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# Confidence Bands for Local Non-Stationarity Detection †

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**Abstract**

This article develops an asymptotic theory for the sample (time-averaged) variance of the locally stationary wavelet packet (LSWP) processes. Our focus is the periodogram and the sample variance, and in particular its average. Under a near-epoch-dependence (NED) condition on the periodogram, and mild moment and summability assumptions, we establish a central limit theorem and a consistent estimator for the average process variance. These results yield asymptotic stationarity bands for the average variance, providing a simple, theoretically grounded benchmark for second-order stationarity in packet-domain models.

**Keywords:** $L_2$ near-epoch dependence; local stationarity; wavelet packet processes

## 1. Introduction

For a stationary time series, second-order properties such as the autocovariance function and spectrum are time-invariant. In many applications (financial volatility, geophysics, biomedical signals), these properties evolve over time: series can appear approximately stationary on short intervals but exhibit clear changes on longer horizons. The locally stationary literature formalizes this idea by allowing a smoothly varying local spectrum; foundational work includes Priestley's evolutionary-spectrum formulation [1] and later treatments of evolutionary spectra [2,3], as well as subsequent general frameworks and reviews [4,5]. Wavelet-based models provide an alternative, multiscale notion of local second-order structure. The locally stationary wavelet (LSW) model of [6] represents a non-stationary process by a superposition of non-decimated wavelet atoms with slowly varying amplitudes, leading to the evolutionary wavelet spectrum (EWS) and associated (bias-corrected) wavelet periodograms. Wavelet packet extensions enrich the time–frequency tiling and can better capture oscillatory features; the locally stationary wavelet packet (LSWP) framework of [7] develops basis selection and model fitting for this class. This paper is a theory note aimed at a single inferential building block: a central limit theorem (CLT) for the average sample variance of LSWP processes. The main results are: (i) an $L_2$ near-epoch-dependence (NED) control for the time-varying periodogram and the average sample variance; (ii) a CLT for the average sample variance; and (iii) consistent estimators of the corresponding average (marginal) variance and its standard error, yielding its asymptotic stationarity bands. Section 2 introduces the relevant definitions and assumptions. Section 3 states the NED bounds and the CLT and describes the resulting stationarity bands. Proofs and technical lemmas are collected in Appendix A.

## 2. Notation, Definitions and Assumptions

We consider a locally stationary wavelet packet representation from [7]. For a nontrivial theory we recall the necessary concepts from that paper, and define some further tools

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and notation. Non-decimated wavelet packets are locally supported oscillatory functions $\psi_{j,i,k}(t)$, where the index $j = 1, 2, \ldots, \log_2(T)$ controls the support width and the scale of oscillation, the index $i = 0, 1, \ldots, 2^j$ controls the oscillation frequency within a fixed scale, and $k$ denotes translation. In what follows, $\mathcal{B}$ denotes a redundant collection of dyadic increasing partitions of the frequency interval $(0, 1/2]$, and each $b \in \mathcal{B}$ is a disjoint cover of that interval. Summation over packet indices is therefore taken over $p, p', q \in b$. A basis $b$ can be selected from $\mathcal{B}$ as shown in [7].

**Definition 1** (Non-decimated wavelet packets)**.** *Let $\{\psi_{p,k}(t)\}_{p\in b, k\in\mathbb{Z}}$ be a non-decimated wavelet packet family (a redundant packet frame), indexed by the flattened packet index $p = (j_p, i_p)$ and translation $k$. Let $\mathcal{L}_p$ denote the support length of packet $p$, with $\mathcal{L}_p \asymp 2^{j_p}$.*

**Definition 2** (Cross-correlation wavelet packets)**.** *For $p, p' \in b$ the cross-correlation wavelet packet is defined by*

$$\Psi_{p,p'}(\tau) = \sum_k \psi_{p,k}\, \psi_{p',k-\tau}, \tag{1}$$

*where $\psi_{p,k}$ are the non-decimated wavelet packets from Definition 1. When $p' = p$, $\Psi_{p,p}(\tau) = \Psi_p(\tau)$ is the corresponding autocorrelation wavelet packet. Let $\mathcal{L}_{p,p'}$ denote the support length of $\Psi_{p,p'}$, with $\mathcal{L}_{p,p'} \asymp 2^{j_p \wedge j_{p'}}$.*

**Definition 3** (Locally stationary wavelet packet process)**.** *A locally stationary wavelet packet (LSWP) process is a triangular array $\{X_{t;T}\}$ with the mean-square representation*

$$X_{t;T} = \sum_{p\in b} \sum_k \omega_{p,k;T}\, \psi_{p,k}(t)\, \varepsilon_{p,k}, \tag{2}$$

