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Proceeding Paper

The Advanced Boundary Integral Equation Method for Modelling Wave Propagation in Layered Acoustic Metamaterials with Arrays of Crack-Like Inhomogeneities †

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† Presented at the 1st International Online Conference on Mathematics and Applications; Available online: https://iocma2023.sciforum.net/.

Abstract: The three-dimensional problem of the modelling of elastic wave propagation in a multi-layered acoustic metamaterial, a periodic elastic composite with periodic arrays of interface cracks or planar voids of arbitrary shape, is considered. The boundary integral equation method is extended for this purpose. The unknown crack-opening displacement vectors for each array are related using the Floquet theorem and solved using the Galerkin method at reference delaminations in the arrays. The developed method provides an efficient tool for fast parametric analysis of the influence of the periodic crack array characteristics on the transmission and diffraction of elastic waves. Two modifications to the boundary integral equation method are proposed and compared for rectangular cracks. To reduce computational costs, a preliminary analytical evaluation of the arising integral representations in terms of the Fourier transform of Green's matrices and the crack-opening displacements are presented.

Keywords: wave propagation; boundary integral equation method; spectral method; acoustic metamaterial; crack; void; periodic composite; diffraction

Citation: Golub, M.V.; Doroshenko O.V.; Fomenko S.I.; Okoneshnikova E.A.; Kozhevnikov V.V. The Advanced Boundary Integral Equation Method for Modelling Wave Propagation in Layered Acoustic Metamaterials with Arrays of Crack-Like Inhomogeneities. *Comput. Sci. Math. Forum* **2023**, *7*, 1. https://doi.org/10.3390/IOCMA2023-14602

Academic Editor: Francisco Chiclana

Published: 15 May 2023

1. Introduction

A novel class of composites, the so-called acoustic metamaterials (AMMs), which provide advanced characteristics has attracted the attention of researchers in recent years [1,2]. AMMs reproduce unique properties that open up prospects for passive and active wave energy manipulation. Currently, various AMMs have already been developed with applications in ultrasonic technology, acoustoelectronics, hydroacoustics, architectural acoustics, and sound absorption [3,4].

AMMs typically have a periodic or quasi-periodic structure, where arrays of inhomogeneities such as holes, voids, or inclusions are embedded in a matrix that can also be a composite. The mathematical modelling is usually performed at the first stages of the design of new AMMs to select the structure parameters that provide the desired wave properties. In this study, multi-layered AMMs with doubly periodic arrays of delaminations/cuts at some interfaces are considered. To describe the dynamic behaviour of the considered AMMs, a modification to the boundary integral equations method (BIEM) proposed by Glushkov and Glushkova [5] is developed. A similar employment of the BIEM was proposed in [6], where the propagation of plane waves through the interface of two elastic media with doubly periodic interface crack arrays was considered. In this study, the advanced BIEM is presented to simulate wave motion in a multi-layered AMM with multiple doubly periodic arrays of cracks or voids.

2. Statement of the Problem

The problem of elastic wave propagation in multi-layered AMMs composed of N periodically arranged unit-cells made from two elastic isotropic layers is considered. M doubly periodic arrays of cracks or infinitesimally thin voids are situated at the interfaces forming a rectangular lattice. It is assumed that the periodic stack of layers is located between two elastic half-spaces and a plane wave comes from the lower half-space at a certain angle to the interfaces. For convenience, the Cartesian coordinates $\{x_1, x_2, x_3\}$ are introduced so that the interfaces are parallel to x_1Ox_2, while the plane and voids are situated along axes Ox_1 and Ox_2. An example of the AMM with $M = 2$ doubly periodic arrays is shown in Figure 1. Accordingly, $V_0 = \{x_3 \leq 0\}$ is the lower half-space and $V_{2N+1} = \{x_3 > h_{2N}\}$ is the upper half-space. The unit-cell consists of two components and, therefore, a total of $2N$ layers $V_k = \{|x_1| < \infty, |x_2| < \infty, h_{k-1} < x_3 \leq h_k\}$ are considered. Each infinite three-dimensional layer V_k is made of homogeneous, isotropic material with the mass density ρ_k, Young's modulus E_k, and Poisson's ratio ν_k.

