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Modified Gravity and a Space Probe–Venus Mission

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Abstract: A comparison of gravitational forces and a space probe's trajectory parameters is made for two different models of the sun's field, expressed in Schwarzschild and isotropic coordinates. It is shown that these two representations of a single Schwarzschild solution give, in the tangent space format, different deflections from classical finite trajectories and, hence, from one other; greatly amplified by a planet's (Venus') gravity assist, this effect renders it possible to experimentally specify the format of the gravity law that dominates the solar system.

Keywords: gravity assist; standard flight; Schwarzschild and isotropic coordinates; tangent space

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1. Introduction

For more than a century, Schwarzschild's solution for the equations of general relativity (GR) seems to have been exhaustively analyzed [1–5], and so revisiting the theme may seem an attempt destined to failure; nonetheless, one essential aspect of this solution deserves attention because of possible experiments that aim to specify the gravity law that dominates the solar system.

The current state of the problem may be characterized as the common belief in GR as a correct math embodiment of physical reality, mostly due to a few observational confirmations; anyway, the effects of relativistic gravity are practically considered in the designs of space missions. We should admit, therefore, that the sun quite probably generates at least Schwarzschild's gravitational field (or Kerr's field, if the spin is considered). The question then arises, as to which sample of the static spherically symmetric solution of Einstein's equation is physically realized: one born in Schwarzschild's coordinates or another one rearranged into the isotropic coordinates?

Until recently, this question sounded senseless, due to the widely perceived impossibility of distinguishing between the solution variants for at least two reasons: (i) in the weak-field approximation both metrics look similar; (ii) the "most convincing" GR effect, Einstein's "perihelion shift" [6], is described in both metrics by apparently the same formula. Other "orbital" effects characterizing the GR-caused distortions of trajectories in the solar system have been assessed to be of the order of the sun's gravitational radius ~ 1.47 km, and hence practically unobservable.

Analytical calculations and numerical modeling, however, have demonstrated that a planet's gravity assist (GA)—a frequent element in current space missions—can amplify the deflection of a spacecraft's orbit sufficiently to make the shift observable [7,8]; moreover, the GA sensitivity—the ability to amplify a small change of the probe–planet impact parameter (IP)—turns out to be great enough to detect the sun's specific GR features in general, as well as to identify the type of Schwarzschild field variant.

The deviation of a space probe ballistic trajectory, caused by the peculiarity of the sun's gravity, greatly enhanced by the "GA instrument", can be considered a basic phenomenon for performing a repeated experiment, the results of which can be recorded in the observer's laboratory: this will enable, with a high degree of reliability, clarification of the format of the law of gravity that dominates the solar system.

In this study, we aim to highlight the difference between the sun's GR gravity laws represented in Schwarzschild's and isotropic coordinates, as they must be evaluated by a physical observer, and to suggest a comparative analytical calculation of the parameters of the respective finite-motion trajectories of a space probe, demonstrating the existence and observability of the difference in a virtual experiment.

Section 2 offers a concept of the probe's standard flight, which is given with an explanation of how a planet's GA amplifies a small deflection of the probe's motion from an assigned trajectory; we also deal here with two of Schwarzschild's versions of the sun's gravity law, and suggest the motivation to regard the probe's motion in tangent space. In Section 3 we briefly present all the requisite analytical equations in general metric form, we reduce the motion equations to a Newton-type dynamic shape for both GR gravity laws, and we compare respective types of gravitational radial forces. In addition, we deduce here the analytical formulas of the probe's quasi-elliptic trajectories for two metrics, we calculate the respective differences of the basic trajectory parameters, and we demonstrate that the mismatch can be experimentally observed. Section 4 concludes the study with a compact discussion.

2. Materials and Methods

2.1. The Standard Flight Concept and GA Sensitivity Function

A type of standard trajectory should be chosen, to compare deflections from it. In previous papers [7,8] we suggested such a trajectory satisfying the following setting conditions: a space probe is supposed to move in an ideal model of a planetary system comprising a spherically symmetric star (the sun) and two planets imitating the Earth and Venus, which orbit the star by coplanar circular trajectories (the masses, velocities, and orbit characteristics of the bodies are close to real physical parameters). The motion is described in polar (R, Φ) coordinates, with the sun placed at the initial point.