*where $b$ is a partition element of the redundant dictionary $\mathcal{B}$. The amplitudes satisfy*

$$\sup_k |\omega_{p,k;T} - W_p(k/T)| \le \frac{C_p}{T}, \qquad \|W_p^2\|_{TV} \le L_p, \qquad L_p \le K\, 2^{-j_p},$$

*for finite constants $C_p$, $L_p$ and $K$, and $||x||_{TV}$ is the total variation norm of $x$.*

More details on this setup and on the packet indexing conventions are given in [7]. The assumptions below are stated at the level needed for the theory that follows. They emphasize (i) moment conditions on the innovations, (ii) summability and tail decay of packet-correlation kernels, and (iii) a multiscale growth regime ensuring uniformity over the packet indices used in the cumulative statistics.

**Assumption 1** (Innovations: moments and weak dependence)**.** *The array $\{\epsilon_{q,m}\}_{q,m}$ satisfies $\mathbb{E}\epsilon_{q,m} = 0$, $\mathbb{E}\epsilon_{q,m}^2 = 1$, and $\sup_{q,m} \mathbb{E}|\epsilon_{q,m}|^{4+\delta} < \infty$ for some $\delta > 0$. Moreover, for each fixed $q$ the sequence $\{\epsilon_{q,m}\}_m$ is $\alpha$-mixing (see [8]), with coefficients $\alpha_\epsilon(r)$ such that $\sum_{r\ge1} \alpha_\epsilon(r)^{\delta/(2+\delta)} < \infty$.*

**Assumption 2** (Weighted tail summability of packet correlations)**.** *There exists $\varepsilon > 0$ such that, uniformly over all relevant indices $(q, p')$,*

$$\sum_{b\in\mathbb{Z}} (1 + |b|)^{1+2\varepsilon}\, \Psi_{q,p'}^2(b) < \infty, \qquad \sum_{b\in\mathbb{Z}} (1 + |b|)^{1+2\varepsilon}\, \Psi_{q,p'}^4(b) < \infty.$$

**Assumption 3** (Uniform bounds on time-varying amplitudes)**.** *There exist finite constants $C_2, C_4$ such that $\sup_{q,m,T} \omega_{q,m;T}^2 \le C_2$ and $\sup_{q,m,T} \omega_{q,m;T}^4 \le C_4$. Whenever sums over packet indices are used, the corresponding scale-energy is summable so that the resulting bounds are finite.*

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**Assumption 4** (Multiscale growth regime)**.** *Inner sums defining the LSWP process and the local second-order structure are taken over $p \in b$, while the cumulative statistics use a finite sample truncation $1 \le p \le P_T$ satisfying $\log_2(T) \le P_T < T$. All bounds involving cumulative statistics are uniform in $(k, T)$ and over $p \le P_T$.*

**Remark 1** (Local spectra and local covariance for LSWP processes)**.** *For $z \in [0,1)$ and $k = [zT]$, and for increasing $T$, we have $[zT]/T \to z$. This is the rescaled time. We then define the evolutionary wavelet packet spectrum (EWPS) by*

$$S_p(z) = |W_p(z)|^2, \tag{3}$$

*where the single index $p$ is a shorthand for the packet pair $(j_p, i_p)$; see [7]. The marginal EWPS is $\bar{S}_p = \int_0^1 S_p(z)\, dz$. The time-localized covariance is*

$$C(z, \tau) = \sum_{p \in b} S_p(z) \Psi_p(\tau), \tag{4}$$

*where $\Psi_p(\tau)$ is the autocorrelation packet from Definition 2. Since $\Psi_p(0) = 1$ for all $p$, the time-localized variance is*

$$\sigma^2(z) = C(z, 0) = \sum_{p \in b} S_p(z). \tag{5}$$

*The corresponding marginal variance is*

$$\sigma^2 = \int_0^1 C(z, 0)\, dz = \sum_{p \in b} \int_0^1 S_p(z)\, dz. \tag{6}$$

**Definition 4** (Packet periodogram and bias-corrected periodogram)**.** *Given a packet family $\{\psi_{p,k}\}$, define the raw packet periodogram by $I_{p,k;T} = \left|\sum_t \psi_p(t-k)\, X_{t;T}\right|^2$. Let $\mathbf{A} = (A_{p,p'})$ with $A_{p,p'} = \sum_\tau \Psi^2_{p,p'}(\tau)$, and let $\mathbf{A}^{-1} = (A^{-1}_{p,p'})$ denote its inverse on the relevant packet set. The bias-corrected (asymptotically unbiased) packet periodogram is*

$$L_{p,k;T} = \sum_{p'=1}^{P_T} A^{-1}_{p,p'} I_{p',k;T}. \tag{7}$$

The bias-corrected packet periodogram satisfies $\mathbb{E} L_{p,k;T} = S_p(z) + O(T^{-1})$ uniformly in $k$.