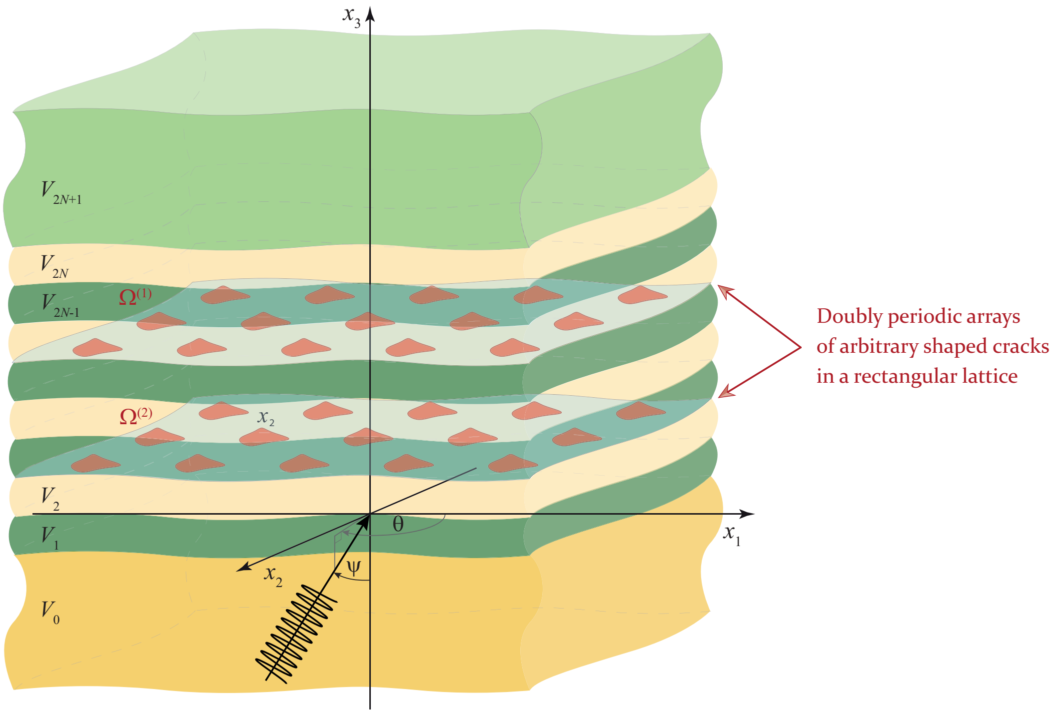

Figure 1. Geometry of the problem: multi-layered elastic periodic composite with two doubly periodic arrays of cracks (with rectangular lattice).

Multiple doubly periodic arrays $\Omega^{(m)}$ of cracks or voids with the same spacing between the crack centres are situated in the planes $x_3 = d^{(m)}$ and $m = \{1, 2\}$. The rectangular lattice corresponding to each doubly periodic array $\Omega^{(m)}$ is based on the vectors $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ with dimensions s_1 and s_2 of the unit-cell, as shown in Figure 2a. In accordance with the location of the cracks, the whole media can be considered as a doubly periodic array of unbounded parallelepipeds $\bigcup_{j_1,j_2} G^{(m)}_{j_1j_2} = \{|x_1| \leq s_1, |x_2| \leq s_2, |x_3| < \infty\}$, which allows to describe scattering by all doubly periodic arrays. The intersection of the parallelepiped

$$G^{(m)}_{00} = \{\boldsymbol{x}|\, (x_1, x_2) = \beta_1\boldsymbol{g}_1 + \beta_2\boldsymbol{g}_2, \quad |x_3| < \infty\}, \quad \beta_i \in [-1/2, 1/2]$$

