

Define a hyperoperation $\circ$ on $\mathcal{P}^3(S)$ by union of curricula:

$$C \circ D = C \cup D,$$

which merges two multi-year programs into a longer curriculum. Then

$$\mathcal{SH}_3 = (\mathcal{P}^3(S), \circ)$$

is a 3-superhyperstructure modeling nested educational plans from courses to semesters to years to full programs.

2.4. Superhypergraph

A superhypergraph generalizes the notion of a hypergraph by embedding recursive, layered structures via iterated power-set constructions [26,73]. In essence, one builds successive “levels” of vertices and edges by repeatedly taking power sets, yielding a rich, hierarchical network.

Definition 11 (*n*-Superhypergraph [26,74]). Let V_0 be a finite “base” vertex set. Define the sequence of iterated power-sets by

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0)), \quad k \geq 0.$$

An *n*-superhypergraph is then any pair

$$\text{SuHyG}^{(n)} = (V, E) \quad \text{with} \quad V \subseteq \mathcal{P}^n(V_0), \quad E \subseteq \mathcal{P}^n(V_0).$$

Here each element of V is called an *n*-supervertex, and each element of E an *n*-superedge.

Example 6 (2-Superhypergraph for Department Collaboration). Imagine an organization with four staff members:

$$V_0 = \{\text{Hiroko, Shinya, Tae, Masahiro}\}.$$

The first power set $\mathcal{P}^1(V_0)$ lists all possible teams. Choose three two-person teams:

$$T_1 = \{\text{Hiroko, Shinya}\}, \quad T_2 = \{\text{Shinya, Tae}\}, \quad T_3 = \{\text{Tae, Masahiro}\}.$$

These teams serve as “supervertices” in $\mathcal{P}^2(V_0)$. Select two departments:

$$v_1 = \{T_1, T_2\}, \quad v_2 = \{T_2, T_3\},$$

so that

$$V = \{v_1, v_2\} \subset \mathcal{P}^2(V_0).$$

Define a single 2-superedge

$$e = \{v_1, v_2\}$$

to indicate cross-department projects. Thus

$$\text{SuHyG}^{(2)} = (V, \{e\})$$

captures both the nested team structure and their collaboration.

Example 7 (3-Superhypergraph for Academic Degree Structures). Consider a university curriculum with a base set of individual lectures:

$$V_0 = \{\ell_1, \ell_2, \ell_3, \ell_4\},$$

where each ℓ_i is a distinct lecture.

Level 1 (Courses). The first power set $\mathcal{P}^1(V_0)$ lists all possible subsets of lectures. Select two courses as “1-supervertices”:

$$C_A = \{\ell_1, \ell_2\}, \quad C_B = \{\ell_3, \ell_4\}.$$

Hence

$$V^{(1)} = \{C_A, C_B\} \subset \mathcal{P}^1(V_0).$$

Level 2 (Departments). The second power set $\mathcal{P}^2(V_0)$ consists of subsets of $\mathcal{P}^1(V_0)$. Choose two departments as “2-supervertices”:

$$D_X = \{C_A\}, \quad D_Y = \{C_A, C_B\}.$$

Thus

$$V^{(2)} = \{D_X, D_Y\} \subset \mathcal{P}^2(V_0).$$

Level 3 (Degree Programs). The third power set $\mathcal{P}^3(V_0)$ comprises subsets of $\mathcal{P}^2(V_0)$. Define two degree programs as “3-supervertices”:

$$P_\alpha = \{D_X\}, \quad P_\beta = \{D_X, D_Y\}.$$

So

$$V^{(3)} = \{P_\alpha, P_\beta\} \subset \mathcal{P}^3(V_0).$$

3-Superedges. To model program-level collaborations or joint curricula, select two “3-superedges” in $\mathcal{P}^3(V_0)$:

$$E^{(3)} = \{\{P_\alpha, P_\beta\}, \{P_\beta\}\}.$$

Altogether, the 3-Superhypergraph

$$\text{SuHyG}^{(3)} = (V^{(3)}, E^{(3)})$$

captures the hierarchical structure of lectures $\rightarrow$ courses $\rightarrow$ departments $\rightarrow$ degree programs, as well as inter-program relationships at the top level.

Definition 12 (*n*-Superhypertree (cf. [75])). An *n*-superhypergraph $\text{SuHyG}^{(n)} = (V, E)$ is an *n*-superhypertree if there exists an ordinary tree $T = (V, E_T)$ (the “host tree”) such that:

1. Each superedge $e \in E$ corresponds to a connected subtree $T_e \subseteq T$.
2. T is acyclic (by definition of a tree).
3. For any two supervertices $v, w \in V$, there is a chain of superedges $e_1, \dots, e_k \in E$ with

$$v \in e_1, \quad w \in e_k, \quad e_i \cap e_{i+1} \neq \emptyset \quad (1 \leq i < k).$$

Example 8 (2-SuperHypertree of Reporting Lines). Using the same base $V_0 = \{\text{Hiroko, Shinya, Tae, Masahiro}\}$, let

$$v_A = \{\{\text{Hiroko}\}\}, v_B = \{\{\text{Shinya}\}\}, v_C = \{\{\text{Tae}\}\}, v_D = \{\{\text{Masahiro}\}\}$$

be the 2-supervertices. Form the host tree T with edges $(v_A, v_C), (v_B, v_C), (v_C, v_D)$, modeling direct reports. Each tree-edge gives a superedge, for example

$$e_{AC} = \{v_A, v_C\}, e_{BC} = \{v_B, v_C\}, e_{CD} = \{v_C, v_D\}.$$

Hence

$$\text{SuHyG}^{(2)} = (\{v_A, v_B, v_C, v_D\}, \{e_{AC}, e_{BC}, e_{CD}\})$$

is a valid 2-superhypertree reflecting the organization's acyclic reporting hierarchy.

2.5. Directed Hypergraphs

Directed hypergraphs generalize directed graphs by permitting each “edge” (often called a hyperarc) to originate from several source vertices and terminate at a single target vertex. This richer structure has found applications in areas such as database dependencies, bioinformatics, and complex network models (see, e.g., [76,77]).

Definition 13 (Directed Graph [46]). A directed graph (or digraph) is a pair $G = (V, E)$, where V is a finite set of vertices and

$$E \subseteq V \times V$$

is a set of ordered pairs. An element $(u, v) \in E$ represents a one-way arc from u (the source) to v (the target).

Definition 14 (Directed Hypergraph [76,78]). A directed hypergraph is a pair $H = (V, E)$, where V is a finite vertex set and E is a collection of hyperarcs. Each hyperarc

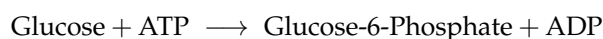
$$e = (\text{Tail}(e), \text{Head}(e))$$

consists of a nonempty subset $\text{Tail}(e) \subseteq V$ and a single vertex $\text{Head}(e) \in V$. It encodes a directed connection from every vertex in the tail set to the head vertex. If $|\text{Tail}(e)| = 1$ for all e , then H reduces to an ordinary digraph.

Example 9 (Chemical Reaction Network). Consider the phosphorylation of glucose in a simple metabolic pathway. Let

$$V = \{\text{Glucose, ATP, Glucose-6-Phosphate, ADP}\}$$

be the set of chemical species. We model the enzyme-catalyzed reaction



as two directed hyperarcs:

$$e_1 = (\{\text{Glucose, ATP}\}, \text{Glucose-6-Phosphate}), \quad e_2 = (\{\text{Glucose, ATP}\}, \text{ADP}).$$

Here, each hyperarc's tail is the pair of reactants and its head is one of the products. Thus,

$$H = (V, \{e_1, e_2\})$$

is a directed hypergraph capturing the one-to-many mapping from the reactant set to each product. If we restricted every tail to size one, we would recover an ordinary directed graph.

To represent nested and hierarchical relationships, the powerset construction can be applied iteratively. The definition of a directed n -SuperHypergraph is presented below.

Definition 15 (Directed n -SuperHypergraph [79]). Starting from a finite base set V_0 , define

$$\mathcal{P}^0(V_0) = V_0, \quad \mathcal{P}^{k+1}(V_0) = \mathcal{P}(\mathcal{P}^k(V_0)) \quad (k \geq 0).$$

A directed n -superhypergraph is then a pair

$$\text{DSH}_n = (V, E),$$

where

$$V \subseteq \mathcal{P}^n(V_0) \quad \text{and} \quad E \subseteq \mathcal{P}(V) \times \mathcal{P}(V).$$

Each n -supervertex in V is an element of $\mathcal{P}^n(V_0)$, and each n -superhyperedge in E is an ordered pair

$$e = (T(e), H(e)),$$

with $T(e) \subseteq V$ (the tail set) and $H(e) \subseteq V$ (the head set).

Example 10 (Document-Approval Flow as a Directed 2-SuperHypergraph). Let $V_0 = \{A, B, C\}$ denote three roles: authors A, B and reviewer C . Form $\mathcal{P}^2(V_0)$ and select these 2-supervertices:

$$v_{\text{Draft}} = \{\{A, B\}\}, \quad v_{\text{Review}} = \{\{C\}\}, \quad v_{\text{Publish}} = \{\{A, B\}, \{C\}\}.$$

Thus $V = \{v_{\text{Draft}}, v_{\text{Review}}, v_{\text{Publish}}\}$. Define three directed 2-superhyperedges:

$$e_1 = (\{v_{\text{Draft}}\}, \{v_{\text{Review}}\}), \quad e_2 = (\{v_{\text{Review}}\}, \{v_{\text{Publish}}\}), \quad e_3 = (\{v_{\text{Review}}\}, \{v_{\text{Draft}}\}),$$

where e_3 captures feedback. Then

$$E = \{e_1, e_2, e_3\} \subseteq \mathcal{P}(V) \times \mathcal{P}(V),$$

and $\text{DSH}_2 = (V, E)$ models the multi-stage approval process from drafting through review to publication.

2.6. Plithogenic and Soft Directed Graphs

The classical notions of fuzzy sets [80–82], intuitionistic fuzzy sets [83–85], neutrosophic sets [86–88], and plithogenic sets [89,90] have inspired a range of corresponding graph models. In particular, one encounters fuzzy graphs [8,91], intuitionistic fuzzy graphs [92,93], neutrosophic graphs [94,95], and plithogenic graphs [35,96]. Their directed analogues—fuzzy directed graphs [97,98], intuitionistic fuzzy directed graphs [99,100], neutrosophic directed graphs [101,102], and plithogenic directed graphs [103]—have also been developed. Below, we recall the formal definitions of these four directed-graph variants.

Definition 16 (Fuzzy Directed Graph [97,104]). A fuzzy directed graph is a quadruple

$$G_F = (V, E, \sigma, \mu),$$

where

- V is the vertex set,
- E is the set of directed edges,
- $\sigma : V \rightarrow [0, 1]$ assigns each vertex a membership degree, and
- $\mu : E \rightarrow [0, 1]$ assigns each edge a membership degree.

Example 11 (Fuzzy Directed Graph for Urban Supply Chain). Consider a small supply network of three warehouses:

$$V = \{W_A, W_B, W_C\}, \quad E = \{e_{AB} = (W_A, W_B), e_{BC} = (W_B, W_C), e_{CA} = (W_C, W_A)\}.$$

We model the operational reliability of each warehouse and the reliability of each delivery route with fuzzy membership degrees.

Vertex Membership. Define

$$\sigma : V \rightarrow [0, 1],$$

by

$$\sigma(W_A) = 0.85, \quad \sigma(W_B) = 0.75, \quad \sigma(W_C) = 0.65.$$

Here, $\sigma(W_i)$ represents the confidence that warehouse W_i is operational.

Edge Membership. Define

$$\mu : E \rightarrow [0, 1],$$

by

$$\mu(e_{AB}) = 0.80, \quad \mu(e_{BC}) = 0.70, \quad \mu(e_{CA}) = 0.60.$$

Each $\mu(e_{ij})$ quantifies the reliability of the directed shipment from W_i to W_j .