**Definition 5** (Locally stationary sample variance)**.** *The locally stationary sample variance (LSSV) is the sequence*

$$s^2_{k;T} = \sum_{p=1}^{P_T} L_{p,k;T}.$$

*We center it by the deterministic local variance curve and define*

$$U_{k;T} = s^2_{k;T} - \sigma^2(k/T), \qquad b_T^2 = \mathbb{E}\left(\sum_{k=0}^{T-1} U_{k;T}\right)^2.$$

## 3. $L_2$ NED Properties of LSWP Statistics

This section studies the dependence structure of the bias-corrected packet periodogram and of the locally stationary sample variance. We use the $L_2$ near-epoch-dependence framework of [8], with the innovation array as the underlying weakly dependent process.

**Definition 6.** *For an array* $\{\{\xi_{k;T}\}_{k=-\infty}^{\infty}\}_{T=1}^{\infty}$ *and* $\mathcal{G}_{k-d;T}^{k+d} = \sigma(\xi_{k-d;T}, \ldots, \xi_{k+d;T})$*, an integrable array* $\{\{X_{k;T}\}\}_T$ *is said to be* $L_2$ *near-epoch-dependent if*

$$\left\| X_{k;T} - \mathbb{E}(X_{k;T} \mid \mathcal{G}_{k-d;T}^{k+d}) \right\|_2 \leq a_T^{(1)} a_{d;T}^{(2)} \tag{8}$$

*uniformly in* $k$*, where* $a_{d;T}^{(2)} \to 0$ *as* $d \to \infty$*. If* $a_{d;T}^{(2)} = O(d^{\lambda-\epsilon})$ *for some* $\epsilon > 0$*, the array is said to be* $L_2$ *NED of size* $\lambda$ *(see [8] for details).*

For the asymptotic arguments below we also define the following:

**Definition 7.** *Let*

$$\kappa_i^{\epsilon} = \mathrm{cum}_i(\epsilon_{q,m}), \qquad \mathcal{G}_{k-d;T}^{k+d} = \sigma(\bar{\epsilon}_{k-d;T}, \ldots, \bar{\epsilon}_{k+d;T}),$$

*where* $\bar{\epsilon}_{m;T} = (\epsilon_{1,m}, \ldots, \epsilon_{P_T,m})'$*, and let* $\widetilde{S}_p = \sup_z S_p(z)$*. Define*

$$\delta_T = \sup_{1 \leq p \leq P_T} \frac{C_p}{\widetilde{S}_p}, \qquad \text{with} \qquad \delta_T / T \in \ell_\infty.$$

Under Assumption 1, the innovation array is $\alpha$-mixing. The next theorem gives the required $L_2$ NED bound for the bias-corrected packet periodogram.

**Theorem 1.** *Let* $X_{t;T}$ *be specified as in Definition 3 and satisfy Assumptions 1–4. Let* $\{L_{p,k;T}\}_k$ *be the bias-corrected packet periodogram sequence for packet* $p$ *from Definition 4. Then for each* $p \in b$*, there exist constants* $\bar{b}_{p;T} = 1$ *and* $\nu_{p,d;T} = O(d^{-1/2-\varepsilon})$ *such that, uniformly in* $k$*,*

$$\left\| L_{p,k;T} - \mathbb{E}\Big(L_{p,k;T} \mid \mathcal{G}_{k-d;T}^{k+d}\Big) \right\|_2 \leq \bar{b}_{p;T} \nu_{p,d;T}. \tag{9}$$

*In particular,* $\{L_{p,k;T}\}_k$ *is* $L_2$ *NED of size* $-1/2 - \varepsilon$*.*

The LSSV sequence inherits this dependence structure by summing over the finitely many packet indices included in the finite sample truncation $P_T$.

**Proposition 1.** *Let* $X_{t;T}$ *be specified as in Definition 3 and satisfy Assumptions 1–4. Consider the sequences* $\{L_{p,k;T}\}_k$ *and* $\{s_{k;T}^2\}_k$ *from Definitions 4 and 5. If, for every* $p \leq P_T$*,* $\{L_{p,k;T}\}_k$ *satisfies an* $L_2$ *NED bound of size* $-1/2 - \varepsilon$ *with constants* $(\bar{b}_{p;T}, \nu_{p,d;T})$*, then* $\{s_{k;T}^2\}_k$ *also satisfies an* $L_2$ *NED bound of size* $-1/2 - \varepsilon$*.*

*Weak Convergence*

We derive a central limit theorem for the centered local sample variance. Since, for all $k$,

$$\mathbb{E}\, s_{k;T}^2 = \sigma^2(k/T) + O(T^{-1}),$$

centering by $\sigma^2(k/T)$ is asymptotically equivalent to centering by the exact mean.

