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Closed-Form Expressions and the Asymptotics to Moments for the Excess Gompertz–Makeham Distribution †

Leonid Ostrer, Vladimir Rusev, Alexander Skorikov * and Sergey Kanygin

The Department of High Mathematics, National University of Oil and Gas «Gubkin University», Moscow 119296, Russia; ostrer.l@gubkin.ru (L.O.); rusev.v@gubkin.ru (V.R.); kanygin.s@gubkin.ru (S.K.)

* Correspondence: skorikov.a@gubkin.ru

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Abstract

The Gompertz law published 200 years ago was the main demographic model. It postulated an exponential increase in human survival risk with age. Since then, this model has been successfully used in biology, actuarial science, and demography to describe survival patterns in many animal species (including humans), define insurance policies, and compile actuarial tables. Probabilistic methods for simulating the lifespan of organisms can be applied, such as those related to reliability theory. This paper provides an exact formula for Gompertz–Makeham residual life expectancy and residual variance that can be used for computational convenience. All the structural properties of moments are expressed in a closed form by using a generalized integro-exponential function or Meyer's G function. This allows direct calculation using standard software. In addition, data analysis of complex systems shows that datasets can often be characterized by the behavior of probability distributions for large argument values. The asymptotic expansions of the first and second residual moments are obtained and the error estimate is given. The paper shows that a high-accuracy approximation can be constructed by asymptotic expansions. The results of this article can serve as a tool for applications in the theory of risk, reliability and extreme events.

Keywords: Gompertz distribution; Makeham distribution; asymptotic expansions; mean excess functions; excess variance

1. Introduction

The distribution was introduced by Gompertz in 1825 with the hazard function

$$h(x) = ae^{bx}$$

This distribution has been proposed in demography to describe the structure of human life expectancy, where a constant b is interpreted as the rate of aging, and a constant a as the initial mortality rate. Makeham in 1860 presented the probability distribution as an extension of the Gompertz distribution with the hazard function

$$h(x) = c + ae^{bx}.$$

Here, the constant c is a background constant unrelated to aging, due to age-independent causes. This distribution is called the Gomperz–Makeham distribution now. Reviews on the use of theoretical models in demography are given in the works of Gavrilov and Gavrilova [1]. The Gompertz–Makeham distribution shows good agreement with

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the datasets of device failures, as highlighted by Gavrilova [1–3] and Adewara, Adeyeye, Thron [4], and the time of detection of software faults (Ohishi, Okamura and Dohi [5]. A simple closed form for expressing the remaining lifetime of Gomperz–Makeham was obtained by Missov and Lenart [6,7], though with a typo, and the correct version by Castellares et al. [8,9]. The research on asymptotic expansions of residual moments, which were obtained by authors Rusev and Skorikov [10] for distributions with heavy tails, was continued.

It is of considerable interest for application in the theory of reliability and demography to have the convenient moment's equations for distributions. This article attempts to clarify and supplement the previously obtained results concerning the moments of the residual lifetime of the Gompertz–Makeham model.

2. Representation of Residual Lifetime Distribution and Residual Moments in Closed Form

Let $F_t(x) = (X - t \leq x | X > t)$ be the residual conditional distribution function of the remaining lifetime at age t of a non-negative random variable X with a cumulative distribution function $F(x)$. The Gompertz–Makeham cumulative distribution function has the form

$$F(x; a, b, c) = 1 - e^{-[cx + \frac{a}{b}(e^{bx} - 1)]}. \tag{1}$$

Proposition 1. *Residual cumulative distribution function at age t equals*

$$E[X_t] = \frac{1}{b} e^{\frac{ae^{bt}}{b}} E_s\left(\frac{ae^{bt}}{b}\right); \tag{2}$$

Thus, the Gompertz–Makeham distribution with parameters a, b, c is preserved for the remaining lifetime similar to the exponential distribution, but with a different first parameter ae^{bt}, b, c. In addition, define

$$X_t = (X - t | X > t). \tag{3}$$

The cumulative distribution function of X_t is given by $F_t(x)$. The closed-form expression for moments derived by Castellares et al. [5] is given by

$$E[X^n] = \frac{n!}{b^n} e^{\frac{a}{b}} E_s^{n-1}\left(\frac{a}{b}\right), \tag{4}$$

where $s = \frac{c}{b} + 1$, $E_s^n(z)$ is the generalized integro-exponential function [11]

$$E_s^n(z) = \frac{1}{n!} \int_1^\infty \frac{(\ln t)^n e^{-zt}}{t^s} dt, \quad E_s^0(z) = E_s(z). \tag{5}$$