The tuple

$$G_F = (V, E, \sigma, \mu)$$

is thus a fuzzy directed graph representing the uncertain reliability of both warehouses and their interconnecting routes in this urban supply chain.

Definition 17 (Intuitionistic Fuzzy Directed Graph [100,105]). Given a digraph $D = (V, E)$, an intuitionistic fuzzy directed graph is a pair

$$G = (A, B),$$

where

- $A = (\lambda_A, \nu_A)$ is an intuitionistic fuzzy set on V ,
- $B = (\lambda_B, \nu_B)$ is an intuitionistic fuzzy relation on E ,

satisfying for each $(y, z) \in E$:

$$\lambda_B(y, z) \leq \min\{\lambda_A(y), \lambda_A(z)\},$$

$$\nu_B(y, z) \geq \max\{\nu_A(y), \nu_A(z)\},$$

$$0 \leq \lambda_B(y, z) + \nu_B(y, z) \leq 1.$$

Example 12 (Intuitionistic Fuzzy Directed Graph for Sensor Network Reliability). Consider a small network of three sensors communicating measurements:

$$V = \{S_1, S_2, S_3\}, \quad E = \{e_{12} = (S_1, S_2), e_{23} = (S_2, S_3)\}.$$

We model each sensor's operational status and each link's quality with intuitionistic fuzzy sets.

Vertex Intuitionistic Fuzzy Set. Let

$$A = (\lambda_A, \nu_A) \quad \text{on } V,$$

with

$$\lambda_A(S_1) = 0.80, \nu_A(S_1) = 0.10;$$

$$\lambda_A(S_2) = 0.70, \nu_A(S_2) = 0.20;$$

$$\lambda_A(S_3) = 0.90, \nu_A(S_3) = 0.05.$$

Here, $\lambda_A(x)$ measures confidence that sensor x is fully operational, and $\nu_A(x)$ captures the degree of non-operational uncertainty.

Edge Intuitionistic Fuzzy Relation. Let

$$B = (\lambda_B, \nu_B) \quad \text{on } E,$$

with

$$\lambda_B(e_{12}) = 0.65, \nu_B(e_{12}) = 0.25;$$

$$\lambda_B(e_{23}) = 0.65, \nu_B(e_{23}) = 0.30.$$

These satisfy for each $(y, z) \in E$:

$$\lambda_B(y, z) \leq \min\{\lambda_A(y), \lambda_A(z)\},$$

$$\nu_B(y, z) \geq \max\{\nu_A(y), \nu_A(z)\},$$

$$0 \leq \lambda_B(y, z) + \nu_B(y, z) \leq 1.$$

Since all conditions hold, the pair

$$G = (A, B)$$

defines an intuitionistic fuzzy directed graph on the sensor network, capturing both the reliability and the uncertainty of sensors and communication links.

Definition 18 (Single-Valued Neutrosophic Directed Graph [101]). *A single-valued neutrosophic directed graph is a 5-tuple*

$$G_{SVN} = (V, E, T, I, F),$$

where

- V is the vertex set,
- $E \subseteq V \times V$ is the arc set,
- $T, I, F: V \cup E \rightarrow [0, 1]$ are the truth, indeterminacy, and falsity functions,

such that for all $x \in V \cup E$,

$$0 \leq T(x) + I(x) + F(x) \leq 3.$$

Example 13 (Single-Valued Neutrosophic Directed Graph for Autonomous Vehicle Map Updates). *Consider a fleet of three autonomous vehicles sharing map-update messages:*

$$V = \{V_A, V_B, V_C\},$$

$$E = \{e_{AB} = (V_A, V_B), e_{BC} = (V_B, V_C), e_{CA} = (V_C, V_A)\}.$$

Each vehicle and each communication link is assigned a neutrosophic triple (T, I, F) reflecting update accuracy (truth), transmission uncertainty (indeterminacy), and risk of erroneous data (falsity).

Vertices (Vehicles).

$$\begin{aligned} T(V_A) &= 0.ninety, & I(V_A) &= 0.05, & F(V_A) &= 0.05, \\ T(V_B) &= 0.80, & I(V_B) &= 0.15, & F(V_B) &= 0.05, \\ T(V_C) &= 0.85, & I(V_C) &= 0.10, & F(V_C) &= 0.05. \end{aligned}$$

Edges (Update Links).

$$\begin{aligned} T(e_{AB}) &= 0.88, & I(e_{AB}) &= 0.07, & F(e_{AB}) &= 0.05, \\ T(e_{BC}) &= 0.75, & I(e_{BC}) &= 0.20, & F(e_{BC}) &= 0.05, \\ T(e_{CA}) &= 0.92, & I(e_{CA}) &= 0.05, & F(e_{CA}) &= 0.03. \end{aligned}$$

Each triple satisfies $0 \leq T(x) + I(x) + F(x) \leq 3$ for all $x \in V \cup E$. Hence,

$$G_{SVN} = (V, E, T, I, F)$$

is a valid single-valued neutrosophic directed graph modeling the reliability, uncertainty, and error-risk of map-update exchanges among the vehicles.

Definition 19 (Plithogenic Directed Graph [103]). Let $G = (V, E)$ be a crisp digraph. A plithogenic directed graph is a pair

$$G_P = (P_M, P_N),$$

where

$$P_M = (M, l, M_l, \text{adf}, \text{acf}) \text{ and } P_N = (N, m, N_m, \text{bdf}, \text{bcf})$$

are plithogenic sets on the vertices and edges respectively. Here:

- $M \subseteq V$ with attribute l taking values in M_l , and $\text{adf} : M \times M_l \rightarrow [0, 1]$, $\text{acf} : M_l \times M_l \rightarrow [0, 1]$ are the degree-of-appurtenance and degree-of-contradiction functions for vertices.
- $N \subseteq E$ with attribute m taking values in N_m , and $\text{bdf} : N \times N_m \rightarrow [0, 1]$, $\text{bcf} : N_m \times N_m \rightarrow [0, 1]$ are the analogous functions for edges.

These must satisfy the usual compatibility and symmetry conditions of plithogenic graphs.

Example 14 (Plithogenic Directed Graph Example: Team Communication). Let us model a directed communication network among three members:

$$V = \{A, B, C\}, \quad E = \{e_{AB} = (A, B), e_{BC} = (B, C)\}.$$

We assign two vertex-level and two edge-level attributes in a plithogenic setting. Vertex Attributes. Suppose each person has a “role” attribute l taking values in

$$M_l = \{\text{admin}, \text{user}\}.$$

Define the vertex appurtenance function by

$$\text{adf}(A, \text{admin}) = 0.90, \quad \text{adf}(A, \text{user}) = 0.10,$$

$$\text{adf}(B, \text{admin}) = 0.20, \quad \text{adf}(B, \text{user}) = 0.80,$$

$$\text{adf}(C, \text{admin}) = \text{adf}(C, \text{user}) = 0.50.$$

The vertex contradiction function satisfies

$$\text{acf}(\text{admin}, \text{user}) = \text{acf}(\text{user}, \text{admin}) = 0.70, \quad \text{acf}(\text{admin}, \text{admin}) = \text{acf}(\text{user}, \text{user}) = 0.$$

Thus the plithogenic vertex set is

$$P_M = (\{A, B, C\}, I, M_I, \text{adf}, \text{acf}).$$

Edge Attributes. Let each directed link have a “priority” value m drawn from

$$N_m = \{\text{high}, \text{low}\}.$$

Assign the edge appurtenance function as

$$\text{bdf}(e_{AB}, \text{high}) = 0.80, \quad \text{bdf}(e_{AB}, \text{low}) = 0.20,$$

$$\text{bdf}(e_{BC}, \text{high}) = 0.30, \quad \text{bdf}(e_{BC}, \text{low}) = 0.70.$$

Set the edge contradiction function by

$$\text{bcf}(\text{high}, \text{low}) = \text{bcf}(\text{low}, \text{high}) = 0.60, \quad \text{bcf}(\text{high}, \text{high}) = \text{bcf}(\text{low}, \text{low}) = 0.$$

Hence the plithogenic edge set is

$$P_N = (\{e_{AB}, e_{BC}\}, m, N_m, \text{bdf}, \text{bcf}).$$

One verifies that for every edge $e = (x, y)$ and attributes $a \in M_I$, $b \in N_m$, $\text{bdf}(e, b) \leq \min\{\text{adf}(x, a), \text{adf}(y, a)\}$ and both acf and bcf are symmetric and zero on identical values. Consequently,

$$G_P = (P_M, P_N)$$

forms a coherent plithogenic directed graph encompassing both role membership and link priority in the team communication model.

Also, we recall the notion of a soft directed graph, which adapts the soft set concept [106–108] to directed graphs.

Definition 20 (Soft Directed Graph [38,39]). Let $D^* = (V, A)$ be a directed graph and P a nonempty parameter set. A soft directed graph is the quadruple

$$D = (D^*, J, L, P),$$

where:

- $D^* = (V, A)$ is the underlying digraph.
- P is a nonempty set of parameters.
- $J: P \rightarrow \mathcal{P}(V)$ makes (J, P) a soft set on the vertices.
- $L: P \rightarrow \mathcal{P}(A)$ makes (L, P) a soft set on the arcs.
- For each $x \in P$, the pair $(J(x), L(x))$ induces a subdigraph of D^* .

Each such subdigraph $(J(x), L(x))$ is called a directed part (or dipart) of D .

Example 15 (Soft Directed Graph). *Let*

$$D^* = (V, A), \quad V = \{v_1, v_2, v_3, v_4\}, \quad A = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\},$$

and parameters $P = \{p_1, p_2\}$. *Define*

$$J(p_1) = \{v_1, v_2, v_3\}, \quad L(p_1) = \{(v_1, v_2), (v_2, v_3)\},$$

$$J(p_2) = \{v_2, v_3, v_4\}, \quad L(p_2) = \{(v_2, v_3), (v_3, v_4)\}.$$

Then, (J, P) is a soft set on V and (L, P) on A , and the resulting soft directed graph is

$$D = \{ (J(p_1), L(p_1)), (J(p_2), L(p_2)) \},$$

comprising two directed parts of D^* .

2.7. Rough Graphs and Rough Digraphs

A rough graph uses equivalence relations on vertices and edges to define lower and upper approximations, capturing ambiguity in undirected network structure [109–112]. A rough digraph extends rough graph concepts to directed networks, approximating vertices and arcs by lower and upper equivalence-based boundaries [36,37]. We now recall the notions of rough sets and their application to graphs and digraphs.

Notation 1 (Rough Set [113,114]). *Let Q be a nonempty universe and φ an equivalence relation on Q . For any subset $A \subseteq Q$, its lower and upper approximations under φ are*

$$\varphi_L(A) = \{x \in Q \mid [x]_\varphi \subseteq A\}, \quad \varphi_U(A) = \{x \in Q \mid [x]_\varphi \cap A \neq \emptyset\},$$

where $[x]_\varphi = \{y \in Q \mid (x, y) \in \varphi\}$ is the equivalence class of x . The pair $(\varphi_L(A), \varphi_U(A))$ is called the rough set of A .

Similarly, if ψ is an equivalence relation on a set $M \subseteq Q \times Q$ and $D \subseteq Q \times Q$, one defines $\psi_L(D)$ and $\psi_U(D)$ analogously for the relation D .

Definition 21 (Rough Graph [109,110]). *Let Q be a nonempty set with equivalence φ , and let $D \subseteq Q \times Q$ be an irreflexive, symmetric relation (an undirected graph). Suppose ψ is an equivalence on a subset of $Q \times Q$. The rough graph is the triple*

$$G = (Q, (\varphi_L(Q), \varphi_U(Q)), (\psi_L(D), \psi_U(D))),$$

where

- (i) $(\varphi_L(Q), \varphi_U(Q))$ approximates the vertex set,
- (ii) $(\psi_L(D), \psi_U(D))$ approximates the edge set.

Thus, G encodes both vertex-level and edge-level uncertainty.