**Theorem 2.** *Let* $X_{t;T}$ *be specified as in Definition 3 and satisfy Assumptions 1–4. Consider* $\{U_{k;T}\}_k$*,* $\{s_{k;T}^2\}_k$ *and* $b_T^2$ *from Definition 5. Then*

$$b_T^{-1} \sum_{k=0}^{T-1} U_{k;T} \rightsquigarrow \mathcal{N}(0,1). \tag{10}$$

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*Moreover, the $O(T^{-1})$ remainder in the centering can be omitted because*

$$\sqrt{T}\sup_k \left|\mathbb{E}\, s^2_{k;T} - \sigma^2(k/T)\right| = O(T^{-1/2}) \to 0.$$

To construct confidence bands to assess departures from stationarity using the average sample variance, we require (i) a consistent estimator of the average variance level and (ii) a consistent estimator of the scaling constant $b_T/T$ appearing in (10).

**Corollary 1.** *Under the conditions of Theorem 2,*

$$\frac{1}{T}\sum_{k=0}^{T-1} s^2_{k;T} - \frac{1}{T}\sum_{k=0}^{T-1} \sigma^2(k/T) \xrightarrow{p} 0.$$

**Proposition 2.** *Assume the conditions of Theorem 2 and, in addition, that $\{U_{k;T}\}_k$ is locally short memory so that $\sup_T \sum_{h\geq 0} \sup_k |\mathrm{Cov}(U_{k;T}, U_{k+h;T})| < \infty$. Let $w$ be bounded with $w(0) = 1$ and $w(x) = 0$ for $x > 1$, and let $L_T \to \infty$ with $L_T/T \to 0$. Define*

$$\hat{b}^2_T = \sum_{k=0}^{T-1} U^2_{k;T} + 2\sum_{h=1}^{L_T} w(h/L_T) \sum_{k=h}^{T-1} U_{k;T}U_{k-h;T}, \qquad \hat{b}_T/T = \frac{\sqrt{\hat{b}^2_T}}{T}.$$

*If $\liminf_{T\to\infty} b^2_T/T > 0$, then $\hat{b}_T/T/(b_T/T) \xrightarrow{p} 1$.*

Let $\bar{s}^2_T = T^{-1}\sum_{k=0}^{T-1} s^2_{k;T}$ and let $z_{1-\alpha/2}$ denote the $(1-\alpha/2)$ quantile of the standard normal distribution. Then, for confidence level $(1-\alpha)$,

$$\bar{s}^2_T \pm z_{1-\alpha/2}\, \hat{b}_T/T$$

are the endpoints of an asymptotic stationarity band for the local variance sequence $\{s^2_{k;T}\}_{k=0}^{T-1}$.

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## Appendix A. Lemmas and Proofs

**Lemma A1.**

$$\sum_{p_s=1}^{P_T} A^{-1}_{q_s,p'_s} \sum_{q_s,p'_s,m_s} O\left(\frac{C_{q_s} + L_{q_s}|k-m_s|}{T}\right)\Psi^2_{q_s,p'_s}(k-m_s) = O\left(\frac{1}{T}\right). \tag{A1}$$

**Proof of Lemma A1.** For $s = 1, 2$ we have

$$\begin{aligned}
&\sum_{p_s=1}^{P_T} \sum_{p'_s,q_s} A^{-1}_{p_s,p'_s} \sum_{m_s} \big(C_{q_s} + L_{q_s}|k-m_s|\big)\Psi^2_{q_s,p'_s}(k-m_s) \\
= &\sum_{p_s=1}^{P_T} \sum_{q_s,p'_s} A^{-1}_{p'_s,p_s} A_{p'_s,q_s} \Big(C_{q_s} + 2L_{q_s}(\mathcal{L}_{q_s,p'_s} - 1)\Big) \\
\leq &\sum_{p_s=1}^{P_T} \sum_{q_s,p'_s} A^{-1}_{p'_s,p_s} A_{p'_s,q_s} \Big(C_{q_s} + 2L_{q_s}\mathcal{L}_{q_s,p'_s}\Big) \\
\leq &\sum_{p_s=1}^{P_T} \sum_{q_s,p'_s} A^{-1}_{p'_s,p_s} A_{p'_s,q_s} \big(C_{q_s} + 2L_{q_s}\mathcal{L}_{q_s}\big) \\
= &\sum_{p_s=1}^{P_T} \sum_{q_s} \delta_{p_s,q_s}\, 2\big(C_{q_s} + L_{q_s}\mathcal{L}_{q_s}\big) \\
= &\sum_{p_s=1}^{P_T} 2\big(C_{p_s} + L_{p_s}\mathcal{L}_{p_s}\big) \leq K. \qquad \text{(A2)}
\end{aligned}$$