Hence, the first and second moments are

$$E[X] = \frac{1}{b^1} e^{\frac{a}{b}} E_s^0\left(\frac{a}{b}\right); \tag{6}$$

$$E\left[X^2\right] = \frac{2}{b^2} e^{\frac{a}{b}} E_s^1\left(\frac{a}{b}\right). \tag{7}$$

Next formulas are obtained by using (6) and (7) for the first and second residual moments and the variance according to Proposition 1:

$$E[X_t] = \frac{1}{b} e^{\frac{ae^{bt}}{b}} E_s\left(\frac{ae^{bt}}{b}\right); \tag{8}$$

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$$E\left[X_t^2\right] = \frac{2}{b^2} e^{\frac{ae^{bt}}{b}} E_s^1\left(\frac{ae^{bt}}{b}\right); \tag{9}$$

$$\begin{aligned} D[X_t] &= E\left[X_t^2\right] - (E[X_t])^2 \\ &= \tfrac{2}{b^2} e^{\frac{ae^{bt}}{b}} E_s^1\left(\tfrac{ae^{bt}}{b}\right) - \left[\tfrac{1}{b} e^{\frac{ae^{b\cdot t}}{b}} E_s\left(\tfrac{ae^{bt}}{b}\right)\right]^2. \end{aligned} \tag{10}$$

3. Asymptotic Expansion of MRL and Residual Variance

It is of significant interest to know the behavior the Gompertz–Makeham residual moments for a high age. The mathematical expectation of random variable X_t (MRL) is given by

$$\mu(t) = E[X_t] = \frac{1}{F(t)} \int_t^\infty F(x)dx,$$

with tail $\overline{F}(x) = 1 - F(x)$ of the Gompertz–Makeham cumulative distribution function.

Let

$$\overline{G}(t) = e^{-ct} e^{-\frac{a}{b} e^{bt}}, \tag{11}$$

then $\overline{F}(t) = e^{\frac{a}{b}} \overline{G}(t)$ and

$$\overline{G}(t)\mu(t) = \int_t^\infty \overline{G}(x)dx. \tag{12}$$

Theorem 1. *Let X be a random variable with the three-parameter Gompertz–Makeham distribution, then*

$$\overline{G}(t)\mu(t) = \overline{G}(t)\frac{e^{-bt}}{a} \sum_{k=0}^{n} (-1)^k (c,b)_k \frac{e^{-kbt}}{a^k} + R_{nG}(t), \tag{13}$$

where

$$\begin{aligned} R_{nG}(t) &= (-1)^{n+1} (c,b)_{(n+1)} I_{n+1}(t); \\ I_{n+1}(t) &= \frac{1}{a^{n+1}} \int_t^\infty \left[-e^{-(c+(n+1)b)x}\right] de^{-\frac{a}{b} e^{bx}}, \end{aligned} \tag{14}$$

and the generalized Pochhammer symbol $(c,b)_k$ is defined as

$$(c,b)_0 = 1, \quad (c,b)_k = (c+b)(c+2b)\ldots(c+b+(k-1)b).$$

The residual variance is given by

$$\overline{F}^2(t)\sigma^2(t) = 2\overline{F}(t) \int_t^\infty \overline{F}(x)\mu(x)dx - \overline{F}^2(t)\mu^2(t). \tag{15}$$

The asymptotic expansion for the residual variance without $\mu^2(t)$ follows from (12) and the following equation

$$\frac{d}{dt}\left(\overline{G}(t)\mu(t)\right)^2 = -2 \int_t^\infty \overline{G}^2(x)\mu(x)dx. \tag{16}$$

Thus, we have

$$\overline{G}^2(t)\sigma^2(t) = 2\overline{G}(t)\, J(t) - 2 \int_t^\infty \overline{G}^2(x)\mu(x)dx, \tag{17}$$

where

$$J(t) = \int_t^\infty \overline{G}(x)\mu(x)\, dx. \tag{18}$$

By (13) one can derive expansion (18) for all n, but we assume n = 4 for simplicity.