Example 16 (Rough Graph). *Take $Q = \{1, 2, 3, 4\}$ with φ partitioning Q into $\{1, 2\}$ and $\{3, 4\}$. For $A = \{1, 3\}$, one finds*

$$\varphi_L(A) = \{3\}, \quad \varphi_U(A) = \{1, 2, 3, 4\}.$$

Let $D = \{(1, 3), (2, 4), (3, 1), (4, 2)\}$ and let ψ be the identity on D . Then $\psi_L(D) = \psi_U(D) = D$. Hence, the rough graph is

$$(Q, (\{3\}, Q), (D, D)).$$

Definition 22 (Rough Digraph [36,37]). Let Q and φ be as above, and let $D \subseteq Q \times Q$ be any directed relation. If ψ is an equivalence on a subset of $Q \times Q$, the rough digraph is

$$G = (Q, (\varphi_L(Q), \varphi_U(Q)), (\psi_L(D), \psi_U(D))),$$

where $(\varphi_L(Q), \varphi_U(Q))$ and $(\psi_L(D), \psi_U(D))$ are the lower/upper approximations of the vertices and arcs, respectively.

Example 17 (Rough Digraph). Let $Q = \{1, 2, 3\}$ with φ the identity relation, so $\varphi_L(A) = \varphi_U(A) = A$ for any A . Take $D = \{(1, 2), (2, 3)\}$ and let ψ also be the identity on D . Then

$$(\varphi_L(Q), \varphi_U(Q)) = (Q, Q), \quad (\psi_L(D), \psi_U(D)) = (D, D).$$

Thus, the rough digraph is $(Q, (Q, Q), (D, D))$, which coincides with the original digraph.

2.8. Multidirected Graph

The multidirected graph unifies features of directed [44,45] and bidirected graphs [115,116] by allowing multiple directed edges between any two vertices. We begin by recalling the bidirected graph.

Definition 23 (Bidirected Graph [115,116]). A bidirected graph $B = (G, \tau)$ consists of:

- A simple undirected graph $G = (V, E)$, with no loops or parallel edges.
- A function $\tau : V \times E \rightarrow \{-1, 0, 1\}$, called the local-orientation map, such that for each edge $e = uv$:
 - $\tau(u, e) = 1$ means e is oriented toward u .
 - $\tau(u, e) = -1$ means e is oriented away from u .
 - $\tau(u, e) = 0$ if u is not incident to e .

Here, G is the underlying graph, and τ assigns an independent direction at each endpoint of every edge.

Example 18 (Traffic Intersection as a Bidirected Graph). Model a three-way junction by $G = (V, E)$ with

$$V = \{1, 2, 3\}, \quad E = \{e_{12}, e_{23}, e_{31}\},$$

where e_{12} links 1 and 2, etc. Define τ by

$$\tau(1, e_{12}) = 1, \tau(2, e_{12}) = -1; \tau(2, e_{23}) = 1, \tau(3, e_{23}) = -1; \tau(3, e_{31}) = 1, \tau(1, e_{31}) = -1,$$

and $\tau(v, e) = 0$ otherwise. This encodes permitted entry and exit directions at each road endpoint. The pair (G, τ) thus captures the asymmetric traffic flow at the intersection.

Definition 24 (Multidirected Graph [44,45]). A Multidirected Graph is a 5-tuple

$$G = (V, E, s, t, m),$$

where:

- V is the set of vertices.
- E is the set of directed edges (allowing repetitions).
- $s : E \rightarrow V$ assigns each edge its source.
- $t : E \rightarrow V$ assigns each edge its target.
- $m : V \times V \rightarrow \mathbb{N}_0$ is the multiplicity function, with $m(u, v)$ counting edges directed from u to v .

Example 19 (Parallel Data Channels as a Multidirected Graph). Let $V = \{A, B, C\}$ denote three servers. Suppose there are:

- Two independent channels $A \rightarrow B$,
- One channel $B \rightarrow C$,
- Three channels $C \rightarrow A$.

Define edges

$$E = \{f_{AB}^1, f_{AB}^2, f_{BC}, f_{CA}^1, f_{CA}^2, f_{CA}^3\},$$

with

$$s(f_{AB}^i) = A, t(f_{AB}^i) = B \quad (i = 1, 2),$$

$$s(f_{BC}) = B, t(f_{BC}) = C,$$

$$s(f_{CA}^j) = C, t(f_{CA}^j) = A \quad (j = 1, 2, 3).$$

The multiplicity map m records

$$m(A, B) = 2, m(B, C) = 1, m(C, A) = 3,$$

and $m(u, v) = 0$ otherwise. This multidirected graph concisely represents both the direction and number of parallel data streams in the network.

Theorem 1 (Multidirected Graph Generalizes Directed and Bidirected Graphs). Let

$$G_D = (V, E, s, t)$$

be a directed graph and

$$B = (G, \tau)$$

be a bidirected graph with underlying simple undirected graph

$$G = (V, E')$$

and bidirection function $\tau : V \times E' \rightarrow \{-1, 0, 1\}$. Then there exists a Multidirected Graph

$$G_{MD} = (V, E_{MD}, s', t', m)$$

such that:

1. (Directed Graph as a Special Case) If each edge $e \in E$ of G_D is identified with a unique edge in E_{MD} having

$$s'(e) = s(e) \quad \text{and} \quad t'(e) = t(e),$$

and if the multiplicity function

$$m : V \times V \rightarrow \{0, 1\}$$

is defined by

$$m(u, v) = \begin{cases} 1, & \text{if } (u, v) \in E, \\ 0, & \text{otherwise,} \end{cases}$$

then G_{MD} is equivalent to the directed graph G_D .

2. (Bidirected Graph as a Special Case) For each edge $e \in E'$ of the bidirected graph B incident with vertices u and v (so that $\tau(u, e) \neq 0$ and $\tau(v, e) \neq 0$), assume without loss of generality that

$$\tau(u, e) = 1 \quad \text{and} \quad \tau(v, e) = -1.$$

Then, associate to e a directed edge $e' \in E_{MD}$ in G_{MD} by setting

$$s'(e') = v \quad \text{and} \quad t'(e') = u.$$

Define the multiplicity function m so that for each such edge,

$$m(v, u) = m(v, u) + 1.$$

In this way, the bidirected graph B is represented as a Multidirected Graph G_{MD} .

Thus, the multidirected graph generalizes both the Directed Graph and the Bidirected Graph.

Proof. (Directed Graph Case): A directed graph $G_D = (V, E, s, t)$ naturally becomes a multidirected graph by letting the vertex set remain the same and by defining the edge set E_{MD} to coincide with E , with the source and target functions given by $s'(e) = s(e)$ and $t'(e) = t(e)$ for each $e \in E$. The multiplicity function is defined by

$$m(u, v) = \begin{cases} 1, & \text{if } (u, v) \in E, \\ 0, & \text{otherwise,} \end{cases}$$

thereby embedding G_D into the framework of Multidirected Graphs.

(Bidirected Graph Case): Consider a bidirected graph $B = (G, \tau)$ with $G = (V, E')$. For each edge $e \in E'$ connecting two vertices u and v , the bidirection function τ assigns local orientations. By the standard convention, if $\tau(u, e) = 1$ then e is considered to be directed towards u , and if $\tau(v, e) = -1$ then e is considered to be directed away from v . In this case, we represent e in the multidirected graph G_{MD} as a directed edge e' with

$$s'(e') = v \quad \text{and} \quad t'(e') = u.$$

The multiplicity function m is then updated so that

$$m(v, u) = m(v, u) + 1.$$

This process, applied to every edge of B , embeds the bidirected structure within the multidirected graph framework.

Since both a directed graph and a bidirected graph can be transformed into a Multidirected Graph by appropriately defining the edge set, source and target functions, and the multiplicity function, the multidirected graph is indeed a generalization of both. $\square$

2.9. Mixed Graph

A mixed graph integrates both undirected and directed edges in a single framework [117–121]. Mixed graphs have been studied in areas such as graph coloring.

Definition 25 (Mixed Graph [117,121]). A mixed graph $G = (V, E, A)$ consists of:

- V : a set of vertices,
- E : a set of undirected edges $\{u, v\}$ linking vertices without orientation,
- A : a set of directed arcs (u, v) indicating an oriented connection from u (tail) to v (head).

In this model, loops are disallowed, and multiple connections between the same pair of vertices may occur.

Example 20 (Hybrid Corporate Network). Model a small corporate LAN of four devices:

$$V = \{D_1, D_2, D_3, D_4\}.$$

Bidirectional connections exist between D_1 – D_2 and D_2 – D_3 , while D_3 sends updates to D_4 unidirectionally. We represent this as

$$G = (V, E, A),$$

with

$$E = \{\{D_1, D_2\}, \{D_2, D_3\}\}, \quad A = \{(D_3, D_4)\}.$$

Any walk in G may traverse the LAN links in either direction or follow the directed update link. For instance,

$$D_1 - D_2 - D_3 \rightarrow D_4$$

is a valid route. Since there are no cycles involving directed arcs, G remains acyclic.

3. Results of This Paper

This section presents the main results of the paper. We extend classical concepts of directed graphs and hypergraphs by introducing *multiple edge structures* (i.e., edge multiplicities), and we further incorporate advanced frameworks including fuzzy, neutrosophic, and plithogenic systems. These generalizations aim to capture various forms of uncertainty, contradiction, and attribute-based reasoning in complex networks.

3.1. Multidirected Hypergraphs

A Multidirected hypergraph generalizes ordinary hypergraphs by permitting directed hyperedges that connect any non-empty set of source vertices (the tail) to a single target vertex (the head), and by recording a positive integer multiplicity for each hyperedge to model parallel occurrences.

Definition 26 (Multidirected Hypergraph). *Let V be a nonempty finite set of vertices. A Multidirected hypergraph is a triple*

$$\mathcal{H} = (V, E, m),$$

where

- E is a set of directed hyperedges, and
- $m: E \rightarrow \mathbb{N}$ is a multiplicity function.

Each hyperedge $e \in E$ is an ordered pair

$$e = (T(e), h(e)),$$

with $T(e) \subseteq V$ non-empty (the tail of e) and $h(e) \in V$ (the head of e). The value $m(e)$ indicates how many parallel instances of e occur.

Example 21 (Collaborative Report Workflow). *Consider a small team of three authors—Hiroko, Shinya, and Masahiro—and one manager, Tae. We model the process of submitting draft reports as the Multidirected hypergraph $\mathcal{H} = (V, E, m)$, where*

$$V = \{\text{Hiroko}, \text{Shinya}, \text{Masahiro}, \text{Tae}\}.$$

For the monthly report, Hiroko and Shinya jointly produce two distinct drafts; this is represented by

$$e_1 = (\{\text{Hiroko}, \text{Shinya}\}, \text{Tae}), \quad m(e_1) = 2.$$

For the weekly summary, Shinya and Masahiro collaborate on three drafts:

$$e_2 = (\{\text{Shinya}, \text{Masahiro}\}, \text{Tae}), \quad m(e_2) = 3.$$

Thus, $E = \{e_1, e_2\}$, and m records the number of versions submitted for each report type.

Theorem 2 (Incidence Connectivity). *Let $\mathcal{H} = (V, E, m)$ be a Multidirected hypergraph. If every vertex $v \in V$ lies in the tail or is the head of at least one hyperedge of positive multiplicity, then $\mathcal{H}$ is incident-connected: every vertex is incident to some hyperedge.*

Proof. By assumption, for each $v \in V$ there exists $e \in E$ with $v \in T(e) \cup \{h(e)\}$ and $m(e) \geq 1$. Therefore

$$\bigcup_{e \in E} (T(e) \cup \{h(e)\}) = V,$$

so every vertex is incident to at least one hyperedge. $\square$

Theorem 3 (Generalization of Directed Hypergraphs and Multidirected Graphs). *Let $\mathcal{H}_D = (V, E_D)$ be a directed hypergraph and let $G_M = (V, E_M, s, t, m)$ be a multidirected graph. Define*

$$E_{MD} = \{(T(e), h(e)) : e \in E_D\} \cup \{(\{s(e)\}, t(e)) : e \in E_M\},$$

and let $m' : E_{MD} \rightarrow \mathbb{N}$ be given by

$$m'(e) = \begin{cases} 1, & \text{if } e = (T(e), h(e)) \text{ for some } e \in E_D, \\ m(e), & \text{if } e = (\{s(e)\}, t(e)) \text{ for some } e \in E_M. \end{cases}$$

Then $\mathcal{H}_{MD} = (V, E_{MD}, m')$ is a Multidirected hypergraph, which contains both $\mathcal{H}_D$ and G_M as special cases.