with the plane $x_3 = d^{(m)}$ is chosen as a reference unit-cell in the m-th array containing the reference crack $\Omega^{(m)}_{00}$. The centre of the reference crack $\Omega^{(m)}_{00}$ for each array is assumed to be the origin of the Cartesian coordinates. Geometrical sizes of the unit-cell are denoted as

$$s_1 = |\boldsymbol{g}_1|, \qquad s_2 = |\boldsymbol{g}_2|,$$

whereas the centre of the unit-cell $G_{ij}^{(m)}$ is defined by the vector

$$\boldsymbol{a}_{j_1 j_2}^{(m)} = \{x_{j_1 j_2}^{(m)},\ y_{j_1 j_2}^{(m)},\ d^{(m)}\} = \{s_1 j_1, s_2 j_2, d^{(m)}\}.$$

The centre of the crack-like voids $\Omega_{j_1 j_2}^{(m)}$ is shifted from the centre of the unit-cell by vector $\boldsymbol{b}^{(m)}$, as shown in Figure 2b.

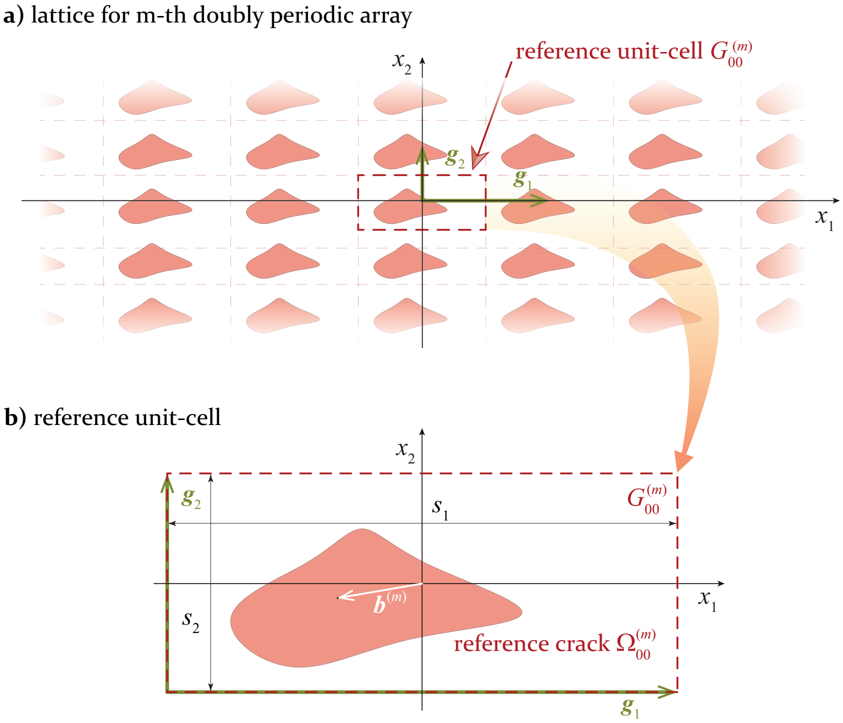

Figure 2. m-th doubly periodic array of rectangular cracks: (**a**) Lattice for m-th doubly periodic array. (**b**) The reference unit-cell.

The steady-state harmonic motion of the multi-layered periodic elastic structure with circular frequency ω is governed by the Lame–Navier equation with respect to the displacement vector $\boldsymbol{u}$. The displacement vector $\boldsymbol{u}$ and the traction vector $\boldsymbol{\tau} = \mathbf{T}_3[\boldsymbol{u}] = (\sigma_{13}, \sigma_{23}, \sigma_{33})$ are assumed continuous outside the voids $\Omega_{i,j}^{(m)}$, while stresses and displacements are related by Hooke's law. The stress-free boundary conditions are assumed at the crack faces such that an unknown crack-opening displacement (COD) function $\Delta\boldsymbol{u}^{(m)}(\boldsymbol{x})$ is introduced for each plane $x_3 = d^{(m)}$ containing an m-th doubly periodic array.