$$\begin{aligned} \overline{G}(x)\mu(x) = I(x) = \overline{G}(x)\Big[&\tfrac{e^{-bx}}{a} - (c+b)\tfrac{e^{-2bx}}{a^2} \\ &+(c+b)(c+2b)\tfrac{e^{-3bx}}{a^3} - (c+b)(c+2b)(c+3b)\tfrac{e^{-4bx}}{a^4}\Big] + R_4(x), \end{aligned} \tag{19}$$

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where

$$R_4(x) \le \frac{(c+b)(c+2b)(c+3b)(c+4b)}{a^5}\overline{G}(x)e^{-5bx}. \tag{20}$$

Using (19) we see after integration

$$\begin{aligned} J(t) &= \overline{G}(t)\Big[\tfrac{e^{-2bt}}{a^2} - (2c+3b)\tfrac{e^{-3bt}}{a^3} \\ &+((2c+3b)(c+3b)+(c+b)(c+2b))\tfrac{e^{-4bt}}{a^4}\Big] + R_J(t) \end{aligned} \tag{21}$$

with

$$\begin{aligned} \left|R_J(t)\right| &< \left[((2c+3b)(c+3b)+(c+b)(c+2b))(c+4b)\right. \\ &\left.+(c+b)(c+2b)(c+3b)\right]\tfrac{e^{-5bt}}{a^5}\overline{G}(t) \\ &= \left[2(c+b)(c+3b)^2+2(c+4b)(c+2b)^2\right]\tfrac{e^{-5bt}}{a^5}\overline{G}(t). \end{aligned} \tag{22}$$

Thus, the second residual moment has the following exponential expansion

$$\begin{aligned} E(X_t^2) &= 2\Big[\tfrac{e^{-2bt}}{a^2} - (2c+3b)\tfrac{e^{-3bt}}{a^3} \\ &+((2c+3b)(c+3b)+(c+b)(c+2b))\tfrac{e^{-4bt}}{a^4}\Big] + R_E(t), \end{aligned} \tag{23}$$

and the inequality of $R_E(t)$ follows from (22).

Hence, it suffices to prove the expansion of the second term in Equation (17).

Following procedures, such as those above for the first term (17), we derive the expansion

$$\begin{aligned} M(t) &= \int_t^\infty \overline{G}^2(x)\mu(x)dx \\ &= \left[\frac{e^{-2bt}}{2a^2} - 2(c+b)\frac{e^{-3bt}}{2a^3} + (c+b)(3c+5b)\frac{e^{-4bt}}{2a^4}\right]\overline{G}^2(t) + R_M(t), \end{aligned} \tag{24}$$

where

$$\begin{aligned} &R_M(t) \\ &= -M_4(t)\left[(c+b)(c+2b)(3c+5b)+(c+b)(c+2b)(c+3b)\right] \\ &+(c+b)(c+2b)(c+3b)(c+4b)\int_t^{+\infty}\overline{G}(x)I_5(x)dx, \quad I_5(x) < \frac{e^{-5bx}}{a^5}. \end{aligned} \tag{25}$$

Thus, for the four term we have the following asymptotic expansion for the residual variance

$$\begin{aligned} \sigma^2(t) &= \tfrac{e^{-2bt}}{a^2} - 2(c+2b)\tfrac{e^{-3bt}}{a^3} \\ &+\left[2(c+3b)(2c+3b)-(c+b)^2\right]\tfrac{e^{-4bt}}{a^4} + \Delta\sigma^2(t), \\ \Delta\sigma^2(t) &= \tfrac{1}{(G(t))^2}\left(2G(t)R_J(t) - 2R_M(t)\right) < 0. \end{aligned} \tag{26}$$

Numerical Analysis

The function (27) can be computed using Wolfram (Champaign, Illinois, USA) Mathematica (version 13) package. The numerical outcome produced by the software package might include rounding errors for large t. So there is an oscillation of the values calculated by (10). For example, we give some numerical values in Table 1. But there is no such problem for asymptotic expansion (26) (Figure 1).