Proof. Every hyperedge $e = (T(e), h(e)) \in E_D$ appears in E_{MD} with multiplicity 1, embedding $\mathcal{H}_D$. Each directed edge $(u, v) \in E_M$ is represented by $(\{u\}, v) \in E_{MD}$ with multiplicity $m(u, v)$, embedding G_M . Hence $\mathcal{H}_{MD}$ generalizes both structures. $\square$

3.2. Multidirected Superhypergraph

A Multidirected superhypergraph is a directed structure of n -th power-set vertices and hyperedges with assigned multiplicities, enabling multi-level, weighted relations. The definition of a Multidirected superhypergraph is provided below.

Definition 27 (Multidirected Superhypergraph). *Let $n \geq 1$ be a fixed positive integer and let V_0 be a finite base set. A Multidirected superhypergraph is a triple*

$$S\mathcal{H} = (V, E, m),$$

where:

- (i) $V \subseteq \mathcal{P}^n(V_0)$ is a set of n -supervertices;
- (ii) $E \subseteq (\mathcal{P}(V) \times \mathcal{P}(V))$ is a set of directed n -superhyperedges. Each superhyperedge $e \in E$ is an ordered pair

$$e = (T(e), H(e)),$$

where:

- (a) $T(e) \subseteq V$ is the non-empty tail set,
- (b) $H(e) \subseteq V$ is the non-empty head set.
- (iii) $m : E \rightarrow \mathbb{N}$ is a multiplicity function, assigning each superhyperedge e a positive integer $m(e)$.

Example 22 (Research Committees and Councils as a Multidirected Superhypergraph). Let four researchers—Hiroko, Shinya, Tae and Masahiro—form the base set

$$V_0 = \{\text{Hiroko}, \text{Shinya}, \text{Tae}, \text{Masahiro}\}.$$

First, we form three two-member committees in $\mathcal{P}^1(V_0)$:

$$C_1 = \{\text{Hiroko}, \text{Shinya}\}, \quad C_2 = \{\text{Shinya}, \text{Tae}\}, \quad C_3 = \{\text{Tae}, \text{Masahiro}\}.$$

Next, we assemble these into two study councils in $\mathcal{P}^2(V_0)$: Council I comprises committees C_1 and C_2 , and Council II comprises C_2 and C_3 . Denote the corresponding 2-supervertices by

$$v_I = \{C_1, C_2\}, \quad v_{II} = \{C_2, C_3\},$$

so that

$$V = \{v_I, v_{II}\} \subseteq \mathcal{P}^2(V_0).$$

We now model the directed exchange of research proposals between these councils. Council I sends five proposals to Council II, while both councils jointly send two collaborative proposals back to Council I. These interactions define two directed 2-superhyperedges:

$$e_1 = (\{v_I\}, \{v_{II}\}), \quad e_2 = (\{v_I, v_{II}\}, \{v_I\}),$$

with multiplicities

$$m(e_1) = 5, \quad m(e_2) = 2.$$

Thus, the Multidirected superhypergraph $\mathcal{SH} = (V, \{e_1, e_2\}, m)$ captures both the hierarchical grouping of researchers into committees and councils (as 2-supervertices) and the directed, multi-version flow of proposals between those councils (as 2-superhyperedges with multiplicity).

Theorem 4 (Projection Theorem). Let $\mathcal{SH} = (V, E, m)$ be a Multidirected superhypergraph defined on a base set V_0 . Define a projection map

$$\pi : V \rightarrow V_0, \quad \pi(v) = \min\{x \in V_0 : x \in v\},$$

where a suitable ordering on V_0 is assumed. Then, for each superhyperedge

$$e = (T(e), H(e)) \in E,$$

the projected hyperedge

$$e' = (\{\pi(v) : v \in T(e)\}, \pi(v'))$$

(for some chosen $v' \in H(e)$) defines a directed hyperedge on V_0 with multiplicity $m(e') = m(e)$. Hence, the projection induces a Multidirected hypergraph on (a subset of) V_0 .

Proof. Since each n -supervertex $v \in V \subseteq \mathcal{P}^n(V_0)$ is a nested subset of V_0 , the projection π selects a representative element in V_0 . Applying π to all members of $T(e)$ and to one element from $H(e)$ yields a valid directed hyperedge (as defined in Definition 26). The multiplicity is preserved by definition. The surjectivity of π (onto the image) guarantees that the incidence structure is maintained. $\square$

Theorem 5 (Generalization of Directed Superhypergraph and Multidirected Hypergraph). Let $\mathcal{SH}_D = (V, E_{SD})$ be a directed superhypergraph (with $V \subseteq \mathcal{P}^n(V_0)$ for some base set V_0) and

let $\mathcal{H}_{MD} = (V, E_{MD}, m)$ be a Multidirected hypergraph as in Theorem 3. Then, there exists a Multidirected superhypergraph

$$\mathcal{SH}_{MD} = (V, E_{SDM}, m')$$

with

$$E_{SDM} = E_{SD} \cup E_{MD},$$

and the multiplicity function $m' : E_{SDM} \rightarrow \mathbb{N}$ defined by

$$m'(e) = \begin{cases} 1, & \text{if } e \in E_{SD}, \\ m(e), & \text{if } e \in E_{MD}. \end{cases}$$

Thus, $\mathcal{SH}_{MD}$ generalizes both the directed superhypergraph and the Multidirected hypergraph.

Proof. Every edge $e \in E_{SD}$ of the directed superhypergraph is assigned multiplicity 1 in $\mathcal{SH}_{MD}$. Meanwhile, every edge from the Multidirected hypergraph $\mathcal{H}_{MD}$ (which may have multiplicities greater than 1) is included in E_{SDM} with its original multiplicity. Since the vertex set $V \subseteq \mathcal{P}^n(V_0)$, the Multidirected hypergraph is a special case (when supervertices degenerate to single elements). Thus, $\mathcal{SH}_{MD}$ encompasses both types of structures. $\square$

3.3. Fuzzy Multidirected Graph

A fuzzy multidirected graph permits multiple directed edges between vertices, vertices and edges assigned their fuzzy membership degrees in $[0, 1]$. The definition of a fuzzy multidirected graph is provided below.

Definition 28 (Fuzzy Multidirected Graph). A fuzzy multidirected graph is a tuple

$$G_F = (V, E, \sigma, \mu),$$

where:

- (i) V is a nonempty set of vertices.
- (ii) E is a multiset of directed edges. Each edge $e \in E$ is an ordered pair $e = (u, v)$ with $u, v \in V$; note that multiple edges between the same ordered pair are allowed.
- (iii) $\sigma : V \rightarrow [0, 1]$ is a vertex membership function that assigns each vertex a degree of membership.
- (iv) $\mu : E \rightarrow [0, 1]$ is an edge membership function that assigns each directed edge a degree of membership.

Example 23 (Fuzzy Multidirected Graph for Uncertain Communication Network). Consider a small communication network of three servers:

$$V = \{S_1, S_2, S_3\}.$$

Each server's operational reliability is modeled by a membership degree:

$$\sigma(S_1) = 0.90, \quad \sigma(S_2) = 0.75, \quad \sigma(S_3) = 0.60.$$

The network has multiple directed channels (edges) between servers, captured by the multiset.

$$E = \{e_1 = (S_1, S_2), e_2 = (S_1, S_2), e_3 = (S_2, S_3), e_4 = (S_3, S_1)\}.$$

Each channel's quality (membership) is given by:

$$\mu(e_1) = 0.85, \quad \mu(e_2) = 0.65, \quad \mu(e_3) = 0.90, \quad \mu(e_4) = 0.50.$$

Thus,

$$G_F = (V, E, \sigma, \mu)$$

is a fuzzy multidirected graph in which two parallel channels exist from S_1 to S_2 with different reliability levels, a high-quality channel from S_2 to S_3 , and a moderate-quality feedback channel from S_3 back to S_1 . This structure models uncertain connectivity and allows analysis of redundant communication paths under reliability constraints.

Theorem 6 (Fuzzy Connectivity). Let $G_F = (V, E, \sigma, \mu)$ be a fuzzy multidirected graph. Assume there exist thresholds $\alpha, \beta \in (0, 1]$ such that:

$$\sigma(v) \geq \alpha \quad \text{for all } v \in V,$$

and for every edge $e \in E$ joining vertices u and v ,

$$\mu(e) \geq \beta.$$

Then, the induced fuzzy subgraph on

$$V_\alpha = \{v \in V \mid \sigma(v) \geq \alpha\}$$

has the property that for any two vertices $u, v \in V_\alpha$ there exists a fuzzy path connecting them (i.e., the fuzzy connectivity is guaranteed).

Proof. Since every vertex in V_α has a membership of at least α and every edge has a membership of at least β , the induced fuzzy subgraph has sufficiently high membership values on both vertices and edges. Standard results in fuzzy graph theory (see, e.g., [80] for fuzzy sets and their applications) ensure that if all vertices and edges exceed a given threshold, a fuzzy path exists between any two vertices. $\square$

Theorem 7 (Generalization of Fuzzy Directed Graph and Multidirected Graph). Let $G_F^D = (V, E, \sigma, \mu)$ be a Fuzzy Directed Graph and

$$G_M = (V, E, s, t, m)$$

a Multidirected Graph. Then, there exists a Fuzzy Multidirected Graph

$$G_{FM} = (V, E_{FM}, \sigma, \mu)$$

where the edge set E_{FM} is a multiset satisfying:

- If each edge appears with multiplicity 1, then G_{FM} is identical to G_F^D .
- If the membership functions σ and μ take only the values in $\{0, 1\}$ (i.e., are crisp), then G_{FM} is equivalent to G_M .

Thus, the fuzzy multidirected graph generalizes both the fuzzy directed graph and the multidirected graph.

Proof. Starting from the Fuzzy Directed Graph $G_F^D = (V, E, \sigma, \mu)$, if we allow the edge set E to be a multiset (i.e., permit multiple copies of the same edge) we obtain the structure of a fuzzy multidirected graph G_{FM} . In the special case where every edge occurs exactly once,

G_{FM} coincides with G_F^D . Moreover, if the membership functions σ and μ are restricted to $\{0, 1\}$, then the fuzziness is removed and the graph becomes equivalent to a multidirected graph G_M . Hence, G_{FM} is a proper generalization. $\square$

3.4. Neutrosophic Multidirected Graph

A Neutrosophic multidirected graph allows multiple directed edges and assigns to each vertex and edge respective truth–indeterminacy–falsity values in $[0, 1]$. The definition of a neutrosophic multidirected graph is provided below.

Definition 29 (Neutrosophic Multidirected Graph). A neutrosophic multidirected graph is a tuple

$$G_{ND} = (V, E, T, I, F),$$

where:

- (i) V is a nonempty set of vertices.
- (ii) E is a multiset of directed edges, each edge $e \in E$ being an ordered pair (u, v) with $u, v \in V$ (allowing multiple edges between the same pair).
- (iii) $T: V \cup E \rightarrow [0, 1]$ is the truth membership function.
- (iv) $I: V \cup E \rightarrow [0, 1]$ is the indeterminacy membership function.
- (v) $F: V \cup E \rightarrow [0, 1]$ is the falsity membership function.