Here $K < \infty$ depends on the envelope $L_p \leq K2^{-j_p}$, the support relation $\mathcal{L}_p \asymp 2^{j_p}$, the fixed finiteness of the outer truncation $P_T$, and the uniform control of $C_p/\widetilde{S}_p$ through $\delta_T$. □

**Lemma A2.** *Let* $b_T^2 = \mathbb{E}(\sum_k U_{k;T})^2$. *Then we have* $b_T \asymp T^{1/2}$.

**Proof of Lemma 2.** Based on the integer sequence $\tau = -k, \ldots, 0, \ldots, T-k-1$, define $\tau' = \tau + k$. For $P_T < \infty$ we have

$$\begin{aligned}
&b_T^2 \\
= &\sum_{k}^{T-1} \sum_{\tau} \prod_{s=1}^{2} \left[ \sum_{p_s=1}^{P_T} \sum_{p'_s,q_s} A^{-1}_{p_s,p'_s} \sum_{m_s} (1 + \kappa_4^{\epsilon}\, \mathbb{I}_{s=1})\, f_{q_s,p'_s,k+\tau,m_s} \right] \\
= &\sum_{k}^{T-1} \sum_{\tau} \prod_{s=1}^{2} \left[ \sum_{p_s=1}^{P_T} \sum_{p'_s,q_s} A^{-1}_{p_s,p'_s} \sum_{m_s} (1 + \kappa_4^{\epsilon}\mathbb{I}_{s=1})\omega^2_{q_s,m_s;T}\, \Psi^2_{q_s,p'_s}(k-m_s+\tau) \right] \\
= &\sum_{k}^{T-1} \sum_{\tau} \prod_{s=1}^{2} \sum_{p_s=1}^{P_T} \sum_{p'_s,q_s} A^{-1}_{p_s,p'_s} (1 + \kappa_4^{\epsilon}\, \mathbb{I}_{s=1}) \left[ S_{q_s}\left(\frac{k}{T}\right) + \sum_{m_s} O\left(\frac{C_{q_s} + L_{q_s}(k-m_s)}{T}\right) \right] \times \\
\times &\ \Psi^2_{q_s,p'_s}(k-m_s+\tau) \\
= &\sum_{k}^{T-1} \left\{ \prod_{s=1}^{2} \left[ \sum_{p_s=1}^{P_T} \sum_{p'_s,q_s} A^{-1}_{p_s,p'_s} (1 + \kappa_4^{\epsilon}\, \mathbb{I}_{s=1}) S_{q_s}\left(\frac{k}{T}\right) \sum_{m_s} \sum_{\tau'} \Psi^2_{q_s,p'_s}(\tau' - m_s) \right] \right\} + O\left(\frac{1}{T}\right)
\end{aligned}$$

We use the finite support of cross-correlation wavelet packets in order to bound the double summation. Since in general $\sum_\tau \Psi^2_{q,p'}(\tau) = A_{q,p'}$ we have

$$\sum_{m_s} \sum_{\tau'} \Psi^2_{q_s,p'_s}(\tau' - m_s) = A_{q_s,p'_s} \mathcal{B}_{q_s,p'_s}$$

where

$$\mathcal{B}_{q_s,p'_s} = \begin{cases} 1, & \text{if } s = 1; \\ \mathcal{L}_{q_s,p'_s}, & \text{if } s = 2. \end{cases} \qquad \text{(A3)}$$