The standard flight is determined by the following conditions. At the initial time moment $T = 0$ the probe is launched from the Earth's orbit, with coordinates and the velocity components (the velocity is less than the Earth's):

$$R_0 = R_E,\ \Phi_0 = 0;\ V_{R0} = 0,\ V_{\Phi 0} = V_0 < V_E. \tag{1}$$

Under condition (1), the probe starts falling freely in the sun's Newtonian (subscript N) gravity towards Venus by an elliptic trajectory:

$$R = p_N/(1 - e_N \cos \Phi), \tag{2}$$

with the focal parameter p_N and eccentricity e_N determined by the sun's mass and condition (1). Within the framework of the "Patched Conic Approximation" (PCA) method (see, e.g., [9] and many references therein) the probe is subject to GA at the time moment $T_{N.GA}$ in a point $R_{GA} = R_V = p_N$, $\Phi_{N.GA} = 3\pi/2$; the first trajectory part "launch—GA" will be referred to as trek 1. Within the standard flight scheme, the probe approaches Venus 'behind it' thus making the planet's GA accelerate the probe so that the second ballistic trajectory part (trek 2), "GA—final point", returns the probe to the Earth's orbit radius, which determines the trajectory final point (FP) $R_{N.FP} = R_E$, azimuth $\Phi_{N.FP}$, and flight-time T_{FP}, with all these values being found from the equation of the type (2) with trek 2 parameters.

A set of standard flight trajectories is parametrized by the principal GA characteristic, the probe–Venus impact parameter (IP), whose value h must be set (with high precision) in the design of this space mission. To get another standard trajectory, we can variate the h value by a virtual choice of Venus' initial position, while leaving conditions (1) intact. With the help of this scheme, we managed to build an empiric GA sensitivity function $S(h)$ [7,8] which gave the values of the probe's FP deflection under a relatively small IP change dh. The results were surprising: for $h = (16 - 10) \cdot 10^4$ km, $S(h) \cong 10^4 - 10^5$; this means that, e.g., for $h = 10{,}000$ km (4000 km over Venus' surface) an IP change $dh = 1$ km entails the

FP shift of ~82,000 km, which can be detected from Earth (for simplicity, the Earth's gravity is not taken into consideration).

There are many physical factors able to distort trek 1, thus changing a pre-set probe–Venus IP: among them are the sun's oblateness, the pressure of solar radiation, the gravitational impact of celestial bodies, and probably other random factors. The primary candidate to change the classical probe's trajectory, however, is non-Newtonian gravity; Schwarzschild field variants are plausible models. Having built trek 1 in both GR fields, we can therefore evaluate their contribution to changing the classical IP value, and by experimentally measuring the FP shift we can judge with more arguments which type of gravity physically dominates the solar system. In Table 1, we give the basic physical data for the considered planetary model and standard flight; here, we use a system of units (kg, km, s) that is convenient for describing the flight of a small probe in near space.

Table 1. Physical constants and parameters of the planetary model and "standard flight".

Magnitude	Symbol	Units	Value
Gravitational constant	G	$\mathrm{km}^3/(\mathrm{kg}\cdot\mathrm{s}^2)$	$6.6741\cdot10^{20}$
Fundamental velocity	c	km/s	299,792
Sun's mass	M_S	kg	$1.989\cdot10^{30}$
Sun's gravitational radius	r_S	km	1.477
Mass of Venus	M_V	kg	$4.876\cdot10^{24}$
Earth's orbit radius	R_E	km	149,587,816
Earth's orbital velocity	V_E	km/s	29.790
Venus orbit radius	R_V	km	108,207,679
Venus orbital velocity	V_V	km/s	35.026
Probe's initial velocity	$V_0 = V_{0\Phi}$	km/s	25.337
Eccentricity Newtonian	e_N	-	0.276627

2.2. The Gravity Models, Motivation for the Tangent Space, and the Universal Radial Map

In the virtual space experiment, the solar system space–time is modeled by the spherically symmetric interval $ds^2 = g_{00}c^2dt^2 + g_{11}dr^2 + g_{22}d\varphi^2 + g_{33}d\theta^2$ (Schwarzschild solution), the probe moving in the ecliptic plane $\theta = \pi/2$, $d\theta = 0$. We consider two versions of the sun's GR gravity, in Schwarzschild coordinates and in isotropic coordinates; both metrics are given here in exact-solution form and in the weak-field approximation (g_{00} down to the second small order if it exists) under condition $r_S/r \sim 10^{-8} << 1$.