These membership functions satisfy the condition

$$0 \leq T(x) + I(x) + F(x) \leq 3, \quad \forall x \in V \cup E.$$

Example 24 (Neutrosophic Multidirected Graph for Uncertain Transportation Network). Consider a transportation network connecting three hubs:

$$V = \{\text{Hub}_A, \text{Hub}_B, \text{Hub}_C\}.$$

Multiple directed connections exist: two parallel bus routes from Hub_A to Hub_B , one high-speed train from Hub_B to Hub_C , and one ferry service from Hub_C back to Hub_A . Denote these edges by the multiset

$$E = \{e_1 = (\text{Hub}_A, \text{Hub}_B), e_2 = (\text{Hub}_A, \text{Hub}_B), e_3 = (\text{Hub}_B, \text{Hub}_C), e_4 = (\text{Hub}_C, \text{Hub}_A)\}.$$

We assign neutrosophic membership degrees to each vertex and edge to model reliability (truth), uncertainty (indeterminacy), and risk (falsity). For the hubs:

$$\begin{aligned} T(\text{Hub}_A) &= 0.85, I(\text{Hub}_A) = 0.10, F(\text{Hub}_A) = 0.05, \\ T(\text{Hub}_B) &= 0.75, I(\text{Hub}_B) = 0.15, F(\text{Hub}_B) = 0.10, \\ T(\text{Hub}_C) &= 0.90, I(\text{Hub}_C) = 0.05, F(\text{Hub}_C) = 0.05. \end{aligned}$$

For the two bus routes from Hub_A to Hub_B :

$$\begin{aligned} T(e_1) &= 0.70, I(e_1) = 0.20, F(e_1) = 0.10, \\ T(e_2) &= 0.60, I(e_2) = 0.30, F(e_2) = 0.10, \end{aligned}$$

reflecting that the first bus is more reliable than the second. The high-speed train and ferry are assigned:

$$T(e_3) = 0.95, I(e_3) = 0.03, F(e_3) = 0.02, \quad T(e_4) = 0.50, I(e_4) = 0.25, F(e_4) = 0.25.$$

Each triple of values sums to at most 3, satisfying $0 \leq T(x) + I(x) + F(x) \leq 3$ for all $x \in V \cup E$.

Thus

$$G_{ND} = (V, E, T, I, F)$$

is a neutrosophic multidirected graph that captures both the multiplicity of routes and the nuanced degrees of reliability, uncertainty, and risk for every hub and connection.

Theorem 8 (Neutrosophic Consistency). Let $G_{ND} = (V, E, T, I, F)$ be a neutrosophic multidirected graph. If there exist constants $\gamma \in (0, 1]$ and $\delta \in [0, 1)$ such that for every edge $e \in E$ it holds that

$$T(e) \geq \gamma \quad \text{and} \quad I(e) + F(e) \leq \delta,$$

then the neutrosophic structure behaves similarly to a crisp multidirected graph in terms of connectivity.

Proof. The condition $T(e) \geq \gamma$ ensures that each edge has a strong truth value, while $I(e) + F(e) \leq \delta$ bounds the uncertainty and falsity. Thus, the effective contribution of each edge is predominantly determined by its truth-membership, making the overall structure similar to that of a crisp graph. Standard techniques in neutrosophic graph theory then imply that connectivity properties of G_{ND} follow those of a crisp multidirected graph. $\square$

Theorem 9 (Generalization of Neutrosophic Directed Graph and Fuzzy Multidirected Graph). Let

$$G_{ND}^D = (V, E, T, I, F)$$

be a Neutrosophic Directed Graph and let $G_{FM} = (V, E, \sigma, \mu)$ be a fuzzy Multidirected Graph. Then, there exists a neutrosophic Multidirected Graph

$$G_{NMD} = (V, E, T, I, F)$$

(with E a multiset) such that:

- If the edge multiplicity is restricted to 1, then G_{NMD} is a neutrosophic directed graph.
- If the neutrosophic membership functions are chosen to satisfy

$$T(x) = \sigma(x), \quad I(x) = 0, \quad F(x) = 1 - \sigma(x) \quad \text{for all } x \in V \cup E,$$

then G_{NMD} coincides with the fuzzy multidirected graph G_{FM} .

Thus, G_{NMD} generalizes both the neutrosophic directed graph and the fuzzy Multidirected graph.

Proof. A neutrosophic directed graph $G_{ND}^D = (V, E, T, I, F)$ is defined with single copies of edges. By allowing E to be a multiset, we extend the model to a neutrosophic multidirected graph G_{NMD} . Furthermore, if we set

$$I(x) = 0 \quad \text{and} \quad F(x) = 1 - \sigma(x) \quad \text{with} \quad T(x) = \sigma(x),$$

for all $x \in V \cup E$, then the neutrosophic parameters reduce to a fuzzy membership scheme. Therefore, G_{NMD} includes both the neutrosophic and the fuzzy Multidirected models. $\square$

3.5. Plithogenic Multidirected Graph

A plithogenic multidirected graph is a complex multigraph enriched with vertex and edge appurtenance and contradiction functions for uncertainty modeling. The definition of a plithogenic multidirected graph is provided below.

Definition 30 (Plithogenic Multidirected Graph). A plithogenic multidirected graph extends the plithogenic directed graph framework to allow multiple directed edges between vertices. It is defined as a triple

$$G_{PM} = (P_M, P_N, m),$$

where:

(i) The plithogenic Vertex Set is given by

$$P_M = (M, l, M_l, \text{adf}, \text{acf}),$$

where:

- M is a set of vertices.
- l is an attribute associated with the vertices.
- M_l is the range of possible values for l .
- $\text{adf}: M \times M_l \rightarrow [0, 1]^s$ is the Degree of Appurtenance Function.
- $\text{acf}: M_l \times M_l \rightarrow [0, 1]^t$ is the Degree of Contradiction Function.

(ii) The Plithogenic Edge Set is given by

$$P_N = (N, m', N_m, \text{bdf}, \text{bcf}),$$

where:

- N is a set of directed edges.
 - m' is an attribute associated with the edges.
 - N_m is the range of possible values for m' .
 - $\text{bdf}: N \times N_m \rightarrow [0, 1]^s$ is the Degree of Appurtenance Function for edges.
 - $\text{bcf}: N_m \times N_m \rightarrow [0, 1]^t$ is the Degree of Contradiction Function for edges.
- (iii) $m: M \times M \rightarrow \mathbb{N}_0$ is a multiplicity function assigning to each ordered pair $(x, y) \in M \times M$ the number of directed edges from x to y . (Here, $\mathbb{N}_0$ denotes the set of nonnegative integers.)

Example 25 (International Flight Network with Plithogenic Attributes). Consider three airports $M = \{A, B, C\}$. We assign each airport a “risk level” attribute l taking values in $M_l = \{\text{low}, \text{medium}, \text{high}\}$. The vertex Degree of Appurtenance Function adf is given by

$$\text{adf}(A, \text{low}) = 0.8, \text{adf}(A, \text{medium}) = 0.2, \text{adf}(B, \text{medium}) = 0.7, \text{adf}(B, \text{high}) = 0.3,$$

$$\text{adf}(C, \text{low}) = 0.6, \text{adf}(C, \text{high}) = 0.4,$$

with $\text{adf}(x, y) = 0$ for all other (x, y) . The vertex Degree of Contradiction Function acf satisfies

$$\text{acf}(\text{low}, \text{high}) = \text{acf}(\text{high}, \text{low}) = 0.9, \text{acf}(\text{low}, \text{medium}) = \text{acf}(\text{medium}, \text{low}) = 0.5,$$

$$\text{acf}(\text{medium}, \text{high}) = \text{acf}(\text{high}, \text{medium}) = 0.7, \text{acf}(x, x) = 0 \text{ for all } x \in M_l.$$

We model three directed flight routes $N = \{f_{AB}, f_{BC}, f_{CA}\}$ with a “reliability” attribute $m' \in N_m = \{\text{on-time}, \text{delayed}\}$. The edge Degree of Appurtenance Function bdf is

$$\text{bdf}(f_{AB}, \text{on-time}) = 0.9, \text{ bdf}(f_{AB}, \text{delayed}) = 0.1,$$

$$\text{bdf}(f_{BC}, \text{on-time}) = 0.6, \text{ bdf}(f_{BC}, \text{delayed}) = 0.4,$$

$$\text{bdf}(f_{CA}, \text{on-time}) = 0.2, \text{ bdf}(f_{CA}, \text{delayed}) = 0.8,$$

with all other values zero. The edge Degree of Contradiction Function bcf has

$$\text{bcf}(\text{on-time}, \text{delayed}) = \text{bcf}(\text{delayed}, \text{on-time}) = 0.7,$$

$$\text{bcf}(x, x) = 0 \text{ for all } x \in N_m.$$

Finally, the multiplicity function $m: M \times M \rightarrow \mathbb{N}_0$ records daily flight counts:

$$m(A, B) = 2, \quad m(B, C) = 1, \quad m(C, A) = 1,$$

and $m(x, y) = 0$ otherwise. This plithogenic multidirected graph thus captures airport risk profiles, flight reliability degrees, and multiple daily connections in a unified model.

Theorem 10 (Plithogenic Simplification). Let $G_{PM} = (P_M, P_N, m)$ be a plithogenic multidirected graph. Suppose that for every vertex $x \in M$ and every edge $e \in P_N$:

$$\text{adf}(x, \cdot) \quad \text{and} \quad \text{bdf}(e, \cdot)$$

are sufficiently high (i.e., near 1) and the corresponding contradiction functions

$$\text{acf} \quad \text{and} \quad \text{bcf}$$

are sufficiently low (i.e., near 0). Then, G_{PM} can be reduced to a crisp multidirected graph

$$\tilde{G} = (M, E, m),$$

where

$$E = \{(x, y) \in M \times M \mid m(x, y) > 0\}.$$

Proof. High values of the Degree of Appurtenance Functions ensure that the vertices and edges strongly satisfy their attribute conditions, while low values of the Degree of Contradiction Functions guarantee minimal conflict between attributes. Under these circumstances, the plithogenic parameters effectively act as binary (crisp) conditions. By thresholding the appurtenance and contradiction values, one obtains a crisp edge set E from P_N such that (x, y) is an edge if and only if $m(x, y) > 0$. This completes the reduction. $\square$

Theorem 11 (Generalization of Plithogenic Directed Graph, Fuzzy Multidirected Graph, and Neutrosophic Multidirected Graph). Let a plithogenic multidirected graph be given by

$$G_{PMD} = (P_M, P_N, m),$$

where

- $P_M = (M, l, M_l, \text{adf}, \text{acf})$ is the plithogenic vertex set,
- $P_N = (N, m', N_m, \text{bdf}, \text{bcf})$ is the plithogenic edge set, and
- $m: M \times M \rightarrow \mathbb{N}_0$ is the edge multiplicity function.

Then:

- (a) If $m(x, y) = 1$ for all $(x, y) \in M \times M$, then G_{PMD} reduces to a Plithogenic Directed Graph.
- (b) If the plithogenic functions adf, acf, bdf, bcf are chosen so that they mirror crisp or fuzzy/neutrosophic membership values (for example, by taking them to be $\{0, 1\}$ -valued or by an appropriate reduction), then G_{PMD} reduces to a fuzzy multidirected graph or a neutrosophic multidirected graph.

Thus, the plithogenic multidirected graph generalizes the frameworks of plithogenic directed graphs, fuzzy multidirected graphs, and neutrosophic multidirected graphs.

Proof. In a plithogenic multidirected graph $G_{PMD} = (P_M, P_N, m)$, the multiplicity function m permits multiple directed edges between vertices. (a) By imposing $m(x, y) = 1$ for every pair (x, y) , the Multidirected aspect is removed, and one recovers a plithogenic directed graph. (b) The additional plithogenic parameters (the appurtenance and contradiction functions) encode attribute-based membership. By choosing these functions appropriately—so that they return binary or fuzzy/neutrosophic values—one obtains models equivalent to either a fuzzy multidirected graph or a neutrosophic multidirected graph. Hence, G_{PMD} serves as a unifying and general framework. $\square$

3.6. Soft Multidirected Graph

A soft multidirected graph uses soft set parameters to select vertex and edge subsets, forming subgraphs within the Multidirected network. The definition of a soft multidirected graph is provided below.