By using this result and Lemma 1, we bound the constants $b_T^2$ as

$$\begin{aligned} & b_T^2 \\ \leq & \sum_k^{T-1}\left\{\prod_{s=1}^{2}\left[\sum_{p_s=1}^{P_T}\sum_{q_s,p_s'} A^{-1}_{p_s,p_s'}A_{q_s,p_s'}\mathcal{B}_{q_s,p_s'}(1+\kappa_4^{\epsilon}\,\mathbb{I}_{s=1})S_{q_s}\left(\frac{k}{T}\right)\right]\right\}+O\left(\frac{1}{T}\right) \\ \leq & \sum_k^{T-1}\left\{\prod_{s=1}^{2}\left[\sum_{p_s=1}^{P_T}\sum_{q_s}\sum_{p_s'} A^{-1}_{p_s,p_s'}A_{q_s,p_s'}\mathcal{B}_{q_s}(1+\kappa_4^{\epsilon}\,\mathbb{I}_{s=1})S_{q_s}\left(\frac{k}{T}\right)\right]\right\}+O\left(\frac{1}{T}\right) \\ = & \sum_k^{T-1}\left\{\prod_{s=1}^{2}\left[\sum_{p_s=1}^{P_T}\sum_{q_s}\delta_{p_s,q_s}\mathcal{B}_{q_s}(1+\kappa_4^{\epsilon}\,\mathbb{I}_{s=1})S_{q_s}\left(\frac{k}{T}\right)\right]\right\}+O\left(\frac{1}{T}\right) \\ = & \sum_k^{T-1}\left\{\prod_{s=1}^{2}\left[(1+\kappa_4^{\epsilon}\,\mathbb{I}_{s=1})\sum_{p_s=1}^{P_T}\mathcal{B}_{p_s}S_{p_s}\left(\frac{k}{T}\right)\right]\right\}+O\left(\frac{1}{T}\right) \end{aligned} \tag{A4}$$

This result shows that the constants $b_T^2$ are of order $O(T)$. □

**Proof of Proposition 1.** From Definition 5, $s^2_{k;T} = \sum_{p=1}^{P_T} L_{p,k;T}$. Hence, for every $d \geq 0$,

$$\left\| s^2_{k;T} - \mathbb{E}\Big(s^2_{k;T} \mid \mathcal{G}^{k+d}_{k-d;T}\Big)\right\|_2 \leq \sum_{p=1}^{P_T}\left\| L_{p,k;T} - \mathbb{E}\Big(L_{p,k;T} \mid \mathcal{G}^{k+d}_{k-d;T}\Big)\right\|_2.$$

If each packet–periodogram sequence satisfies an $L_2$ NED bound of size $-1/2-\varepsilon$, then

$$\left\| s^2_{k;T} - \mathbb{E}\Big(s^2_{k;T} \mid \mathcal{G}^{k+d}_{k-d;T}\Big)\right\|_2 \leq \sum_{p=1}^{P_T}\bar{b}_{p;T}\nu_{p,d;T}.$$

Since $P_T < \infty$ for each $T$ by Assumption 4, the finite sum on the right-hand side preserves the rate $O(d^{-1/2-\varepsilon})$. Therefore, $\{s^2_{k;T}\}_k$ satisfies an $L_2$ NED bound of size $-1/2-\varepsilon$. □

**Proof of Theorem 1.** Using the notation introduced in Definition 7, let

$$\begin{aligned} Q_{p',k,d;T} &= \sum_{|m-k|>d}\sum_{q}\Psi_{q,p'}(k-m)\,\omega_{q,m;T}\,\epsilon_{q,m} \\ \kappa_4^{Q_{p',d}} &= \sup_k \operatorname{cum}_4(Q_{p',k,d;T}) \end{aligned}$$

We also define

$$\begin{aligned} L_{p,k;T} &= \sum_{p'} A^{-1}_{p,p'} I_{p',k;T} \\ &= \sum_{p'} A^{-1}_{p,p'}\left|\sum_{q,m}\Psi_{q,p'}(k-m)\omega_{q,m;T}\epsilon_{q,m}\right|^2 \\ &= \sum_{p'} A^{-1}_{p,p'}\sum_{q_1,q_2}\sum_{m_1,m_2} f_{p',q_1,k,m_1}\, f_{p',q_2,k,m_2}\,\epsilon_{q_1,m_1}\,\epsilon_{q_2,m_2} \end{aligned} \tag{A5}$$

along with the sequence

$$L_{p,k;T} - \mathbb{E}(L_{p,k;T} \mid \mathcal{G}^{k+d}_{k-d;T}) = \sum_{p'} A^{-1}_{p,p'}\left(J^2_{p',k;T} - Z^2_{p',k,d;T} - \bar{Q}^{(2)}_{p',k,d;T}\right) \tag{A6}$$

where $J^2_{p',k;T} = Z^2_{p',k,d;T} + Q^2_{p',k,d;T}$ such that $\mathbb{E}Q^2_{p',k,d;T} = \bar{Q}^{(2)}_{p',k,d;T}$ so that $\mathbb{E}(J^2_{p',k;T}|\mathcal{G}^{k+d}_{k-d;T}) = Z^2_{p',k,d;T} + \bar{Q}^{(2)}_{p',k,d;T}$ and

$$\begin{aligned} Z^2_{p',k,d;T} &= \left| \sum_{|m-k|\le d} \sum_q \Psi_{q,p'}(k-m)\, \omega_{q,m;T}\, \epsilon_{q,m} \right|^2 \\ \bar{Q}^{(2)}_{p',k,d;T} &= \sum_{|m-k|>d} \sum_q \Psi^2_{q,p'}(k-m)\, \omega^2_{q,m;T} \end{aligned} \tag{A7}$$