The space–time metric in Schwarzschild coordinates is

$$g_{00} = 1 - \frac{2r_S}{r},\ |g_{11}| = \left(1 - \frac{2r_S}{r}\right)^{-1} \cong 1 + \frac{2r_S}{r},\ |g_{22}| = r^2, \tag{3}$$

and the metric in isotropic coordinates is (see, e.g., [5])

$$g_{00} = \left(\frac{1 - r_S/2r}{1 + r_S/2r}\right)^2 \cong 1 - \frac{2r_S}{r} + \frac{2r_S{}^2}{r^2},\ |g_{11}| = \left(1 + \frac{r_S}{2r}\right)^4 \cong 1 + \frac{2r_S}{r},\ |g_{22}| = \left(1 + \frac{r_S}{2r}\right)^4 r^2 \cong \left(1 + \frac{2r_S}{r}\right)r^2. \tag{4}$$

Equations (3) and (4) have close weak-field approximations, but their space–times have different topology; we look for the relevant difference in the gravitational forces and the probe's motion, analyzing solutions of the geodesic equation. To perform the analysis, many studies have been done only in holonomic variables [1,5,6]; however, one readily demonstrates that a specific (additional) GR term appearing in the dynamic equation written in holonomic variables $(t,\ r,\ \varphi)$ *weakens* the classical gravitational force for both metrics, which appears to contradict the GR perihelion shift *forward*.

Having in mind a real experiment, we can therefore accept here the technique (close to the tetrad approach) of implying the use of physical magnitudes observed and measured experimentally in the locally flat tangent space (TS). The non-holonomic differentials of TS variables are $dT \equiv \sqrt{g_{00}}dt$, $dR \equiv \sqrt{|g_{11}|}dr$, $d\Phi \equiv \sqrt{|g_{22}|}d\varphi$. Apart from the differentials, there is a radial variable that needs an appropriate mapping from the holonomic to the tangential embodiment. For metrics (3) and (4) the tangent radial coordinate is determined similarly:

$$R = \int \sqrt{|g_{11}|}dr \cong \int (1 + r_S/r)dr \cong r[1 + (r_S/r)ln(r/r_0)],\ r_0 = const. \tag{5}$$

L'Hospital's rule indicates that for $r \to \infty$ the small term in Equation (5) drops as r_S/r. With this fact in mind, we consider the following motivation for a simplified "tangent—holonomic radial map", acceptable on the experiment scale. The probe travels between the orbits of Mercury and Mars; Equation (5) gives the difference between the tangent W and holonomic w width of the ring $W - w = r_S ln(R_{Mar}/R_{Merc}) \cong 1.46 r_S$. For an even smaller inner ring radius, the tangent and holonomic widths differ only in units of r_S in agreement with the earlier estimates (see, e.g., [2,5]); therefore, we suggest an approximate tangent–holonomic *universal radial map*:

$$R \cong r(1 + r_S/r) \to r \cong R(1 - r_S/R),\ 1/r \cong (1/R)(1 + r_S/R) \tag{6}$$

which is identical for both metrics; thus, we intentionally "minimize" the GR gravity impact on the whole experimental area.

2.3. Dynamic and Trajectory Equations in General Form

In this section, we represent in general form the probe's dynamic equations, motion integrals, and trajectory equation in holonomic and TS formats (without detailed calculations which are elementary). The geodesic equation is used in the form convenient to solve

$$\frac{d}{ds}\left(g_{\mu\nu}\frac{dx^\nu}{ds}\right) = \frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}. \tag{7}$$