Definition 31 (Soft Multidirected Graph). Let

$$G_{MD}^* = (V, E, s, t, m)$$

be a multidirected graph, where V is a nonempty set of vertices, E is a set of edges, $s : E \rightarrow V$ and $t : E \rightarrow V$ are the source and target functions, and

$$m : V \times V \rightarrow \mathbb{N}_0$$

is the multiplicity function that assigns to each ordered pair of vertices the number of directed edges between them. Let P be a nonempty set of parameters. A soft multidirected graph is defined as a 4-tuple.

$$D = (G_{MD}^*, J, L, P),$$

where:

1. $G_{MD}^* = (V, E, s, t, m)$ is the underlying multidirected graph.
2. P is a nonempty set of parameters.
3. $J : P \rightarrow \mathcal{P}(V)$ is a mapping such that (J, P) is a soft set over V ; that is, for each $x \in P$, $J(x) \subseteq V$.
4. $L : P \rightarrow \mathcal{P}(E)$ is a mapping such that (L, P) is a soft set over E ; that is, for each $x \in P$, $L(x) \subseteq E$.
5. For each parameter $x \in P$, the pair

$$M(x) = (J(x), L(x))$$

forms a sub-multidirected graph of G_{MD}^* ; that is, the directed edges in $L(x)$ (with their associated multiplicities as inherited from m) connect only vertices in $J(x)$.

The sub-multidirected graph $M(x)$ corresponding to a parameter x is called a directed part (or dipart) of the soft multidirected graph D .

Example 26 (Soft Multidirected Graph for Time-Dependent Communication). Consider a network of three servers:

$$V = \{S_1, S_2, S_3\}.$$

The underlying multidirected graph $G_{MD}^* = (V, E, s, t, m)$ has the following multiset of directed edges:

$$E = \{e_1, e_2, e_3, e_4, e_5, e_6\},$$

with

$$s(e_1) = s(e_2) = S_1, \quad t(e_1) = t(e_2) = S_2, \quad s(e_3) = S_2, \quad t(e_3) = S_3,$$

$$s(e_4) = s(e_5) = s(e_6) = S_3, \quad t(e_4) = t(e_5) = t(e_6) = S_1.$$

The multiplicity function is

$$m(S_1, S_2) = 2, \quad m(S_2, S_3) = 1, \quad m(S_3, S_1) = 3.$$

Let the parameter set be

$$P = \{\text{Peak}, \text{OffPeak}\}.$$

Define the soft-set mappings J on vertices and L on edges as follows:

$$J(\text{Peak}) = \{S_1, S_2\}, \quad L(\text{Peak}) = \{e_1, e_2\},$$

$$J(\text{OffPeak}) = \{S_1, S_2, S_3\}, \quad L(\text{OffPeak}) = \{e_3, e_4, e_5, e_6\}.$$

For each parameter x , the pair

$$M(x) = (J(x), L(x))$$

forms a sub-multi-directed graph of G_{MD}^* . Specifically, during Peak hours only the two channels from S_1 to S_2 operate, and during OffPeak hours the channel from S_2 to S_3 , and all three channels from S_3 to S_1 are active.

Thus, the quadruple

$$D = (G_{MD}^*, J, L, P)$$

is a soft multidirected graph whose directed parts reflect time-dependent network configurations.

Theorem 12 (Generalization of Soft Directed Graph and Multidirected Graph). Let

$$D = (G_{MD}^*, J, L, P)$$

be a soft Multidirected graph.

- (a) If, for every $x \in P$ and for all $u, v \in J(x)$, the multiplicity $m(u, v)$ is either 0 or 1, then D is equivalent to a soft directed graph.
- (b) If the parameter set P is a singleton (i.e., $P = \{p\}$), then D reduces to the underlying multidirected graph G_{MD}^* (or, more precisely, to its sub-multidirected graph $M(p)$).

Proof. (a) Suppose that for every $x \in P$ and every pair $u, v \in J(x)$, the multiplicity $m(u, v) \in \{0, 1\}$. In this case, each edge in $L(x)$ is unique, and no multiple edges occur between any two vertices in $J(x)$. Consequently, the soft multidirected graph D becomes a soft directed graph according to the standard definition—each dipart $M(x) = (J(x), L(x))$ is a subdigraph (without edge multiplicities) of the underlying directed graph D^* .

(b) If $P = \{p\}$ is a singleton, then the soft set (J, P) over V simply reduces to the single set $J(p)$, and similarly (L, P) reduces to $L(p)$. In this scenario, the soft multi-

directed graph D is represented by the single dipart $M(p) = (J(p), L(p))$, which is a sub-multidirected graph of G_{MD}^* . Thus, D is essentially identical to a (possibly proper) multidirected graph, thereby showing that the soft multidirected graph generalizes the classical Multidirected graph.

In both cases, the soft multidirected graph D encompasses the structures of a soft directed graph (when edge multiplicities are trivial) and a multidirected graph (when the parameter set is trivial). Therefore, D indeed generalizes both concepts. $\square$

3.7. Rough Multidirected Graph

A rough multidirected graph applies equivalence-based lower and upper approximations to vertices and edges, capturing uncertainty in Multidirected structures. The definition of a rough multidirected graph is provided below.

Definition 32 (Rough Multidirected Graph). *Let $G = (V, E, s, t, m)$ be a multidirected graph. Let φ be an equivalence relation on V and ψ be an equivalence relation on E . The Rough Multidirected Graph is defined as the tuple*

$$G_{RMD} = (V, (\varphi_L(V), \varphi_U(V)), E, (\psi_L(E), \psi_U(E)), m).$$

In this structure:

- (i) $(\varphi_L(V), \varphi_U(V))$ is the rough approximation of the vertex set V .
- (ii) $(\psi_L(E), \psi_U(E))$ is the rough approximation of the edge set E .
- (iii) The multiplicity function m is inherited from the underlying multidirected graph.

Example 27 (Rough Multidirected Graph for Fault Propagation Analysis). *Consider a small fault-monitoring network of three components:*

$$V = \{C_1, C_2, C_3\}.$$

Component pairs exchange status messages via directed channels, with two parallel channels from C_1 to C_2 and one channel from C_2 to C_3 . We represent this as a Multidirected Graph $G = (V, E, s, t, m)$ with

$$E = \{e_1, e_2, e_3\},$$

where $s(e_1) = s(e_2) = C_1$, $t(e_1) = t(e_2) = C_2$, $s(e_3) = C_2$, $t(e_3) = C_3$ and multiplicities

$$m(C_1, C_2) = 2, \quad m(C_2, C_3) = 1.$$

To handle incomplete knowledge of fault-prone elements, define an equivalence φ on V by grouping $\{C_1, C_2\}$ together (similar behavior) and $\{C_3\}$ separately. On edges, define ψ by grouping the two $C_1 \rightarrow C_2$ channels $\{e_1, e_2\}$ and the single $C_2 \rightarrow C_3$ channel $\{e_3\}$.

Suppose our current warning set of possibly faulty components is

$$X = \{C_1, C_3\}.$$

The rough lower and upper approximations under φ are

$$\varphi_L(X) = \{C_3\}, \quad \varphi_U(X) = \{C_1, C_2, C_3\}.$$

Similarly, if our flagged edges are

$$Y = \{e_2, e_3\},$$

then under ψ we obtain

$$\psi_L(Y) = \{e_3\}, \quad \psi_U(Y) = \{e_1, e_2, e_3\}.$$

Therefore, the Rough Multidirected Graph is

$$G_{RMD} = (V, (\varphi_L(X), \varphi_U(X)), E, (\psi_L(Y), \psi_U(Y)), m),$$

where m is inherited from G . This structure captures both the incomplete classification of faulty components and uncertain identification of problematic communication channels.

Theorem 13 (Rough Multidirected Graph Generalizes Rough Directed Graph). Assume that $G = (V, E, s, t, m)$ is a multidirected graph in which the multiplicity function m is such that $m(x, y) \in \{0, 1\}$ for all $x, y \in V$ (i.e., there are no multiple edges, so that G is essentially a directed graph). Then, with φ and ψ as defined, the rough multidirected graph

$$G_{RMD} = (V, (\varphi_L(V), \varphi_U(V)), E, (\psi_L(E), \psi_U(E)), m)$$

reduces to the rough directed graph defined by

$$G_{RD} = (V, (\varphi_L(V), \varphi_U(V)), E, (\psi_L(E), \psi_U(E))).$$

In other words, the rough multidirected graph generalizes the rough directed graph.

Proof. If $m(x, y) \in \{0, 1\}$ for all $x, y \in V$, then each directed edge is unique, and the structure $G = (V, E, s, t, m)$ is identical to a directed graph. Therefore, the rough approximations $(\varphi_L(V), \varphi_U(V))$ and $(\psi_L(E), \psi_U(E))$ capture the uncertainty in the vertex and edge sets exactly as in the classical definition of a rough directed graph. Hence, G_{RMD} reduces to G_{RD} . $\square$

Example 28 (Rough Directed Graph Example). Let $V = \{1, 2, 3, 4\}$ with the equivalence relation φ partitioning V into $\{1, 2\}$ and $\{3, 4\}$. Consider the directed graph $G = (V, E, s, t, m)$ with

$$E = \{(1, 3), (2, 4)\},$$

$s(1, 3) = 1$, $t(1, 3) = 3$, $s(2, 4) = 2$, $t(2, 4) = 4$, and m being the indicator function. Let ψ be the identity on E . Then, for the vertex set, if we take $A = \{1, 3\}$, the lower approximation $\varphi_L(A)$ would be empty (since $\{1, 2\} \not\subseteq A$ and $\{3, 4\} \not\subseteq A$), while the upper approximation $\varphi_U(A) = V$ (since each equivalence class intersects A). With the identity equivalence on E , we have $\psi_L(E) = \psi_U(E) = E$. Hence, the rough directed graph is

$$G_{RD} = (V, (\varphi_L(V), \varphi_U(V)), E, (\psi_L(E), \psi_U(E))),$$

which is a special case of G_{RMD} .

Theorem 14 (Rough Multidirected Graph Generalizes Multidirected Graph). If the equivalence relations φ on V and ψ on E are the identity relations (i.e., every equivalence class is a singleton), then the lower and upper approximations satisfy:

$$\varphi_L(V) = \varphi_U(V) = V \quad \text{and} \quad \psi_L(E) = \psi_U(E) = E.$$

Consequently, the Rough Multidirected Graph

$$G_{RMD} = (V, (\varphi_L(V), \varphi_U(V)), E, (\psi_L(E), \psi_U(E)), m)$$

reduces to the classical Multidirected Graph (V, E, s, t, m) .

Proof. If φ and ψ are the identity relations, then for any subset $A \subseteq V$ we have:

$$\varphi_L(A) = \{x \in V \mid \{x\} \subseteq A\} = A \quad \text{and} \quad \varphi_U(A) = \{x \in V \mid \{x\} \cap A \neq \emptyset\} = A.$$

Similarly, for any $D \subseteq E$, $\psi_L(D) = \psi_U(D) = D$. Thus, the rough approximations yield the exact original sets, and

$$G_{RMD} = (V, (V, V), E, (E, E), m),$$

which is equivalent to the classical multidirected graph (V, E, s, t, m) . $\square$

Example 29 (Multidirected Graph Example). Let $V = \{a, b, c\}$ and let the multidirected graph be given by $E = \{r_1, r_2, r_3\}$ with:

$$s(r_1) = a, \quad t(r_1) = b, \quad m(a, b) = 2;$$

$$s(r_2) = b, \quad t(r_2) = c, \quad m(b, c) = 1;$$

$$s(r_3) = a, \quad t(r_3) = c, \quad m(a, c) = 3.$$

If we take the identity equivalence relations on V and E , then the rough multidirected graph reduces to the classical multidirected graph.

3.8. Bimixed Graph and Multimixed Graph

We introduce two new graph structures. A Bimixed Graph combines bidirected edges—each locally oriented at both endpoints—and directed arcs, unifying bidirectional and unilateral connections. A multimixed Graph supports multiple directed arcs and bidirected edges between vertices, with explicit multiplicities and local endpoint orientations.