3. The Advanced Boundary Integral Equation Method

Let us consider plane wave scattering propagating in the composite by M arrays. In this case, the wave-field $\boldsymbol{u}^0$ incident by a plane wave incoming from the lower half-space V_0 can be simulated using the transfer matrix method [7]. The total wave-field in the composite is the sum of the incident wave-field $\boldsymbol{u}^0$ propagating in the layered structure in the absence of inhomogeneities and the wave-fields $\tilde{\boldsymbol{u}}_{j_1,j_2}^{m}$ scattered by each crack in the doubly periodic arrays $\boldsymbol{\Omega}_m = \bigcup\limits_{j_1,j_2} \Omega_{j_1 j_2}^{(m)}$.

The mutual effect of the cracks on each other can be taken into account using the Floquet theorem. Therefore, the two-dimensional Fourier transform of the COD $\Delta\boldsymbol{U}$ with parameter $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2\}$ has the following representation

$$\Delta\boldsymbol{U}^{(m)}(\boldsymbol{\alpha}) = \sum_{j_1,j_2=-\infty}^{\infty} \Delta\boldsymbol{U}^{(m)}_{j_1 j_2}(\boldsymbol{\alpha}) = \sum_{j_1,j_2=-\infty}^{\infty} \Delta\boldsymbol{U}^{(m)}_{00}(\boldsymbol{\alpha})\, \mathrm{e}^{\mathrm{i}\boldsymbol{\alpha}^{(m)}_{j_1 j_2}\cdot\boldsymbol{\alpha}_p}, \tag{1}$$

where $\boldsymbol{\alpha}_p = k_0\boldsymbol{p} + \boldsymbol{\alpha}$, $\boldsymbol{p} = \{p_1, p_2\}$ is the two-dimensional projection of the unit vector of the wave propagation vector on the plane $x_3 = d^{(m)}$, and k_0 is the wave-number of the incident plane wave with polar and azimuthal incidence angles θ and ϕ, respectively.

On the other hand, the scattered field can be expressed in terms of the two-dimensional Fourier transform in accordance with the BIEM [5,6] as contour integrals along the contours Γ_i bending poles and branch points of the two-dimensional Fourier transform of Green's matrix for the whole structure (see [8]).Notice that the Fourier transform of the unknown traction vector can be expressed in terms of the COD.

Substitution of the integral representation of the total wave-field into the stress-free boundary conditions, accounting for Hooke's law and Floquet's theorem, gives the following boundary integral equation for the reference cracks in the m-th array:

$$\frac{1}{4\pi^2}\sum_{i=1}^{M}\int_{\Gamma_1}\int_{\Gamma_2} \tilde{\mathbf{S}}^{(i)}(\boldsymbol{\alpha}^{j_1 j_2}, d^{(m)}) \sum_{j_1,j_2=-\infty}^{\infty} \Delta\boldsymbol{U}^{(i)}_{00}(\boldsymbol{\alpha}^{j_1 j_2})\, \mathrm{e}^{\mathrm{i}\boldsymbol{\alpha}^{(m)}_{j_1 j_2}\cdot\boldsymbol{\alpha}_p} \mathrm{e}^{-\mathrm{i}\boldsymbol{\alpha}\cdot\boldsymbol{y}} \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 = -\boldsymbol{\tau}^0(\boldsymbol{x}), \boldsymbol{x} \in \Omega^{(m)}_{00} \tag{2}$$

$$\boldsymbol{\alpha}^{j_1 j_2} = \left\{-k_0 p_1 + \frac{2\pi j_1}{s_1}, -k_0 p_2 + \frac{2\pi j_2}{s_2}\right\}, \qquad \boldsymbol{y} = \{x_1, x_2\}.$$

For more details related to the derivation of the boundary integral equation and $\tilde{\mathbf{S}}^{(i)}$ see [9].