Since $J$ and $Z$ are square-integrable, the following elementary bound applies by setting $Q^2 = \mathbb{E}(J-Z)^2$; in the present proof this corresponds to $Q^2 = \bar{Q}^{(2)}_{p',k,d;T}$. Then, using $(x-c)^2 \le 2x^2 + 2c^2$ and $(J^2 - Z^2)^2 = (J-Z)^2(J+Z)^2$ with Cauchy–Schwarz, we obtain

$$\mathbb{E}(J^2 - Z^2 - Q^2)^2 \le 2\,\mathbb{E}(J^2 - Z^2)^2 + 2Q^4 \le 4Q^2\,\mathbb{E}(J^2 + Z^2) + 2Q^4. \tag{A8}$$

Define $\widetilde{J}^{(2)}_{p';T} = 2(1+\delta_T/T)\sum_q A_{p',q}\widetilde{S}_q$ and $\widetilde{Q}^{(2)}_{p',d;T} = 2(1+\delta_T/T)\sum_q \widetilde{S}_q \sum_{|g|>d} \Psi^2_{q,p'}(g)$. By construction, $\widetilde{J}^{(2)}_{p';T}$ bounds $\sup_k \mathbb{E}(J^2_{p',k;T} + Z^2_{p',k,d;T})$, and $\widetilde{Q}^{(2)}_{p',d;T}$ bounds $\sup_k \bar{Q}^{(2)}_{p',k,d;T}$. Moreover, Ref. [7] showed that $\{A^{-1}_{p,p'}\}_{p'}$ is a summable sequence. We therefore bound the $L_2$ norm as

$$\begin{aligned} & \mathbb{E}\left| \sum_{p'} A^{-1}_{p,p'} (J^2_{p',k;T} - Z^2_{p',k,d;T} - \bar{Q}^{(2)}_{p',k,d;T}) \right|^2 \\ \le\ & \sum_{p'} (A^{-1}_{p,p'})^2 \sum_{p'} \mathbb{E}(J^2_{p',k;T} - Z^2_{p',k,d;T} - \bar{Q}^{(2)}_{p',k,d;T})^2 \\ \le\ & \sum_{p'} (A^{-1}_{p,p'})^2 \left\{ \sum_{p'} \kappa_4^{Q_{p',d}} + 4\sum_{p'} \bar{Q}^{(2)}_{p',k,d;T} \mathbb{E}(J^2_{p',k;T} + Z^2_{p',k,d;T}) + 2\sum_{p'} (\bar{Q}^{(2)}_{p',k,d;T})^2 \right\} \\ \le\ & \sum_{p'} (A^{-1}_{p,p'})^2 \left\{ \sum_{p'} \kappa_4^{Q_{p',d}} + 4\sum_{p'} \bar{Q}^{(2)}_{p',k,d;T} \widetilde{J}^{(2)}_{p';T} + 2\sum_{p'} (\bar{Q}^{(2)}_{p',k,d;T})^2 \right\} \\ \le\ & \sum_{p'} (A^{-1}_{p,p'})^2 \left\{ \sum_{p'} \kappa_4^{Q_{p',d}} + 4\sum_{p'} \widetilde{Q}^{(2)}_{p',d;T} \widetilde{J}^{(2)}_{p';T} + 2\sum_{p'} (\widetilde{Q}^{(2)}_{p',d;T})^2 \right\} = \nu^2_{p,d;T} \end{aligned} \tag{A9}$$

Using standard cumulant properties, we obtain

$$\kappa_4^{Q_{p',d}} = 2\kappa_4^{\epsilon}(1+\delta_T/T)\sum_q \widetilde{S}^2_q \sum_{|g|>d} \Psi^4_{q,p'}(g) \ge \sum_{|m-k|>d} \sum_q \Psi^4_{q,p'}(k-m)\, \omega^4_{q,m;T}\, \kappa_4^{\epsilon} \tag{A10}$$