Definition 33 (Bimixed Graph). A Bimixed graph is a quadruple

$$G_{BM} = (V, E, A, \tau)$$

where:

1. V is a non-empty set of vertices.
2. E is a set of (undirected) edges intended to be endowed with bidirectional structure. For each edge $e \in E$, there exist exactly two distinct vertices $u, v \in V$ such that

$$\tau(u, e), \tau(v, e) \in \{-1, 1\},$$

and for every $w \in V \setminus \{u, v\}$, $\tau(w, e) = 0$.

3. A is a set of directed arcs, where each arc is an ordered pair (u, v) with $u, v \in V$.
4. $\tau : V \times E \rightarrow \{-1, 0, 1\}$ is the bidirection function that assigns a local orientation to each vertex–edge pair.

In this framework:

- A Bidirected Graph is obtained when $A = \emptyset$.
- A Mixed Graph is recovered when the bidirected edges in E are interpreted as undirected edges (e.g., by consistently setting $\tau(u, e) = 1$ and $\tau(v, e) = -1$ for each edge e joining u and v).

Example 30 (Bimixed Graph Representing a Hybrid Network). Consider a small communication network of three routers:

$$V = \{R_1, R_2, R_3\}.$$

The set of bidirected edges models fiber-optic links that carry traffic in both directions, but with local orientation at each endpoint:

$$E = \{e_{12}, e_{23}\},$$

where e_{12} connects R_1 and R_2 , and e_{23} connects R_2 and R_3 . The bidirection function $\tau : V \times E \rightarrow \{-1, 0, 1\}$ is defined by:

$$\tau(R_1, e_{12}) = 1, \quad \tau(R_2, e_{12}) = -1, \quad \tau(R_3, e_{12}) = 0,$$

$$\tau(R_1, e_{23}) = 0, \quad \tau(R_2, e_{23}) = 1, \quad \tau(R_3, e_{23}) = -1.$$

In addition, there is a unidirectional wireless link from R_3 back to R_1 , represented by the set of arcs

$$A = \{a = (R_3, R_1)\},$$

where each arc is an ordered pair. No bidirection assignment applies to arcs, so $\tau(R_i, a) = 0$ for all i in V .

Thus, the quadruple

$$G_{BM} = (V, E, A, \tau)$$

forms a bimixed graph: The fiber links e_{12} and e_{23} carry local orientations at both ends via τ , while the wireless connection a is a standard directed arc from R_3 to R_1 . This model captures the coexistence of bidirected physical links and unidirectional wireless channels in a single network structure.

Definition 34 (Multimixed Graph). A Multimixed Graph is a structure

$$G_{MM} = (V, E_d, E_b, s, t, \tau, m_d, m_b)$$

where:

1. V is a non-empty set of vertices.
2. E_d is a set of directed arcs.
3. E_b is a set of bidirected edges.
4. $s : E_d \rightarrow V$ and $t : E_d \rightarrow V$ are the source and target functions for each directed arc in E_d .
5. $\tau : V \times E_b \rightarrow \{-1, 0, 1\}$ is the bidirection function for the bidirected edges; that is, for every $e \in E_b$ there exist exactly two distinct vertices $u, v \in V$ with

$$\tau(u, e), \tau(v, e) \in \{-1, 1\}$$

and for every $w \in V \setminus \{u, v\}$, $\tau(w, e) = 0$.

6. $m_d : V \times V \rightarrow \mathbb{N}_0$ is a multiplicity function for directed arcs, so that $m_d(u, v)$ is the number of directed arcs from u to v . (The set E_d may be viewed as a multiset according to m_d .)
7. $m_b : \{\{u, v\} \mid u, v \in V, u \neq v\} \rightarrow \mathbb{N}_0$ is a multiplicity function for bidirected edges, where $m_b(\{u, v\})$ denotes the number of bidirected edges connecting u and v (with their corresponding bidirectional orientations given by τ).

Example 31 (Hybrid Communication Network as a Multimixed Graph). Consider a small network of three stations:

$$V = \{S_1, S_2, S_3\}.$$

There are both unidirectional wireless links and bidirectional wired cables. We model the wireless links as directed arcs and the cables as bidirected edges.

Directed arcs. Two parallel wireless channels run from S_1 to S_2 , and one channel runs from S_3 back to S_1 . Let

$$E_d = \{w_{12}^{(1)}, w_{12}^{(2)}, w_{31}\},$$

with source and target functions

$$s(w_{12}^{(i)}) = S_1, \quad t(w_{12}^{(i)}) = S_2 \quad (i = 1, 2), \quad s(w_{31}) = S_3, \quad t(w_{31}) = S_1.$$

The directed-multiplicity function is

$$m_d(S_1, S_2) = 2, \quad m_d(S_3, S_1) = 1,$$

and all other pairs have multiplicity zero.

Bidirected edges. Two fiber-optic cables connect S_2 and S_3 , and a single copper cable connects S_1 and S_3 . Let

$$E_b = \{c_{23}^{(1)}, c_{23}^{(2)}, c_{13}\}.$$

The bidirection function $\tau: V \times E_b \rightarrow \{-1, 0, 1\}$ is defined by

$$\tau(S_2, c_{23}^{(i)}) = 1, \quad \tau(S_3, c_{23}^{(i)}) = -1 \quad (i = 1, 2), \quad \tau(S_1, c_{13}) = 1, \quad \tau(S_3, c_{13}) = -1,$$

with τ zero on all other vertex–edge pairs. The bidirected-multiplicity function is

$$m_b(\{S_2, S_3\}) = 2, \quad m_b(\{S_1, S_3\}) = 1.$$

Putting these together, the multimixed Graph

$$G_{MM} = (V, E_d, E_b, s, t, \tau, m_d, m_b)$$

captures the coexistence of multiple wireless channels (directed arcs) and multiple bidirectional cables in a unified network model.

Theorem 15 (Multimixed Graph Generalizes Multidirected and Bimixed Graphs). *Let*

$$G_{MM} = (V, E_d, E_b, s, t, \tau, m_d, m_b)$$

be a multimixed graph. Then:

1. *If $E_b = \emptyset$, then G_{MM} reduces to a Multidirected Graph.*
2. *If $E_d = \emptyset$ and the bidirection function τ satisfies the conditions of a bidirected graph on E_b , then G_{MM} is equivalent to a bimixed graph.*

Thus, the multimixed graph is a common generalization of both multidirected graphs and bimixed graphs.

Proof. (Case 1: $E_b = \emptyset$)

If $E_b = \emptyset$, then

$$G_{MM} = (V, E_d, \emptyset, s, t, \tau, m_d, m_b).$$

In this case, the structure consists solely of directed arcs E_d with associated source and target functions s and t , and a multiplicity function m_d . This is precisely the definition of a multidirected graph. Hence, G_{MM} generalizes the Multidirected graph.

(Case 2: $E_d = \emptyset$)

If $E_d = \emptyset$, then

$$G_{MM} = (V, \emptyset, E_b, s, t, \tau, m_d, m_b).$$

In this situation, the graph consists exclusively of bidirected edges E_b equipped with the bidirection function τ and a multiplicity function m_b . By the definition of a bimixed graph, this structure is equivalent to a bimixed graph.

Since in both cases G_{MM} reduces to a known structure—either a multidirected graph (when E_b is empty) or a bimixed graph (when E_d is empty)—the multimixed graph indeed generalizes both. $\square$

3.9. Hyperdirected Graph

Hyperdirected graph uses subsets of a base set as nodes, directed hyperarcs between them, and a hyperoperation composing matching hyperarcs. The definition of a hyperdirected graph is provided below.

Definition 35 (Hyperdirected Graph). *Let S be a nonempty set. A hyperdirected graph is a triple*

$$\text{HDG} = (\mathcal{P}(S), E, \odot),$$

where:

(i) **Vertex Set:** *The vertex set is the powerset $\mathcal{P}(S)$, which endows the structure with a hyperstructure.*

(ii) **Directed Hyperarcs:**

$$E \subseteq \mathcal{P}(S) \times \mathcal{P}(S)$$

is a set of directed hyperarcs. An element $e \in E$ is an ordered pair

$$e = (A, B) \quad \text{with } A, B \in \mathcal{P}(S).$$

(iii) **Hyperoperation on Arcs:** *A binary composition hyperoperation*

$$\odot : E \times E \rightarrow \mathcal{P}(E)$$

is defined by

$$(A, B) \odot (C, D) = \begin{cases} \{(A, D)\} & \text{if } B = C, \\ \emptyset & \text{otherwise.} \end{cases}$$

This operation generalizes the standard composition of directed edges.

Remark 1. *Note that in a classical directed hypergraph, the hyperedges are unordered subsets (or multisets) of vertices. In contrast, here the hyperarcs are ordered pairs of subsets, and the structure $(\mathcal{P}(S), \odot)$ is a hyperstructure (not just a hypergraph).*

Example 32 (Hyperdirected Graph). *Let*

$$S = \{a, b\}.$$

Then the powerset of S is

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

We define a hyperdirected graph $G = (\mathcal{P}(S), E)$ where the edge set E is a collection of ordered pairs (hyperarcs) chosen from $\mathcal{P}(S) \times \mathcal{P}(S)$. For instance, let

$$E = \{(\{a\}, \{a, b\}), (\{a, b\}, \{b\}), (\{b\}, \{a\}), (\{a\}, \{a\})\}.$$

A natural composition hyperoperation $\odot$ on the edges is defined by

$$(A, B) \odot (C, D) = \begin{cases} \{(A, D)\}, & \text{if } B = C, \\ \emptyset, & \text{otherwise.} \end{cases}$$

For example, we have

$$(\{a\}, \{a, b\}) \odot (\{a, b\}, \{b\}) = \{(\{a\}, \{b\})\}.$$

Theorem 16 (Hyperdirected Graph as a Hyperstructure). *Let S be a nonempty set and*

$$\text{HDG} = (\mathcal{P}(S), E, \odot)$$

a hyperdirected graph as in Definition 35. Define two hyperoperations:

$$\begin{aligned} \star : \mathcal{P}(S) \times \mathcal{P}(S) &\longrightarrow \mathcal{P}(\mathcal{P}(S)), & A \star B &= \{A \cup B\}, \\ \odot : E \times E &\longrightarrow \mathcal{P}(E), & (A, B) \odot (C, D) &= \begin{cases} \{(A, D)\}, & B = C, \\ \emptyset, & \text{otherwise.} \end{cases} \end{aligned}$$

Then:

- (a) $(\mathcal{P}(S), \star)$ is a classical hyperstructure since $\star$ maps any two subsets to a (possibly singleton) subset of $\mathcal{P}(S)$.
- (b) $(E, \odot)$ is also a hyperstructure because $\odot$ sends each ordered pair of hyperarcs to a subset of E .

In particular, neither $\mathcal{P}(S)$ nor E are merely ordinary graphs—their binary operations take values in the powersets of their domains, satisfying the definition of hyperstructure.

Proof. (a) The vertex hyperstructure. By definition, a hyperstructure on a set X is any binary map $X \times X \rightarrow \mathcal{P}(X)$. Here we take $X = \mathcal{P}(S)$ and define

$$A \star B = \{A \cup B\},$$

which clearly lies in $\mathcal{P}(\mathcal{P}(S))$. Hence $(\mathcal{P}(S), \star)$ satisfies the hyperstructure axioms.

(b) The hyperarc hyperstructure. Similarly, for the set of hyperarcs $E \subseteq \mathcal{P}(S) \times \mathcal{P}(S)$ we have

$$\odot : E \times E \longrightarrow \mathcal{P}(E)$$

given by

$$(A, B) \odot (C, D) = \begin{cases} \{(A, D)\} & \text{if } B = C, \\ \emptyset & \text{if } B \neq C. \end{cases}$$

Since (A, D) always belongs to E when $B = C$, and otherwise the result is the empty subset, $\odot$ indeed lands in $\mathcal{P}(E)$. Thus $(E, \odot)$ is a hyperstructure.