For any static spherically symmetric metric, Equation (7) has the relativistic energy and angular momentum integrals

$$\varepsilon \equiv 1 + E/c^2 = g_{00}\frac{cdt}{ds} = \sqrt{\frac{g_{00}}{1 - V^2/c^2}} = const,\ l \equiv L/c = |g_{22}|\frac{d\varphi}{ds} = \varepsilon\sqrt{\frac{|g_{22}|}{g_{00}}}\frac{d\Phi}{dT} = const. \tag{8}$$

where $E,\ L$ are the physical constants, the probe's specific (per unit mass) energy, and angular momentum, and V is the probe's velocity modulus. The radial and azimuthal components of Equation (7) are

$$\frac{d^2R}{dT^2} = -\frac{\partial_r g_{00}c^2}{2g_{00}\sqrt{|g_{11}|}} + \frac{\partial_r|g_{22}|}{2|g_{22}|\sqrt{|g_{11}|}}R^2\left(\frac{d\Phi}{dT}\right)^2,\ \frac{d\Phi}{dT} = \frac{cl}{\varepsilon R}\sqrt{\frac{g_{00}}{|g_{22}|}}. \tag{9}$$

The trajectory equation in holonomic and TS formats is

$$\frac{d\varphi}{dr} = \frac{l}{-g_{22}}\left[\frac{1}{-g_{11}}\left(\frac{\varepsilon^2}{g_{00}} - 1 + \frac{l^2}{g_{22}}\right)\right]^{-1/2},\ \frac{d\Phi}{dR} = \frac{l}{R\sqrt{|g_{22}|}\sqrt{|g_{11}|}}\left[\frac{1}{|g_{11}|}\left(\frac{\varepsilon^2}{g_{00}} - 1 - \frac{l^2}{|g_{22}|}\right)\right]^{-1/2}. \tag{10}$$

We note that the integration of the first Equation (10) with the subsequent transition to the TS format gives, in the accepted approximation, the same result as the straight integration of the second Equation (10); it can be verified by direct computation.

3. Results

3.1. Motion Integrals and Comparison of GR Forces

Firstly, for cases (3) and (4) we need to compute the integrals (8) entering the second dynamic Equation (9); then, for each case, we can represent Equation (10) in the Newtonian form and compare the respective radial forces acting on the probe. We also use the map (6) everywhere.

3.1.1. Schwarzschild Coordinates

In case (3), the probe's energy and angular momentum constants (8) have the TS-format (to the first order of infinitesimals):

$$E_{Sch} = (\varepsilon - 1)c^2 \cong E_N + \frac{1}{2c^2}\left(\frac{3V_0{}^4}{4} - \frac{3r_S{}^2c^2}{R_E{}^2} - \frac{r_S V_0{}^2}{R_E}\right),\ L_{Sch} = lc \cong L_N\left(1 + \frac{E_N}{c^2}\right), \quad (11)$$

the respective classical characteristics being $E_N = V_0{}^2/2 - GM_S/R_E$, $L_N = R^2 d\Phi/dT = R_E V_{\Phi 0}$. All quantities are evaluated at the launch point (1). The radial motion equation is the azimuth time derivative replaced by angular momentum:

$$\left.\frac{d^2R}{dT^2}\right|_{Sch} \cong -\frac{r_S c^2}{R^2} - \frac{3r_S{}^2c^2}{R^3} + \frac{L_N{}^2}{R^3}. \quad (12)$$

We stress that in Equation (12) $L_N = const$ as in the second Equation (11) $E_N \cong const$.

3.1.2. Isotropic Coordinates

In the second case (4) the motion constants have TS format:

$$E_{iso} \equiv (\varepsilon - 1)c^2 \cong E_N + \frac{1}{2c^2}\left(\frac{3V_0{}^4}{4c^2} - \frac{r_S{}^2c^2}{R_E{}^2} - \frac{V_0{}^2 r_S}{R_E}\right),\ L_{iso} = lc \cong L_N\left(1 + \frac{E_N}{c^2} + \frac{r_S}{R_E}\right); \quad (13)$$

note that here $L_N \neq const$. The radial component of the respective motion equation is

$$\left.\frac{d^2R}{dT^2}\right|_{iso} \cong -\frac{r_S c^2}{R^2} - \frac{r_S{}^2c^2}{R^3} + \frac{L_N{}^2}{R^3}\left(1 - \frac{r_S}{R}\right). \quad (14)$$