It turns out that $\sum_{p'} \kappa_4^{Q_{p',d}} < \infty$, the sequence $\omega^4_{q,m;T}$ being uniformly bounded as a consequence of Assumption 3, and the positive sequence $\{\Psi^4_{q,p'}(k-m)\}_{p'}$ being bounded above by the converging sequence $\{A_{q,p'}\}_{p'}$. Looking at Equations (A6) and (A7), by Assumption 2 we have the uniform tail bounds $\sum_{|g|>d} \Psi^4_{q,p'}(g) = O(d^{-1-2\varepsilon})$ and $\sum_{|g|>d} \Psi^2_{q,p'}(g) = O(d^{-1-2\varepsilon})$. Therefore, $\kappa_4^{Q_{p',d}} = O(d^{-1-2\varepsilon})$ and $\widetilde{Q}^{(2)}_{p',d;T} = O(d^{-1-2\varepsilon})$. The term $2\sum_{p'}(\widetilde{Q}^{(2)}_{p',d;T})^2$ is then $O(d^{-2-4\varepsilon})$ and does not affect the leading rate for $\nu^2_{p,d;T}$. In particular, $\nu^2_{p,d;T} = O(d^{-1-2\varepsilon})$; hence $\nu_{p,d;T} = O(d^{-1/2-\varepsilon})$ and $L_{p,k;T}$ satisfies the $L_2$–NED bound of size $-1/2-\varepsilon$. □

**Proof of Theorem 2.** Set $\widetilde{U}_{k;T} = U_{k;T} - \mathbb{E}U_{k;T}$. Since $\sigma^2(k/T)$ is deterministic, $\widetilde{U}_{k;T}$ inherits the $L_2$–NED bound of the LSSV sequence from Proposition 1. Moreover, by Assumption 1

the innovation array is $\alpha$–mixing with sufficient moments. Therefore, the CLT for NED functions of strong mixing processes (Davidson [8], Theorem 24.6 with Lemma 24.7) applies to $b_T^{-1}\sum_{k=0}^{T-1}\widetilde{U}_{k;T}$ once the standard conditions are verified.

**(a)** By construction, $\mathbb{E}\widetilde{U}_{k;T}=0$ and with $b_T^2=\mathbb{E}\left(\sum_{k=0}^{T-1}\widetilde{U}_{k;T}\right)^2$ we have

$$\mathbb{E}\left(\sum_{k=0}^{T-1}\frac{\widetilde{U}_{k;T}}{b_T}\right)=0,\qquad \mathbb{E}\left(\sum_{k=0}^{T-1}\frac{\widetilde{U}_{k;T}}{b_T}\right)^2=1.$$

The centering error is negligible because $\sup_k|\mathbb{E}U_{k;T}|=O(T^{-1})$ uniformly in $k$ and $b_T\asymp\sqrt{T}$ (Lemma A2); hence, $b_T^{-1}\sum_{k=0}^{T-1}\mathbb{E}U_{k;T}=O(T^{-1/2})\to 0$.

**(b)** From Assumptions 1–4 and standard continuity arguments, for some $r>2$ we have $\sup_{k;T}\|\widetilde{U}_{k;T}/b_T\|_r<\infty$.

**(c)** $\widetilde{U}_{k;T}$ is $L_2$–NED of size $-1/2-\varepsilon$ as a consequence of Proposition 1, and the base innovation process is $\alpha$-mixing by Assumption 1.

**(d)** Lemma A2 implies $b_T\asymp T^{1/2}$.

This completes the proof. □

**Proof of Proposition 2.** Write $b_T^2=\sum_{k=0}^{T-1}\mathbb{E}\,U_{k;T}^2+2\sum_{h\geq 1}\sum_{k=h}^{T-1}\mathbb{E}(U_{k;T}U_{k-h;T})$. Decompose $\hat{b}_T^2-b_T^2=R_{T,1}+R_{T,2}+R_{T,3}$, where $R_{T,1}$ is the empirical-minus-mean part over $1\leq h\leq L_T$, $R_{T,2}$ is the kernel bias from $(w(h/L_T)-1)$ over $1\leq h\leq L_T$, and $R_{T,3}$ is the covariance tail over $h>L_T$. By the $L_2$-NED law of large numbers for the uniformly short-memory quadratic array (see [8]), uniformly over $1\leq h\leq L_T$, $T^{-1}\sup_{1\leq h\leq L_T}\left|\sum_{k=h}^{T-1}(U_{k;T}U_{k-h;T}-\mathbb{E}(U_{k;T}U_{k-h;T}))\right|=o_p(1)$; hence, $R_{T,1}=o_p(T)$ since $L_T/T\to 0$. By local short memory and bounded $w$, both $R_{T,2}=o(T)$ and $R_{T,3}=o(T)$ as $L_T\to\infty$. Therefore, $T^{-1}(\hat{b}_T^2-b_T^2)\xrightarrow{p}0$, and with $\liminf_T b_T^2/T>0$ we conclude $(\hat{b}_T/T)/(b_T/T)\xrightarrow{p}1$ by continuity. □

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