The boundary integral in Equation (2) is solved using the Galerkin scheme. The unknown COD for the crack $\Omega^{(m)}_{00}$ in the reference unit-cell $G^{(m)}_{00}$ is approximated by the complete set of basis functions $\varphi_k(x_1, x_2)$:

$$\Delta\boldsymbol{u}^{(m)}_{00}(x_1, x_2) = \sum_{k=1}^{\infty} \boldsymbol{c}^{(m)}_k \varphi^{(m)}_k(x_1, x_2). \tag{3}$$

The choice of basis and projection functions depends on the cracks shape. In the case of rectangular cracks, the CODs can be expanded in terms of the Chebyshev polynomials $\mathrm{U}_n(x)$ of the second kind with the square root weight $p_k(x) = \mathrm{U}_{k-1}(x)\sqrt{1-x^2}$ for each coordinate. For arbitrary-shaped cracks/voids the unknown COD vector is expanded in terms of axisymmetric basis functions

$$\phi(x_1, x_2) = \begin{cases} \left(1 - x_1^2 - x_2^2\right)^{\pi-1}, & x_1^2 + x_2^2 < 1; \\ 0 & \text{otherwise.} \end{cases}$$

Though convergence of the COD is not guaranteed in a continuous metric, the COD convergence of the solution at the nodal points for all $h > 0$ is guaranteed [5] .

As a result of applying the Bubnov–Galerkin scheme to (2), keeping N terms after reduction, the following system is obtained:

$$\sum_{m=1}^{M}\sum_{k=1}^{N} \mathbf{A}^{(m)}_{jk} \cdot \boldsymbol{c}_k = \boldsymbol{f}_j, \qquad j = 1, 2 \cdots M. \tag{4}$$

The right-hand side of system of (4) is the projection of the wave-field $\boldsymbol{\tau}^0$ onto the projection function $\psi_j(x_1, x_2)$, whereas the double series

$$\mathbf{A}_{jk}^{(m)} = \frac{1}{s_1 s_2} \sum_{j_1=-M_1}^{M_1} \sum_{j_2=-M_2}^{M_2} \tilde{\mathbf{S}}^{(i)}(\boldsymbol{\alpha}^{j_1 j_2}, d^{(m)}) \mathrm{e}^{-\mathrm{i}\boldsymbol{\alpha}^{j_1 j_2} \cdot \boldsymbol{b}^{(m)}} \Phi_j^*((\boldsymbol{\alpha}^{j_1 j_2})^*) \Psi_j^*((\boldsymbol{\alpha}^{j_1 j_2})^*) \quad (5)$$

describes the scattering of the wave-field by the m-th array, induced due to the presence of j-th array. Here, Φ_j and Ψ_j are the two-dimensional Fourier transforms of the basis and projection functions, respectively.

The calculation of the left-hand side of the system (4) demands computations of double series (5), which exhibit a low convergence rate for rectangular cracks if Chebyshev polynomials are employed as basis and projection functions due to the Fourier transform $\Phi_k(\alpha_j) \sim \alpha_j^{-3/2}$ and kernel $\tilde{\mathbf{S}}^{(i)}(\alpha_1, \alpha_2) \sim \alpha$ at $\alpha \to \infty$. Thus, the double series summarize the products of four Bessel and power functions. The convergence of the series (5) is shown to estimat the absolute values. Moreover, such analytic evaluations allow for the direction to be determined in the $\boldsymbol{\alpha}$-plane, where the slowest convergence is observed (along axes $O\alpha_1$ and $O\alpha_2$). It is shown that the terms with $\{\alpha_1, \alpha_2\}$ lying inside a certain asteroid and along the coordinate axes provide the largest contribution to the sum, which is used to calculate the double series.