3.1.3. Analysis and Comparison

For both metrics, Equations (12) and (14) are the Newton-type dynamic equations written in the *chasing frame*, i.e., in a rotating (non-inertial) frame, where its radial vector always points onto the probe; therefore, apart from pure gravitational forces (the first two terms on the right-hand side of the radial dynamic equations), the centrifugal force terms appear. We state also that in both cases the additional GR terms make gravity forces *greater* than the classical attraction. The classical gravitation (specific) force and its GR additions, "Schwarzschild's" and "isotropic", are evaluated at the launch point as follows:

$$f_N = \frac{r_S c^2}{R_E{}^2} = 5.93 \cdot 10^{-6}\mathrm{km/s^2},$$

$$\delta f_{Sch} = \frac{3r_S{}^2c^2}{R_E{}^3} = 1.75 \cdot 10^{-13}\mathrm{km/s^2},\ \delta f_{iso} = \frac{r_S{}^2c^2}{R_E{}^3} + \frac{r_S L_N{}^2}{R_E{}^4} = 1.01 \cdot 10^{-13}\mathrm{km/s^2}.$$

The most precise (and visualized) difference between gravities (3) and (4) can be assessed in the numerical construction of the probe's treks under the action of forces f_{Sch} and f_{iso}, which deserves a separate study.

3.2. Trajectory Equations and Comparison of GR Shifts of the Gravity Assist Points

3.2.1. Schwarzschild Coordinates

The first Equation (10) in holonomic coordinates has the integral form:

$$\varphi_{Sch} = \int \frac{l dr}{r^2\sqrt{\varepsilon^2 - 1 + \frac{2r_S}{r} - \frac{l^2}{r^2} + \frac{2r_S l^2}{r^3}}}.$$

Transition to the tangent–space azimuth $\varphi_{Sch} \cong (1 - r_S/r)\Phi_{Sch}$, and change of the integration variable $r = R - r_S$ (here coinciding with the map (6)), yields the tabular integral in TS format:

$$\Phi_{Sch} \cong \left(1 + \frac{r_S{}^2}{l^2}\right) \int \frac{\left(1 + \frac{r_S}{R}\right) dR}{R^2 \sqrt{\frac{\varepsilon^2 - 1}{l^2 - 2r_S{}^2} + \frac{2r_S}{(l^2 - 2r_S{}^2)R} - \frac{1}{R^2}}}; \tag{15}$$

with an appropriate choice of the integration constant, the integral (15) gives a quasi-elliptic trajectory equation:

$$\Phi_{Sch} \cong \left(1 + \frac{2r_S}{p_N}\right) \arccos \frac{1 - p_{Sch}/R}{e_{Sch}} + r_S \sqrt{\frac{\varepsilon^2 - 1}{l^2} + \frac{2r_S}{l^2 R} - \frac{1}{R^2}}, \tag{16}$$

where the focal parameter and eccentricity are expressed through the motion constants (11):

$$p_{Sch} \equiv \frac{l^2}{r_S} - 2r_S,\ e_{Sch} \cong \sqrt{1 + \frac{(\varepsilon^2 - 1)(l^2 - 2r_S{}^2)}{r_S{}^2}}. \tag{17}$$

One can show that at the gravity-assist point, $R_{GA} = R_V = p_N$, the last term in Equation (16) (an extra rotation of the ellipse) is approximately a 1.5 order smaller than the other GR corrections; therefore, it can be neglected here. The azimuth of the gravity-assist point (16) then breaks down into a sum:

$$\Phi_{GA.Sch} \cong \frac{2r_S}{p_N} \frac{3\pi}{2} + \arccos \frac{1 - p_{Sch}/p_N}{e_{Sch}} \Phi_{sq} \equiv \Delta\Phi_{Ein} + \Phi_{sq}. \tag{18}$$