Because both $\mathcal{P}(S)$ and E admit operations into their respective powersets, the Hyperdirected Graph HDG is genuinely a pair of hyperstructures rather than an ordinary graph. $\square$

Theorem 17 (Hyperdirected Graph Generalizes Directed, Bidirected, and Multidirected Graphs). Assume that in a Hyperdirected Graph $\text{HDG} = (\mathcal{P}(S), E, \odot)$ the base set S and the hyperoperation $\odot$ are restricted as follows:

- When one restricts to atomic vertices (i.e., the singleton subsets of S), then $\mathcal{P}(S)$ contains these as distinguished elements. The composition $\odot$ then reduces to the standard composition of directed edges, yielding a directed graph.
- Allowing the tail or head of a hyperarc to be a non-singleton recovers a bidirected graph structure (or more generally a hypergraph with directed edges).
- Defining a multiplicity function on the number of edges between given vertices (or subsets thereof) recovers a Multidirected Graph.

Proof. Let S be a nonempty set. Observe:

- a. If we consider the atomic elements $\{s\}$ for $s \in S$, then the subset

$$V_{\text{atomic}} = \{\{s\} : s \in S\} \subset \mathcal{P}(S)$$

serves as the vertex set of a standard directed graph. The composition $\odot$ on pairs $(\{u\}, \{v\})$ works as:

$$(\{u\}, \{v\}) \odot (\{v\}, \{w\}) = \{(\{u\}, \{w\})\},$$

which is exactly the usual composition of edges in a directed graph.

- b. In a bidirected graph the edges have two orientations at their endpoints. By allowing a hyperarc (A, B) with A, B possibly containing more than one element, one may model such phenomena. For example, if A and B are non-singleton, then the ordered pair reflects that there is a directed relation from all elements of A to all elements of B , thus generalizing the notion of a bidirected edge.
- c. Finally, if we introduce a multiplicity function $m : \mathcal{P}(S) \times \mathcal{P}(S) \rightarrow \mathbb{N}_0$ counting the number of hyperarcs from A to B (or if E itself is allowed to have repeated elements), then the Hyperdirected Graph framework naturally generalizes the Multidirected Graph concept.

Thus, under the stated restrictions, the Hyperdirected Graph indeed generalizes the three classical graph concepts. $\square$

3.10. n -Superhyperdirected Graph

An n -Superhyperdirected Graph uses iterated n -th power sets as vertices, directed superhyperarcs between them, and hyperarc composition when endpoints match. The definition of a n -Superhyperdirected Graph is provided below.

Definition 36 (n -Superhyperdirected Graph). Let S be a nonempty set and $n \geq 1$ an integer. An n -Superhyperdirected Graph is a triple

$$n\text{-SHDG} = (\mathcal{P}^n(S), E_n, \odot_n),$$

where:

- (i) **Vertex Set:** The vertex set is $\mathcal{P}^n(S)$, so that the underlying structure is the n -Superhyperstructure $\mathcal{SH}_n$.
- (ii) **Directed Hyperarcs:**

$$E_n \subseteq \mathcal{P}^n(S) \times \mathcal{P}^n(S)$$

is a set of directed n -superhyperarcs. An element $e \in E_n$ is an ordered pair

$$e = (A, B) \quad \text{with } A, B \in \mathcal{P}^n(S).$$

(iii) **Composition Hyperoperation:** A hyperoperation

$$\odot_n : E_n \times E_n \rightarrow \mathcal{P}(E_n)$$

is defined by

$$(A, B) \odot_n (C, D) = \begin{cases} \{(A, D)\} & \text{if } B = C, \\ \emptyset & \text{otherwise.} \end{cases}$$

Remark 2. When $n = 1$ the 1-Superhyperdirected Graph coincides with the Hyperdirected Graph of Definition 35. In this sense the n -Superhyperdirected Graph is a higher-order generalization.

Remark 3. Although both an n -Superhyperdirected Graph and a Directed n -SuperhyperGraph share the same underlying vertex set $\mathcal{P}^n(V_0)$ and employ ordered pairs of supervertices to represent directed connections, they differ in their algebraic structure. A Directed n -SuperhyperGraph $\text{DSH}_n = (V, E)$ specifies only which n -supervertices are connected (via tail and head sets) and does not endow its hyperarcs with any composition operation. In contrast, an n -Superhyperdirected Graph $\text{n-SHDG} = (\mathcal{P}^n(S), E_n, \odot_n)$ additionally carries a binary hyperoperation $\odot_n$ on its hyperarcs, turning the collection of directed superhyperarcs into a true n -superhyperstructure. Thus the latter combines both incidence and composition, whereas the former encodes only incidence.

Example 33 (n -Superhyperdirected Graph). Let the base set be

$$V_0 = \{1, 2\}.$$

Then the first iterated powerset is

$$\mathcal{P}(V_0) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}.$$

The second iterated powerset (i.e., the 2nd powerset) is

$$\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0)),$$

which consists of all subsets of $\mathcal{P}(V_0)$.

We now choose two concrete elements (2-supervertices) from $\mathcal{P}^2(V_0)$:

$$X = \{\{1\}\} \quad \text{and} \quad Y = \{\{2\}\}.$$

Note that $X, Y \in \mathcal{P}^2(V_0)$ since each is a subset of $\mathcal{P}(V_0)$.

Next, define a 2-superedge by choosing a subset of $\mathcal{P}(V_0)$:

$$e = \{\{1\}, \{2\}\} \in \mathcal{P}^2(V_0).$$

Thus, we obtain a 2-superhypergraph (an n -superhypergraph with $n = 2$) given by

$$\text{SuHyG}^{(2)} = (\{X, Y\}, \{e\}).$$

In this example, the vertices are not just elements of V_0 but are nested subsets (i.e., elements of $\mathcal{P}^2(V_0)$) and the hyperedge e is a subset of $\mathcal{P}(V_0)$.

Theorem 18 (*n*-superhyperdirected graph has an *n*-superhyperstructure). Let $n\text{-SHDG} = (\mathcal{P}^n(S), E_n, \odot_n)$ be an *n*-superhyperdirected graph as in Definition 36. Then:

- (a) The vertex set $\mathcal{P}^n(S)$ together with an appropriate hyperoperation forms an *n*-superhyperstructure.
- (b) The set $E_n \subseteq \mathcal{P}^n(S) \times \mathcal{P}^n(S)$ consists of ordered pairs (directed hyperarcs), so that the structure is not an *n*-Superhypergraph (which would have unordered hyperedges).

Proof. (a) By Definition, the *n*th powerset $\mathcal{P}^n(S)$ is constructed recursively from *S*. Hence, the pair

$$\mathcal{SH}_n = (\mathcal{P}^n(S), \circ_n)$$

(with any hyperoperation $\circ_n$ defined on $\mathcal{P}^n(S)$) is by definition an *n*-superhyperstructure.

(b) In *n*-SHDG, each hyperarc is an ordered pair (A, B) with $A, B \in \mathcal{P}^n(S)$. This ordering encodes the direction from *A* (tail) to *B* (head) and hence is not compatible with the concept of an *n*-superhypergraph, where hyperedges are simply (unordered) members of $\mathcal{P}^n(S)$. The composition hyperoperation $\odot_n$ defined by

$$(A, B) \odot_n (C, D) = \begin{cases} \{(A, D)\} & \text{if } B = C, \\ \emptyset & \text{otherwise,} \end{cases}$$

again relies on the order; therefore, *n*-SHDG is not an *n*-superhypergraph. $\square$

Theorem 19 (*n*-Superhyperdirected Graph Generalizes Hyperdirected and Multidirected Graphs). Let

$$n\text{-SHDG} = (\mathcal{P}^n(S), E_n, \odot_n)$$

be an *n*-superhyperdirected graph as in Definition. Then:

- (i) Recovery of the Hyperdirected Graph. If $n = 1$ then $\mathcal{P}^1(S) = \mathcal{P}(S)$, $E_1 = E \subseteq \mathcal{P}(S) \times \mathcal{P}(S)$ and $\odot_1 = \odot$ coincide exactly with those of the hyperdirected graph of Definition 35.
- (ii) Recovery of a Multidirected Graph. Suppose we are given an ordinary multidirected graph $G_M = (V, E_M, s, t, m)$. Choose any injective “nesting” map

$$\iota: V \longrightarrow \mathcal{P}^n(S) \quad \text{so that each } v \in V \text{ is represented by a unique } \iota(v) \in \mathcal{P}^n(S).$$

Then define a subset of superhyperarcs

$$E'_n = \{ (\iota(u), \iota(v)) \mid (u, v) \in E_M \} \subseteq E_n,$$

and a new multiplicity function

$$m': E'_n \longrightarrow \mathbb{N}_0, \quad m'((\iota(u), \iota(v))) = m(u, v).$$

Then (V, E_M, s, t, m) is recovered by the *n*-superhyperdirected graph $(\iota(V), E'_n, \odot_n)$ together with m' . In other words, when one restricts vertices to the images $\iota(V) \subset \mathcal{P}^n(S)$, hyperarcs to $E'_n \subset E_n$, and records multiplicities via m' , the resulting structure is exactly the original multidirected graph.

Proof. (i) $n = 1$. By construction $\mathcal{P}^1(S) = \mathcal{P}(S)$ and the definition of an *n*-Superhyperdirected Graph reduces to that of Definition 35. In particular

$$E_1 = E \subseteq \mathcal{P}(S) \times \mathcal{P}(S), \quad \odot_1 = \odot,$$

so 1-SHDG and the Hyperdirected Graph coincide.

(ii) Recovering a Multidirected Graph. Let $G_M = (V, E_M, s, t, m)$ be any Multidirected Graph. We embed its vertex set V into $\mathcal{P}^n(S)$ by an injective map ι . For each directed edge $(u, v) \in E_M$, we form the superhyperarc

$$e' = (\iota(u), \iota(v)) \in \mathcal{P}^n(S) \times \mathcal{P}^n(S),$$

and collect these into $E'_n = \{(\iota(u), \iota(v)) : (u, v) \in E_M\} \subseteq E_n$. The original source and target maps are recovered by

$$s(e') = u, \quad t(e') = v \iff e' = (\iota(u), \iota(v)).$$

Finally, we define

$$m'(e') = m(u, v),$$

so that multiple directed edges in G_M appear as multiplicities of superhyperarcs. Since $\odot_n$ on these atomic superhyperarcs agrees with ordinary edge composition (matching head to tail), the restricted structure $(\iota(V), E'_n, \odot_n)$ with m' is isomorphic to G_M .

Thus the n -superhyperdirected graph unifies both the hyperdirected graph (case $n = 1$) and any multidirected graph (by suitable restriction and introduction of a multiplicity function). $\square$

4. Conclusions and Future Work

In this paper, we have extended traditional graph-theoretic models by introducing the *Multidirected hypergraph*, which unifies the multi-way connectivity of hypergraphs with the hierarchical power-set layers of superhypergraphs. Building on this, we formulated five uncertainty-aware Multidirected variants—fuzzy, neutrosophic, plithogenic, rough, and soft multidirected graphs—each equipped with precise definitions and concrete examples. We also presented four additional graph families (bimixed, multimixed, hyperdirected, and superhyperdirected graphs) and surveyed their key structural properties.

Despite these theoretical advances, practical toolkits and algorithms for Multidirected and related hyper-structures are still in their infancy. Future work will focus on:

- **Algorithmic development:** Designing efficient routines for traversal, querying, and analysis of Multidirected and hyper-directed networks.
- **Software implementations:** Creating open-source libraries that support construction, visualization, and manipulation of these new graph classes.
- **Machine learning integration:** Embedding uncertainty-aware multidirected graphs into graph neural networks and other learning architectures to handle complex, hierarchical, or imprecise data.
- **Applications and case studies:** Validating the proposed models in domains such as bioinformatics, social network analysis, and decision support systems, where multi-headed and uncertain relationships naturally arise.

We anticipate that combining deeper theoretical insights with practical tools will unlock a rich set of applications and inspire further research on advanced graph structures under uncertainty.

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