Figure 3 illustrates the convergence of several non-zero components of the matrices $\mathbf{A}_{jk}^1$ at lower and higher frequencies $k_0 s_2 = 2$ (Figure 3a) and $k_0 s_2 = 10$ (Figure 3b), where k_0 is the wave-number of incoming plane longitudinal waves. The variation in the relative error during the double series calculation

$$\epsilon_{\mathrm{r}}\left(A_{jk;ij}\right) = \frac{A_{jk;ij} - A_{jk;ij}^{(\mathrm{exact})}}{|A_{jk;ij}^{(\mathrm{exact})}|}$$

with respect to the number of terms M_i is presented here. $A_{jk_1k_2;ij}^{(\mathrm{exact})}$ is calculated numerically setting $M_1 = M_2 = 2 \times 10^4$. The higher the frequency, the greater the convergence rate of the double series. The latter can be explained by the fact that the Fourier transform of the kernel of the boundary integral equation $\mathbf{S}(\alpha_1, \alpha_2)$ decreases slowly at lower frequencies ω. For non-square rectangular cracks ($l_1 \neq l_2$), the ratio between the number of terms N_2/N_1 should be approximately equal to the ratio l_2/l_1.

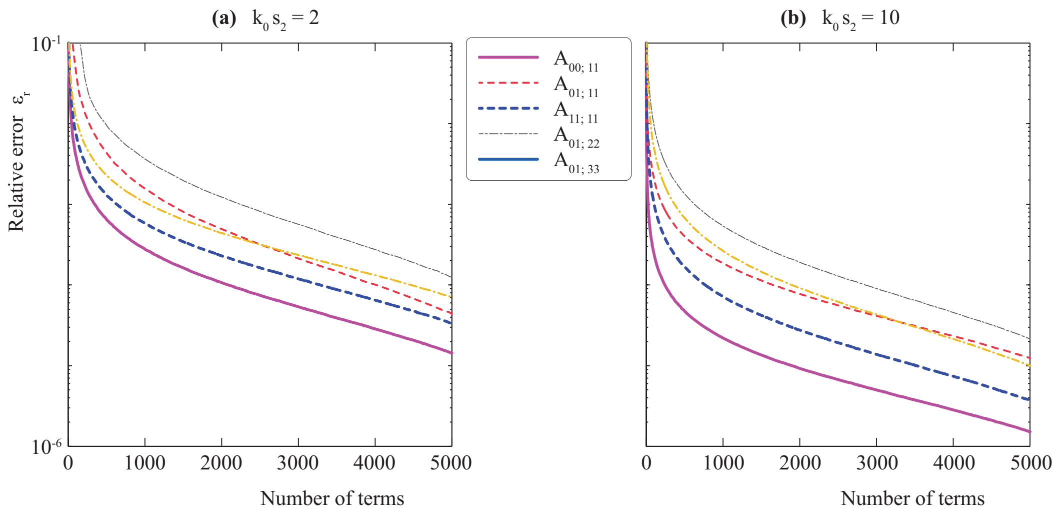

Figure 3. The convergence of the double series.

4. Conclusions

Two choices of basis and projection functions have been compared. In the case of the same basis and projection functions in the Bubnov–Galerkin method, a slow convergence of the arising double series is observed. If axisymmetric functions, proposed by Glushkov and Glushkova [10], are used as projection functions in the Petrov–Galerkin method, guaranteeing fast convergence in the series, then more basis functions are required for better accuracy in the crack-opening displacements. The results of the numerical analysis show good accuracy and convergence rate of the proposed method. The authors believe that the proposed advanced BIEM will be further employed for experimental and theoretical studies of wave propagation in acoustic metamaterials with doubly periodic arrays of crack-like voids, see, e.g., [11].

Author Contributions: Conceptualization, M.V.G., O.V.D. and S.I.F.; methodology, M.V.G. and O.V.D.; software, S.I.F., M.V.G. and E.A.O.; validation, M.V.G., S.I.F. and O.V.D.; formal analysis, M.V.G., V.V.K. and O.V.D.; investigation, M.V.G. and O.V.D.; resources, M.V.G.; data curation, E.A.O. and S.I.F.; writing—original draft preparation, M.V.G., O.V.D. and E.A.O.; writing—review and editing, M.V.G. and O.V.D.; visualization, M.V.G.; project administration, M.V.G. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Russian Science Foundation (project 22-11-00261).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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