The first small term is the Einstein-type precession ("perihelion shift"):

$$\Delta\Phi_{Ein} = 3\pi r_S/p_N \tag{19}$$

and the second term is the GA azimuth on only the GR-distorted (in fact, squeezed) ellipse; we expect this to be close to the classical GA azimuth $\Phi_{sq} = 3\pi/2 + \Delta\Phi_{sq}$, so the geometric distortion adds to the GA azimuth a small angle:

$$\Delta\Phi_{sq} \cong \cos(3\pi/2 + \Delta\Phi_{sq}) \cong \frac{p_N - p_{Sch}}{p_N e_N}, \tag{20}$$

the focal parameter p_{Sch} being found from the second Equations (11) and the first Equation (17); within standard flight data given above, it turns out a few km shorter than the classical focal parameter $\Delta p_{Sch} \equiv p_N - p_{Sch} \approx 4$km. Both GR corrections (19) and (20) are positive, i.e., the probe (approaching behind Venus) crosses its trajectory closer to the planet; hence, the GR-affected impact parameter is shorter by the value:

$$\Delta h_{Sch} = \left(\Delta\Phi_{Ein} + \Delta\Phi_{sq}\right) p_N = 3\pi r_S + \frac{\Delta p_{Sch}}{e_N} \approx 14\text{ km} + 16\text{ km} = 30\text{ km}. \tag{21}$$

Apart from precession and geometric distortion of the trajectory, there is one more factor making the GR-caused IP shorter than in the classical case: the probe's "early arrival" to the GA point, traveling along a squeezed (hence, shorter) orbit. Using the method given in detail in [7], we find that within the standard flight scheme (in the case of the Schwarzschild coordinate) the probe crosses the Venus orbit in the vicinity of the GA point $\Delta T_{N-Sch} \sim 0.25$ s earlier than in the classical case; hence, the distance of probe–Venus (with the relative azimuthal velocity ~35 km/s) additionally is contracted in $\Delta h_{T.Sch} = V_V \Delta T_{N-Sch} \cong 8$ km. Together with the shift (21), we get nearly 38 km of the GR-caused IP contraction; at the IP value $h \sim 10{,}000$ km, this should entail more than 3.1 million km deflection of the probe's final point from its classical position.

3.2.2. Isotropic Coordinates

In this case, the holonomic azimuth is equal to that of the tangent space $r(1 + r_S/r)d\varphi \cong Rd\Phi$, so it is convenient to use the TS trajectory formula—the second Equation (10) and the integral following from it:

$$\Phi_{iso} \cong \int \frac{l\left(1 - \frac{r_S}{R}\right)dR}{R^2\sqrt{\varepsilon^2 - 1 + \frac{2r_S}{R} - \frac{l^2}{R^2}\left(1 - \frac{2r_S}{R}\right)}}.$$

The variable substitution $R = R' - r_S$ makes it integrable (precisely as in Equation (15)):

$$\Phi_{iso} \cong \left(1 + \frac{{r_S}^2}{l^2}\right)\int \frac{(1 + r_S/R')dR'}{R'^2\sqrt{\frac{(\varepsilon-1)^2}{l^2 - 2{r_S}^2} + \frac{2r_S}{R'(l^2 - 2{r_S}^2)} - \frac{1}{R'^2}}}. \tag{22}$$

This yields the trajectory equation (written already in original TS radial coordinate R):

$$\Phi_{iso} \cong \left(1 + \frac{2r_S}{p_N}\right)\arccos\frac{1 - p_{iso}/R}{e_{iso}} + r_S\sqrt{\frac{\varepsilon^2 - 1}{l^2} + \frac{2r_S}{l^2 R} - \frac{1}{R^2}}, \tag{23}$$

where the last term in Equation (23) is again too small (and ignored); the eccentricity is given by the second Equation (17) but expressed through the motion constants (13). The Einstein-type precession here is the same as in the previous case, but the focal parameter is quite different: here, it turns out variable $p_{iso} \equiv l^2/r_S - 2r_S - p_N r_S/R$, and (to minimize as usual the GR impact) we calculate it at the launch point (the angular momentum determined by the second Equation (13)); it appears to be longer than p_{Sch} but still shorter than the classical focal parameter $\Delta p_{iso} \equiv p_N - p_{iso} \approx 3$ km, i.e., in the isotropic coordinates the elliptic trajectory is less squeezed than in the case of the Schwarzschild coordinates. The full GA azimuth shift is found by formulas similar to Equations (19) and (20), and the respective GR-caused contraction of the impact parameter is

$$\Delta h_{iso} = \left(\Delta\Phi_{Ein} + \Delta\Phi_{sq}\right)p_N = 3\pi r_S + \Delta p_{iso}/e_N \approx 14\text{ km} + 12\text{ km} = 26\text{ km}. \tag{24}$$