Table 1. Numerical outcome of closed-form (10) residual variance produced by the software package if $a = 1$, $b = 1$, $c = 0$.

t	3.2	3.3
$D[X_t]$	0.0012	0.0008

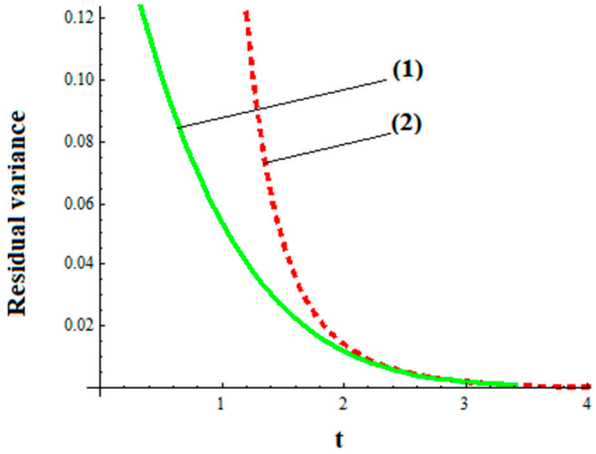

Figure 1. Residual variance graphics for different analytical representation if parameters of distribution are: $a = 1$, $b = 1$, $c = 0$. (**1**)—Residual variance by (10); (**2**)—residual variance using asymptotic expansion (26).

4. Parameter Estimation

In this section the question of the existence and uniqueness of Fischer's classical Maximum likelihood estimates (MLEs) of unknown Gompertz–Makeham distribution parameters is investigated.

Let us denote by a^*, b^*, c^* the true values the Gompertz–Makeham distribution, and $\hat{a}, \hat{b}, \hat{c}$ are MLE parameters determined by sample t_1, t_2,...,t_n. Let $c^* = 0$ (Gompertz case). To obtain the MLE, we write the log-likelihood function:

$$\begin{aligned} l &= \sum_{i=1}^{n}\left[\ln(ae^{bt_i} - \frac{a}{b}(e^{bt_i} - 1)\right] \\ &= n\ln a + b\sum_{i=1}^{n} t_i - \frac{a}{b}\sum_{i=1}^{n} e^{bt_i} + n\frac{a}{b} \end{aligned} \tag{27}$$

and likelihood equations

$$\frac{\partial l}{\partial a} = \frac{n}{a} - \frac{1}{b}\sum_{i=1}^{n}(e^{bt_i} - 1) = 0, \tag{28}$$

$$\frac{\partial l}{\partial b} = \sum_{i=1}^{n} t_i - \frac{a}{b}\sum_{i=1}^{n} t_i e^{bt_i} + \frac{a}{b^2}\sum_{i=1}^{n}(e^{bt_i} - 1) = 0. \tag{29}$$

We have from the first equation

$$a = \frac{nb}{\sum_{i=1}^{n}(e^{bt_i} - 1)}, \tag{30}$$

but the resulting equation

$$\sum_{i=1}^{n} t_i - \frac{n\sum_{i=1}^{n} t_i e^{bt_i}}{\sum_{i=1}^{n}(e^{bt_i} - 1)} + \frac{n}{b} = 0 \tag{31}$$

has no closed-form solution. Thus, the complexity of ML estimates through the numerical method is due to the question of the existence of a solution of the likelihood equations.

Theorem 2. *For parameters $a^* > 0$ and $b^* > 0$, there exists a unique ML estimate; i.e., there is only one extremum of the logarithm of the maximum likelihood function.*

The theorem's proof relies on analyzing the function, which is equal to the left-hand side of Equation (32).

Theorem 3. *Let the parameter $a^* > 0$ be known, and the parameter $b^* > 0$ be unknown. There is a unique solution to the likelihood equation if the condition*

$$t_n > \frac{a^*}{2} t_n^2$$

is true for the sample $t_1, t_2, \ldots, t_n$. Otherwise, there are no solutions.

The proof of the theorem is based on the inequality

$$MX_i > \frac{a^*}{2} MX_i^2 \tag{32}$$

and the strong law of large numbers (also called Kolmogorov's law).

Three parameters of the Gompertz–Makeham model can be estimated numerically with use of the EM algorithm [12].

5. Conclusions

In model selection it is advisable to study the residual moments for a relatively small and large t. The asymptotic expansions and analytical expressions in terms of the special functions of the first residual moment and the residual variance for the Gompertz–Makeham distribution were obtained.

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Abbreviations

The following abbreviations are used in this manuscript:

MLE	Maximum Likelihood Estimate
MRL	Mean Residual Life

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