The "early arrival" time difference, in this case, is about $\Delta T_{N-iso} \sim 0.12$ s: hence, an additional "IP-loss" of $\Delta h_{T.iso} = V_V \Delta T_{N-iso} \cong 4$ km, the total IP contraction thus being ~30 km. In the standard flight scheme (with $h \sim 10{,}000$ km) this must cause about 2.4 million km deflection of the probe's classical FP position. This deflection seems to be quite well experimentally detected by an Earth observer. Similarly, a smaller but also great distance, of nearly 650,000 km, is experimentally observable, distinguishing the final point positions of the probe moving under identical physical standard flight conditions in different GR gravities modeled by the Schwarzschild field in the different coordinate systems discussed here. Some results of the calculations are in Table 2.

Table 2. Analytically calculated differences of the probe's trajectories parameters in the sun's GR gravities, modeled by the metric in Schwarzschild and isotropic coordinates. The final points distance is evaluated for the probe's altitude of ~4000 km over Venus' surface (h = 10,000 km).

Magnitude	Units	Equation/Source	Value
Focal parameter iso-Sch difference	km	$\Delta p = p_{iso} - p_{Shc}$	1.07
Eccentricity iso-Sch difference	–	$\Delta e = e_{iso} - e_{Shc}$	$-0.714 \cdot 10^{-8}$
IP geometric iso-Sch difference	km	$\Delta h_{geom} = \Delta h_{iso} - \Delta h_{Sch}$	3.86
GA-time iso-Sch difference	s	$\Delta T = T_{T.iso} - T_{T.Sch}$, [7]	0.13
IP GA-time iso-Sch difference	km	$\Delta h_T = \Delta h_{T.iso} - \Delta h_{T.Sch}$	4.2
IP total iso-Sch difference	km	$\Delta h = \Delta h_{geom} + \Delta h_T$	8.06
Sensitivity function (h = 10,000 km)	–	$S_{10{,}000}$ [8]	81,857
Final points iso-Sch distance	km	$\Delta l_{FP} = S_{10{,}000}\Delta h$	659,767

4. Conclusions and Discussion

Summarizing the obtained results, we make the following conclusion. Three principal GR gravity factors change (decrease) the probe–Venus classical IP in the standard flight model: a quasi-elliptic orbit precession (Einstein's "perihelion shift"); the ellipse compression (squeeze); and the early arrival at the GA point. Table 2 demonstrates the importance of the last two factors changing trek 1 in the sun's GR gravity modeled by the Schwarzschild solution in Schwarzschild and isotropic coordinates, respectively. Venus' gravity assist greatly amplifies (~10^5 times) emerging relatively small IP changes, so that the distance between the probe's final points (achieved by equal times) in different gravities can reach more than a half million *km*, an experimentally observable length; moreover, since this result is obtained under the assumption of minimal GR effect in the holonomic—TS mapping, the observed effect may be even stronger. A thoroughly arranged space experiment (obligatory comprising GA) may therefore not only detect a difference between the classical and GR gravities of the sun, but may also indicate the type of GR gravity; formerly, this seemed an unsolvable problem. In his fundamental work [5] Weinberg enumerates several physical reasons that affect a body's orbit, causing, e.g., precession much bigger than GR: some of them are mentioned in Section 2 of this paper (except the Earth-based observation error); fortunately, however, all these effects can be precomputed with the help of well-known methods of classical mechanics. As to the error of observations from the Earth laboratory, one can note that the suggested experiment is focused on the results of the gravitational interaction of the probe–Venus, and there is no need to bother about the observation of the small deviations of the probe's orbit; one needs to observe only the last great deflection of the probe's final point (not far from the Earth's orbit).

It is necessary, nonetheless, to note that the analytical method used here (including PCA) gives only approximate results. As is mentioned in in Section 3 (Section 3.1.3), greater precision—and visualization—is provided by a point-by-point construction of probe trajectories on the basis of a fine numerical solution of Newton-type equations, with an adequately defined force function for each type of GR gravity; we expect this task to be realized. We also note that the discussed model of the gravitational experiment may help to specify the parameters of the generalized theories of gravity (e.g., Brans–Dicke theory).

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